Applications of the Contraction Mapping Theorem: Toral actions

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Γ is a finitely generated group.

Question. What are the "essential" actions of Γ on a compact manifold M? On a compact metric space X?

Strategy. For an essential action, find relations between

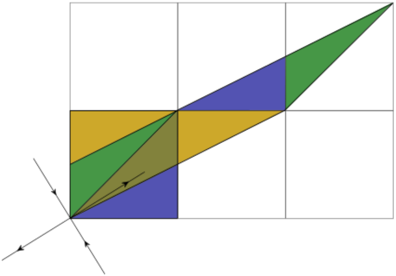
- The algebraic properties of Γ (e.g. nilpotent, higher rank, etc)
- The dynamical properties of the action (e.g. minimal, ergodic, expansive, positive entropy, etc)
- The cohomological properties of the action.

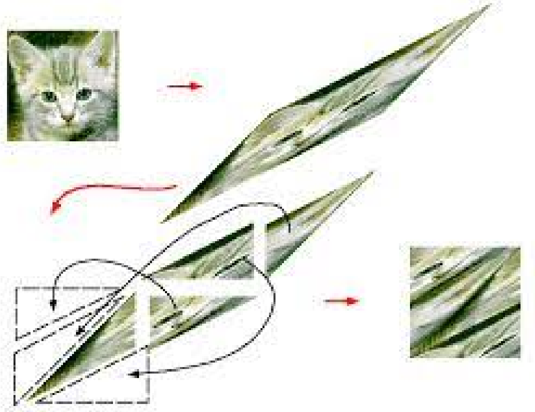
Two cases that are better understood:

- The action is by isometries
- The action has some aspects of hyperbolic behavior

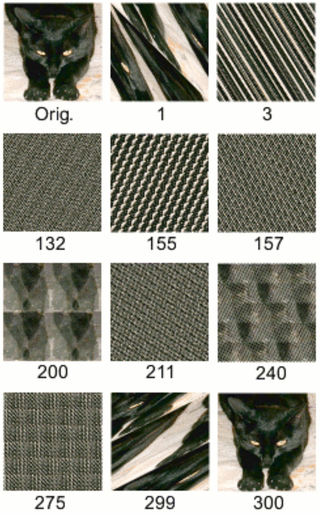
For $A \in SL(n, \mathbb{Z})$ get transformation $\phi_A \colon \mathbb{R}^n \to \mathbb{R}^n$ which restricts to $\phi_A \colon \mathbb{Z}^n \to \mathbb{Z}^n$. The quotient $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the standard n-torus, and we get induced map $\phi_A \colon \mathbb{T}^n \to \mathbb{T}^n$.

Arnold's Cat Map. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\phi_A \colon \mathbb{T}^2 \to \mathbb{T}^2$





From order to chaos and back. Use a sample mapping on a picture of 150x150 pixels. The number shows the iteration step; after 300 iterations, the original image returns.



Proposition. The periodic points of $\phi_A \colon \mathbb{T}^n \to \mathbb{T}^n$ are dense.

Proof. Let $\mathbb{Z}[1/m] \subset \mathbb{Q}$ denote the rational points with denominator $1/m$. Then we obtain an induced map

$$\phi_A \colon \mathbb{Z}[1/m]^n/\mathbb{Z}^n \to \mathbb{Z}[1/m]^n/\mathbb{Z}^n$$

ϕ_A acts as permutation of the finite set $\mathbb{Z}[1/m]^n/\mathbb{Z}^n$.

$\Rightarrow$ a finite power of ϕ_A fixes this set.

$\mathbb{T}^n_{\mathbb{Q}} = \mathbb{Q}^n/\mathbb{Z}^n \subset \mathbb{T}^n$ is dense

$\Rightarrow$ periodic points of ϕ_A are dense.

Definition. A diffeomorphism $f: M \to M$ is *Anosov* if there exists a direct sum decomposition $TM = E^+ \oplus E^-$ where

- Df uniformly expands the distribution E^+
- Df uniformly contracts the distribution E^-

That is, there exists $\lambda > 1$ such that

- for all $\vec{v} \in E^+$, $Df(\vec{v}) \in E^+$ and $\|Df(\vec{v})\| \geq \lambda \|\vec{v}\|$
- for all $\vec{v} \in E^-$, $Df(\vec{v}) \in E^-$ and $\|Df(\vec{v})\| \leq \lambda^{-1} \|\vec{v}\|$

$\Rightarrow$ the distributions E^+ and E^- are uniquely integrable, defining foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ of M. The leaves of these foliations are *smoothly immersed submanifolds*.

Definition. $A \in SL(n, \mathbb{Z})$ is hyperbolic $\Leftrightarrow$ all eigenvalues $|\lambda| \neq 1$.

Observation. $A \in SL(n, \mathbb{Z})$ hyperbolic $\Leftrightarrow$ $\phi_A: \mathbb{T}^n \to \mathbb{T}^n$ is Anosov.

Definition. A smooth action $\phi\colon \Gamma \times M \to M$ is *Anosov* if there exists $\gamma \in \Gamma$ such that $\phi(\gamma)$ is Anosov diffeomorphism of M.

Revised Problem. Classify the Anosov actions of Γ on M.

Conjecture. Let $f : M \to M$ be Anosov action, then M is an infra-nil manifold. That is, $\Lambda = \pi_1(M, x)$ has a nilpotent subgroup of finite index and the universal covering $\widetilde{M}$ is contractible.

Definition. $x \in M$ is *wandering* for action $f : M \to M$ if there is an open neighborhood $x \in U$ such that the translates of U are disjoint, and is *non-wandering* otherwise.

Theorem. If $f : M \to M$ be Anosov action and the non-wandering set $\Omega(f) = M$, then the conjecture is true.

This motivates the working assumption that $\phi \colon \Gamma \times M \to M$ is an Anosov action and M is a nil-manifold. In fact, let $M = \mathbb{T}^n$.

For $\gamma \in \Gamma$, we have $\phi(\gamma)_* \in \mathbf{Aut}\{H_1(\mathbb{T}^n;\mathbb{Z})\} \subset \mathbf{Aut}\{H_1(\mathbb{T}^n;\mathbb{R})\}$ which gives an affine representation

$$\rho \colon \Gamma \to \mathbf{Aut}\{H_1(\mathbb{T}^n;\mathbb{R})/H_1(\mathbb{T}^n;\mathbb{Z})\} \subset \mathbf{Homeo}(\mathbb{T}^n)$$

This is called the *standard action* for ϕ.

Problem. Find conditions on an Anosov action $\phi \colon \Gamma \times M \to M$, for M a nil-manifold, which are sufficient to imply there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that the restriction of ϕ to Γ' is conjugate to the standard action of Γ'.

Theorem. Let $f\colon \mathbb{T}^n \to \mathbb{T}^n$ be Anosov, then f is *topologically* conjugate to a linear hyperbolic automorphism ϕ_A.

⋆ J. Franks, *Anosov diffeomorphisms on tori*, **Trans. Amer. Math. Soc.**, 145:117–124, 1969.

⋆ A. Manning, *There are no new Anosov diffeomorphisms on tori*, **American Jour. Math.**, 96:422–429, 1974.

Assume we have an Anosov action $\phi\colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$, then for some $\gamma \in \Gamma$, the action $\phi(\gamma)$ is conjugate to a hyperbolic $\rho(\gamma) \in \mathbf{Aut}(\mathbb{T}^n)$

Does this imply that the full action ϕ is a standard linear action?

Theorem. [Folkert Tangerman, 1990] There exists an analytic family $\{\varphi_t \mid 0 \leq t \leq 1\}$ of volume-preserving real analytic actions of $SL(2,\mathbb{Z})$ on $\mathbb{T}^2$, with $\varphi_0 = \varphi$ the standard action, such that the *action* φ_t is not topologically conjugate to φ for all $0 < t \leq 1$.

Observe that the deformed actions are Anosov.

Sketch of construction.

Lemma. For the generators of $SL(2,\mathbb{Z})$:

- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$
- A has order 4, B has order 6, and $A^2 = B^3 = -I$.
- $SL(2,\mathbb{Z})$ is isomorphic to the amalgamated product

$$SL(2,\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$$

generated by $\{A, B\}$.

Let $T\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the translation action.

Let $\vec{Z}_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ be the rotational vector field about the origin.

Let $\psi\colon [0,1] \to [0,1]$ be a smooth function such that $\psi(0) = 1$, $\psi(s) \geq 0$ for all s, and $\psi(s) = 0$ for $s \geq 10^{-4}$.

Define the divergence-free vector field $\vec{Z}_\psi = \psi(x^2 + y^2) \cdot \vec{Z}_1$.

Form the translate $Z_+ = DT_{(1/2,0)}(Z_\psi)$ of the vector field $\vec{Z}_\psi$, centered at the point $(1/2, 0) \in \mathbb{R}^2$, and the vector fields

$$Z_- = D(\phi_A^2)(Z_+) = D(\phi_{-I})(Z_+) \quad , \quad Z = Z_+ + Z_-$$

Note that $D(\phi_A^2)(Z) = Z$.

Form the infinite sum

$$\widetilde{Z} = \sum_{(a,b)\in\mathbb{Z}^2} DT_{(a,b)}(Z)$$

which is well-defined since the supports of the translates are disjoint. $\widetilde{Z}$ is invariant under the action of $\mathbb{Z}^2$.

Let $\xi(t)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of $\widetilde{Z}$, then for $(a,b) \in \mathbb{Z}^2$,

$$\xi(t) \circ \phi_A^2 = \phi_A^2 \circ \xi(t) \quad , \quad T_{(a,b))} \circ \xi(t) = \xi(t) \circ T_{(a,b)}$$

Let $\widetilde{\xi}(t)\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the induced flow. Set

$$\phi_t(A) = \widetilde{\xi}(t)^{-1} \circ \phi_A \circ \widetilde{\xi}(t) \quad , \quad \phi_t(B) = \phi_B$$

Then $\phi_t(A)^2 = \phi_{-I}$ so we get an action $\phi_t\colon SL(2,\mathbb{Z}) \times \mathbb{T}^2 \to \mathbb{T}^2$.

Proposition. If there exists a homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ conjugating the action ϕ_t to ϕ_1 then $t = 0$.

The proof uses fundamental domains for the actions of A and B.

There is a companion result:

Theorem. Let $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism which conjugates the actions $\phi, \phi'\colon SL(2,\mathbb{Z}) \times \mathbb{T}^2 \to \mathbb{T}^2$, where ϕ is the standard action. Then h is smooth.

$\star$ Elise Cawley, *The Teichmüller space of the standard action of* $\mathrm{SL}(2,\mathbb{Z})$ *on* $\mathbb{T}^2$ *is trivial*, **Internat. Math. Res. Notices**, International Mathematics Research Notices, 7:135–141, 1992.

The key to the above example is that $H^1(SL(2,\mathbb{Z});\mathbb{R}) \neq 0$, which is used implicitly in the construction of the deformation.

"The notion of hyperbolicity has proved the key to questions of stability in the cases of actions by $\mathbb{Z}$ or $\mathbb{R}$. For actions by other groups, we would hope to find conditions analogous to hyperbolicity in the sense that they facilitate analysis to a comparable extent. However, we should not expect these conditions to resemble hyperbolicity too closely, for they should reflect the algebra of the particular group being studied."

$\star$ D. Stowe *Stable orbits of differentiable group actions*, **Trans. Amer. Math. Soc.**, 277:665–684, 1983.

Theorem: Let $\phi\colon \Gamma \times M \to M$ be a smooth action with isolated fixed-point $x \in M$. Suppose that $H^1(\Gamma; \mathbb{V}_\phi) = 0$ for all finite dimensional modules over the action of $D_x\phi$. Then for a nearby action ϕ' there is an isolated fixed-point x' near to x.

$\star$ D. Stowe, *The stationary set of a group action*, **Proc. Amer. Math. Soc.**, 79:139–146, 1980.

Definition.[SVC] Γ satisfies the *strong vanishing cohomology* condition if $H^1(\Gamma'; \mathbb{R}^N_{\rho'}) = \{0\}$ for every subgroup $\Gamma' \subset \Gamma$ of finite index and representation $\rho'\colon \Gamma' \to GL(N, \mathbb{R})$, $N \geq 1$.

Γ satisfies SVC $\Longrightarrow$ $\Gamma' \subset \Gamma$ finite index satisfies SVC.

Theorem.[Margulis] Let $\Gamma \subset G$ be an irreducible lattice in a connected semi-simple algebraic $\mathbb{R}$-group G. Assume that the $\mathbb{R}$-split rank of each factor of G is at least 2, and that $G^0_{\mathbb{R}}$ has no compact factors. Then Γ satisfies condition SVC.

⋆ Theorem 2.1 in G. A. Margulis, **Discrete Subgroups of Semisimple Lie Groups**. Springer-Verlag, 1991.

Example. For $n \geq 3$, a lattice $\Gamma \subset SL(n, \mathbf{Z})$ satisfies SVC.

Theorem. Let $\phi\colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$, for $n \geq 3$ and $\Gamma \subset SL(n, \mathbb{Z})$ finite index. Then a deformation ϕ_t of ϕ is smoothly conjugate to ϕ_t.

The proof has three parts, and uses:

- Action is Anosov and the periodic points of the action are dense.
- Γ satisfies SVC condition so the periodic points are stable.
- Γ contains maximal abelian semi-simple subgroup, hence the action is *trellised*, and so the conjugation must be smooth.

$\star$ S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, **Annals of Math. (2)**, 135:361-410, 1992.

Step 1. Let $\phi_t \colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$ be a 1-parameter family of actions.

Let $\gamma_0 \in \Gamma$ so that $\phi_0(\gamma_0)$ is Anosov, hence a hyperbolic matrix.

The diffeomorphism $\phi_0(\gamma_0)$ is structurally stable by Anosov, so there exists homeomorphisms $h_t \colon \mathbb{T}^n \to \mathbb{T}^n$ for $0 \leq t \leq \epsilon$ conjugating ϕ_t and ϕ_0.

The periodic points of $\phi_0(\gamma_0)$ are dense, so the same holds for $\phi_t(\gamma_0)$ for $0 \leq t \leq \epsilon$.

Next, must show that h_t conjugates the full action of Γ.

Step 2. Let $y \in \mathbb{T}^n_{\mathbb{Q}}$ be a rational point, so is periodic for $\phi_0(\gamma_0)$.

Set $y_t = h_t(y)$ which is isolated periodic for $\phi_t(\gamma_0)$ for $0 \leq t \leq \epsilon$.

Let $\Gamma_y = \{\gamma \in \Gamma \mid \phi_0(\gamma)(y) = y\}$ a finite index subgroup of Γ.

Γ satisfies the SVC condition, so isolated periodic points of the action of Γ_y are stable.

There exists $0 < \epsilon'_y \leq \epsilon$ such that $\phi_t(\Gamma_y)(y_t) = y_t$ for $0 \leq t < \epsilon'_y$

So $\phi_t(\Gamma_y)(y_t) = y_t$ for $t = \epsilon'_y$

There is $n \geq 1$ such that $\gamma_0^n \in \Gamma_y$ which is again Anosov, hence y_t is isolated fixed point for $\phi_t(\Gamma_y)$.

Hence y_t is isolated fixed point for $\phi_t(\Gamma_y)$ for all $0 \leq t \leq \epsilon$.

Thus, h_t conjugates ϕ_t to ϕ_0 on the dense set $\mathbb{T}^n_{\mathbb{Q}} \subset \mathbb{T}^n$

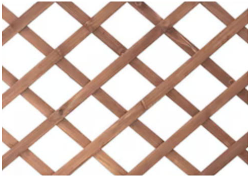
and so conjugates the action of ϕ_t to ϕ_0 for all $0 \leq t \leq \epsilon$.

Step 3. We show that the conjugating homeomorphism is smooth.

That is, the "Teichmüller space" of deformations of the standard action $\phi_0 \colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$ is trivial.

Definition. A C^1-action $\phi \colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$ is *Cartan* if there exists an abelian subgroup $\mathcal{A} \subset \Gamma$ generated by $\{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{A}$ such that each $\phi(\gamma_i)$ is Anosov with 1-dimensional stable foliation $\mathcal{F}_i^-$ and the foliations $\{\mathcal{F}_1^-, \ldots, \mathcal{F}_n^-\}$ are completely transverse.

That is, their tangent spaces define a framing of $T\mathbb{T}^n$ – a trellis



Theorem. Let G be a semi-simple analytic Lie group and $\Gamma \subset G$ a lattice. Let H be a Cartan subgroup of G, then there exists $g \in G$ such that $\Gamma_H = \Gamma \cap g^{-1}Hg$ is a uniform lattice in $g^{-1}Hg$.

$\star$ G. Prasad and M. S. Raghunathan, *Cartan subgroups and lattices in semi-simple groups*, **Annals of Math.**, 96:296–317, 1972.

Corollary. Let $\Gamma \subset SL(n, \mathbb{Z})$ be a lattice. Then the standard action $\phi \colon \Gamma \times \mathbb{T}^n \to \mathbb{T}^n$ is Cartan.

Let $\mathcal{A} = \langle \gamma_1, \ldots, \gamma_n \rangle \subset \Gamma \subset SL(n, \mathbb{Z})$ be a Cartan subgroup.

The stable manifolds of Anosov diffeomorphisms are preserved by a conjugacy, so the conjugacy $h_t \colon \mathbb{T}^n \to \mathbb{T}^n$ preserves the stable foliations $\{\mathcal{F}_1^-, \ldots, \mathcal{F}_n^-\}$. That is, h_t preserves the lines in the trellises for the actions.

Stowe's Theorem implies that the actions of Γ_y are conjugated at a periodic orbit y, which implies that the exponents of contraction are equal for the actions $\phi_0(\gamma_i)$ and $\phi_t(\gamma_i)$ at y and $h_t(y)$, respectively.

The Livsic Theorem then implies that the restriction of h_t to the stable foliations is a smooth map of 1-dimensional manifolds.

We have homeomorphisms $h_t \colon \mathbb{T}^n \to \mathbb{T}^n$ whose restrictions to a transverse family of 1-dimensional foliations of $\mathbb{T}^n$ are leafwise smooth. These 1-manifolds are smoothly embedded in $\mathbb{T}^n$.

Theorem. Let $\mathcal{F}_s$ and $\mathcal{F}_u$ be two transverse foliations with uniformly smooth leaves, of some manifold M. If f is uniformly smooth along the leaves of $\mathcal{F}_s$ and $\mathcal{F}_u$, then f is smooth.

⋆ J.-L Journé, *A regularity lemma for functions of several variables*, **Rev. Mat. Iberoamericana**, 4:187–193, 1988.

Thus, the conjugating maps h_t are smooth, as was to be shown.

The above results apply to a wide variety of other lattice actions:

⋆ Section 7, *Rigidity for Anosov actions of higher rank lattices*.

Let $\phi \colon \Gamma \times M \to M$ be a C^∞-action on a compact n-manifold M. Choose a *measurable* framing $TM \cong M \times \mathbb{R}^n$ then the derivative defines a measurable cocycle, or "virtual homomorphism",

$$D\phi \colon \Gamma \times M \to GL(n, \mathbb{R})$$

$$D\phi(\gamma_2\gamma_1, x) = D\phi(\gamma_2, \gamma_1 \cdot x) \cdot D\phi(\gamma_1, x)$$

This is just the chain rule for group actions.

- *Margulis Rigidity* gives conditions on a lattice $\Gamma \subset G$ which imply that a homomorphism $\rho \colon \Gamma \to H$ extends to a homomorphism $\widehat{\rho} \colon G \to H$.

- *Zimmer Superrigidity* gives conditions on a lattice $\Gamma \subset G$ and measure-preserving action $\phi \colon \Gamma \times M \to M$ which imply that a cocycle $\alpha \colon \Gamma \times M \to H$ is *measurably* conjugate to a constant cocycle, which extends to a homomorphism $\widehat{\rho} \colon G \to H$.

In the 1980's, Zimmer used superrigidity to study volume-preserving actions of higher rank lattices on compact manifolds. He posed the question whether a group action, given by a map $\phi\colon \Gamma \to \mathbf{Diff}(M)$, must behave like its cocycle $D\phi$?

Conjecture. [Zimmer] Let $\Gamma \subset G$ be a higher rank lattice, and suppose there are no non-trivial representations $\widehat{\rho}\colon G \to GL(n, \mathbb{R})$, then an action $\phi\colon \Gamma \times M \to M$ factors through a finite action.

For a discussion of the Zimmer Program, see

⋆ D. Fisher, *Groups acting on manifolds: around the Zimmer program*, in **Group actions in ergodic theory, geometry, and topology—selected papers**, Univ. Chicago Press, Chicago, IL, 2020, pages 609–683.

There has been remarkable progress towards establishing the Zimmer Conjecture in special cases.

Hypothesis. Suppose G is a connected semisimple Lie group with finite center, all of whose noncompact almost-simple factors have $\mathbb{R}$-rank 2 or higher, and suppose Γ is a lattice in G.

Theorem. Let G and Γ be as in the Hypothesis. Let α be a C^∞-action of Γ on a compact nilmanifold $M = N/\Lambda$. Suppose α can be lifted to an action on the universal cover $\widetilde{M}$ of M, and let ρ be the associated linear data of α. If $\alpha(\gamma)$ is hyperbolic for some element $\gamma \in \Gamma$, then there are a finite-index subgroup $\Gamma' \subset \Gamma$ and a C^∞-diffeomorphism $h\colon M \to M$, homotopic to identity, such that $\rho(\gamma) \circ h = h \circ \alpha(\gamma)$ for all $\gamma \in \Gamma'$.

$\star$ A. Brown, F. Rodriguez Hertz and Z. Wang, *Global smooth and topological rigidity of hyperbolic lattice actions*, **Ann. of Math. (2)**, 186:913–972, 2017.

When the dimension m of M is less than n, there are no non-trivial representations $\rho\colon SL(n,\mathbb{Z}) \to GL(m,\mathbb{R})$. The Zimmer Conjecture then becomes a generalized version of Witte's Theorem for $\mathbb{S}^1$.

Theorem. Given a subgroup $\Gamma \subset SL(n,\mathbb{Z})$ of finite index, and a closed manifold M with dimension $m < n-1$, then every C^2-action $\alpha\colon \Gamma \times M \to M$ is finite.

$\star$ A. Brown, D. Fisher and S. Hurtado, *Zimmer's conjecture for actions of $SL(m,\mathbb{Z})$*, **Invent. Math.**, 221:1001–1060, 2020.

There have been many more results in the period between 1990 and today. The introduction to the paper by Brown, Rodriguez Hertz, and Wang gives a nice overview, as of 2015.

Moreover, the action continues with the conference:

- Group Actions and Rigidity: Around the Zimmer Program

Introductory school: Rigidity, Dynamics and Geometric Structures
April 15 to 19, 2024 - CIRM, Marseille Luminy
https://indico.math.cnrs.fr/event/9764/

This is part of a thematic semester

- Group Actions and Rigidity: Around the Zimmer Program, Paris

April 15th to July 5th, 2024
https://indico.math.cnrs.fr/category/619/