

Classifying Foliations

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In Spring 1982, news arrived at the IAS of Gérard Duminy's breakthrough:

THEOREM: [Duminy] Let \mathcal{F} be a C^2 -foliation of codimension one on a compact manifold M . If the Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M)$ is non-trivial, then \mathcal{F} has a resilient leaf, and hence an uncountable set of leaves with exponential growth.

In a seminar that Spring at the IAS, including Paul, Larry Conlon, James Heitsch, the speaker and others, Duminy's hand-written manuscript with the proof was presented and dissected.

This seed inspired 25 years of subsequent work.

Fundamental problems – 1982 & today:

Question 1: How do you construct foliations on a given manifold M ? Must there exist any at all?

Question 2: How do the geometry and topology of the leaves of a foliation \mathcal{F} appear? Are there restrictions on the geometry of a manifold which is (quasi-isometric to) a leaf, imposed by the topology of M and the dynamical properties of \mathcal{F} ?

Question 3: How do you keep track of all the foliations on a given manifold M ? Is it possible to classify foliations on M modulo some natural equivalence relation, like foliation preserving diffeomorphism, or Morita equivalence?

Question 3': Is it possible to classify (almost all) foliations on M based on their dynamical behavior?

Foliation dynamics

- A continuous dynamical system on a compact manifold M is a flow $\varphi: M \times \mathbb{R} \rightarrow M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x, t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M , and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.
- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the dynamics of \mathcal{F} asks for properties of the aggregate and statistical behavior of the collection of its leaves.

Classifying spaces:

Let q denote the codimension of the foliation \mathcal{F} , so $q = m - p$ where p is the leaf dimension.

$B\Gamma_q^r$ denotes the “classifying space” of codimension q , C^r -foliations introduced by André Haefliger in 1970.

Theorem: (Haefliger [1970]) Each C^r -foliation \mathcal{F} on M of codimension q determines a well-defined map $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^r$ whose homotopy class is uniquely defined by \mathcal{F} .

Theorem: (Thurston [1975]) Each “natural” map $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^r \times BO_p$ yields a C^r -foliation \mathcal{F} on M with concordance class determined by $h_{\mathcal{F}}$.

The topological type of $B\Gamma_q^r$ is analyzed using the “linearization” of the normal structure along the leaves – the Bott connection and its invariants.

Secondary classes

Assume \mathcal{F} is C^r -foliation with $r \geq 2$.

Theorem: (Godbillon-Vey [1971]) For each codimension q , there is a secondary invariant $GV(\mathcal{F}) = \Delta(h_1 c_1^q) \in H^{2q+1}(M; \mathbb{R})$.

Theorem: (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972]) For each codimension q , there is a non-trivial space of secondary invariants $H^*(WO_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{array}{ccc} & & H^*(B\Gamma_q; \mathbb{R}) \\ & \nearrow \tilde{\Delta} & \downarrow h_{\mathcal{F}}^* \\ H^*(WO_q) & \xrightarrow{\Delta} & H^*(M; \mathbb{R}) \end{array}$$

The study of these maps has been the principle source of information about the (non-trivial) homotopy type of $B\Gamma_q^r$ for $r \geq 2$.

Homotopy chaos

Again, assume that \mathcal{F} is C^r -foliation with $r \geq 2$.

Theorem: (Bott–Heitsch [1972]) $B\Gamma_q^r$ does not have finite topological type for $q \geq 2$.

Theorem: (Thurston [1972]) $\pi_3(B\Gamma_1^r)$ surjects onto \mathbb{R} .

Theorem: (Heitsch [1978]) There are continuous families of foliations with non-trivial variations of their secondary classes for $q \geq 3$.

Theorem: (Rasmussen [1980]) There are continuous families of foliations with non-trivial variations of their secondary classes for $q = 2$.

Corollary: $B\Gamma_q^r$ has uncountable topological type for all $q \geq 1$.

Theorem: (Hurder [1980]) For $q \geq 2$, $\pi_n(B\Gamma_q^r) \rightarrow \mathbb{R}^{k_n} \rightarrow 0$ where $k_{2q+1} \neq 0$, and in general, k_n has a subsequence $k_{n_\ell} \rightarrow \infty$

Secondary classes measure some uncountable aspect of foliation geometry.

$C^{1+\alpha}$ is essential

In contrast, Takashi Tsuboi proved the following amazing result:

Theorem: (Tsuboi [1989]) The classifying map of the normal bundle $\nu: B\Gamma_q^1 \rightarrow BO(q)$ is a homotopy equivalence.

The proof is a technical tour-de-force, using Mather-Thurston type techniques for the study of $B\Gamma$, along with (to paraphrase) “smearing along orbits in acyclic models”.

Ergodic theory & secondary classes

Mizutani, Morita and Tsuboi [1981], Duminy & Sergiescu [1981], and Duminy [1982] developed techniques of localizing the Godbillon-Vey class to open saturated subsets, first for foliations of depth 1, and then for arbitrary depth. Heitsch & Hurder [1984] extended the localization technique to saturated measurable subsets. Then add two key ideas:

Idea 1: (Heitsch & Hurder) The normal derivative cocycle used to define the forms $\Delta(h_I)$ appearing in the secondary class $\Delta(h_I c_J)$ is only required to be smooth along leaves, and measurable transversally. Thus, the contribution of $\Delta(h_I)$ can be estimated using ergodic theory techniques for the measurable equivalence relation defined by \mathcal{F} .

Idea 2: (Hurder & Katok) Before passing to cohomology or homotopy, “smear along orbits the linearization data” for $B\Gamma_q^r$. More precisely, use the ergodic theory and dynamical data for the foliation to “optimally temper” the normal derivative cocycle.

Pseudogroups & Groupoids

Every foliation admits a discrete model by choosing a section $\mathcal{T} \subset M$, an embedded submanifold of dimension q which intersects each leaf of \mathcal{F} at least once, and always transversally. The holonomy of \mathcal{F} yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on \mathcal{T} .

Definition: A pseudogroup of transformations \mathcal{G} of \mathcal{T} is *compactly generated* if there is

- a relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of \mathcal{F}
- a finite set $\Gamma = \{g_1, \dots, g_k\} \subset \mathcal{G}$ such that $\langle \Gamma \rangle = \mathcal{G}|_{\mathcal{T}_0}$;
- $g_i: D(g_i) \rightarrow R(g_i)$ is the restriction of $\tilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\tilde{g}_i)$.

Definition: The groupoid of \mathcal{G} is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \text{ \& } x \in D(g)\}, \quad \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and \mathcal{T} has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: The normal cocycle $D\varphi: \Gamma_{\mathcal{G}} \times \mathcal{T} \rightarrow \mathbf{GL}(\mathbb{R}^q)$ is defined by

$$D\varphi[g]_x = D_x g: T_x\mathcal{T} \cong_x \mathbb{R}^q \rightarrow T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D[h]_y \cdot D[g]_x$$

Pseudogroup word length

Definition: For $g \in \Gamma_{\mathcal{G}}$, the word length $\| [g] \|_x$ of the germ $[g]_x$ of g at x is the least n such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_n}^{\pm 1}]_x$$

Word length is a measure of the “time” required to get from one point on an orbit to another.

Asymptotic exponent

Definition: The transverse expansion rate function at x is

$$\lambda(\mathcal{G}, n, x) = \max_{\|[\mathcal{g}]\|_x \leq n} \frac{\ln(\max\{\|D_x \mathcal{g}\|, \|D_y \mathcal{g}^{-1}\|\})}{\|[\mathcal{g}]\|_x} \geq 0$$

Definition: The asymptotic transverse growth rate at x is

$$\lambda(\mathcal{G}, x) = \limsup_{n \rightarrow \infty} \lambda(\mathcal{G}, n, x) \geq 0$$

This is essentially the maximum Lyapunov exponent for \mathcal{G} at x .

Expansion classification

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

where each are \mathcal{F} -saturated, Borel subsets of M , defined by:

- 1 Elliptic points: $\mathcal{E} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \forall n \geq 0, \lambda(\mathcal{G}, n, x) \leq \kappa(x)\}$
i.e., “points of bounded expansion”
- 2 Parabolic points: $\mathcal{P} \cap \mathcal{T} = \{x \in \mathcal{T} - (\mathcal{E} \cap \mathcal{T}) \mid \lambda(\mathcal{G}, x) = 0\}$
i.e., “points of slow-growth expansion”
- 3 Partially Hyperbolic points: $\mathcal{H} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}, x) > 0\}$
i.e., “points of exponential-growth expansion” or non-uniformly, partially hyperbolic transverse dynamics

$\mathcal{E} \sim$ measurable Riemannian structure

Theorem: There exists a measurable Riemannian metric on the normal bundle $Q | \mathcal{E}$ which is holonomy invariant.

Proof. Introduce the space of fiberwise Riemannian metrics $\mathcal{S} = GL(Q)/O(q) \rightarrow M$ on which the derivative cocycle $D\varphi$ acts isometrically on the fiberwise symmetric spaces $GL(Q_x)/O(q)$.

A measurable section $\sigma: \mathcal{E} \rightarrow \mathcal{S}$ corresponds to a measurable transverse metric on \mathcal{E} , and the action of $D\varphi$ extends to an action on such sections. Let σ_0 be a smooth metric on Q restricted to \mathcal{E} .

For $x \in \mathcal{E}$ there is an upper bound on the distance between $\sigma_0(x)$ and $[g]_x \cdot \sigma_0(x)$ for all $[g]_x \in \Gamma_{\mathcal{G}}$. Hence we can use a center of mass construction to obtain a section σ which is invariant. \square

Examples with $M = \mathcal{E}$

Example: \mathcal{F} is Riemannian $\Rightarrow M = \mathcal{E}$.

Is this the only example?

Question: If $M = \mathcal{E}$, must \mathcal{F} be Riemannian?

In the case where \mathcal{F} is defined by a smooth measure-preserving action of a higher rank lattice Γ on a compact manifold, this is a well-known (old) question of Robert Zimmer, which has recently been shown true by David Fisher and Gregory Margulis if Γ has Property T.

$\mathcal{P} \sim$ almost invariant metric

Theorem: For all $\epsilon > 0$, there exists a measurable Riemannian metric σ_ϵ on the normal bundle $Q | \mathcal{P}$ which is ϵ -holonomy invariant.

Proof. Much the same as above, but using tempering of cocycles and techniques from the papers:

- [Hurder & Katok 1987] “Ergodic theory and Weil measures for foliations”, Ann. of Math. (2) 126 (1987)
- [Hurder & Langevin 2004] “Dynamics and the Godbillon-Vey Class of C^1 Foliations”, Jour. Diff. Geometry (to appear)

Examples with $M = \mathcal{P}$

Example: Let \mathcal{F} be defined by the suspension of an irrational rotation diffeomorphism of \mathbb{S}^1 which is not C^1 -conjugate to a rotation. ($q = 1$)

Example: Let \mathcal{F} be defined by the suspension of the action of a f.g. subgroup $\Gamma \subset G$ on G/Λ where G is non-abelian simply connected nilpotent Lie group, and G/Λ is compact manifold. ($q > 1$)

Example: A foliation \mathcal{F} is distal if its pseudogroup $(\mathcal{G}_{\mathcal{F}}, \mathcal{T})$ is distal: that is, for all $x \neq y \in \mathcal{T}$ there exists $\epsilon_{x,y} > 0$ such that

$$d_{\mathcal{T}}(g(x), g(y)) \geq \epsilon_{x,y} \text{ for all } g \in \mathcal{G}_{\mathcal{F}}$$

For example, all compact foliations are distal.

Theorem: If \mathcal{F} is distal and $\mathbf{C}^{1+\alpha}$ for some $\alpha > 0$, then $M = \mathcal{P}$.

Examples with $M = \mathcal{P}$

Theorem: (Clark & Hurder [2006]) Suppose that \mathcal{F} has a compact leaf L with $H^1(L, \mathbb{R}) \neq 0$, and there is a saturated open neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation. Then there is an arbitrarily small smooth perturbation \mathcal{F}' of \mathcal{F} such that \mathcal{F}' has a solenoidal minimal set $\mathbf{K} \subset U$, where the leaves of $\mathcal{F}' \mid \mathbf{K}$ all cover L . Moreover, if \mathcal{F} is distal, then \mathcal{F}' is distal.

Problem: If $M = \mathcal{P}$, does there exist a structure theory for the minimal sets of \mathcal{F} ? For example, must such \mathbf{K} admit a topological Lie group structure transversally, or have a factor with this property?

Secondary classes and dynamics

Recall that a secondary class $y_I c_J \in H^*(WO_q)$ is *residual* if c_J has degree $2q$. The two results above then imply:

Theorem: (Hurder) If $y_I c_J \in H^*(WO_q)$ is a residual secondary class (e.g., Godbillon-Vey type) then the localizations $\Delta(y_I c_J)|_{\mathcal{E}} = 0$ and $\Delta(y_I c_J)|_{\mathcal{P}} = 0$. Hence, if $\Delta(y_I c_J)$ non-zero implies that \mathcal{H} has positive Lebesgue measure.

Thus, understanding the dynamical meaning of the residual secondary classes requires understanding the dynamics of foliations which have non-uniformly, partially hyperbolic behavior on a set of positive measure.

$\mathcal{H} \sim$ codimension one

Theorem: [Hurder (2005)] Let \mathcal{G} be a compactly generated $C^{1+\alpha}$ -pseudogroup, where the Hölder exponent $\alpha > 0$, and \mathcal{T} has dimension one. Then for every minimal set $\mathbf{K} \subset \mathcal{T}$ the intersection $\mathbf{K} \cap \mathcal{H}$ has Lebesgue measure zero.

Combining this with results from Poincaré-Bendixson Theory for C^2 -foliations, one gets:

Theorem: [Hurder 2005] Let \mathcal{F} be a C^2 -foliation of codimension $q = 1$ with $GV(\mathcal{F}) \neq 0$. Then there is an open subset $U \subset M$ with

- U is saturated by the leaves of \mathcal{F} ,
- U contains the support of the cohomology class $GV(\mathcal{F})$
- U contains a dense collection of chaotic laminations.
- $\mathcal{F}|_U$ is expansive

Geometric entropy

Given a subset $X \subset \mathcal{T}$, $\mathcal{S} = \{x_1, \dots, x_\ell\} \subset X$ is (n, ϵ) -separated if

$$\forall x_i \neq x_j, \exists g \in \mathcal{G}|X \text{ such that } \|g\|_{x_i} \leq n \text{ \& } d_{\mathcal{T}}(g(x_i), g(x_j)) \geq \epsilon$$

Then set

$$h(X, n, \epsilon) = \max \#\{\mathcal{S} \mid \mathcal{S} \subset X \text{ is } (n, \epsilon) \text{ separated}\}$$

Definition: (Ghys, Langevin, Walczak [1986])

$$h(\mathcal{G}) = \lim_{\epsilon \rightarrow 0} \left\{ \limsup_{n \rightarrow \infty} \frac{\ln h(\mathcal{T}, n, \epsilon)}{n} \right\}$$

The geometric entropy of \mathcal{F} is $h(\mathcal{F}) = h(\mathcal{G}_{\mathcal{F}})$.

Proposition: If \mathcal{G} contains a Markov subpsudogroup, then $h(\mathcal{G}) > 0$.

Local entropy

Local entropy for measure-preserving transformations was introduced by Brin & Katok at a talk in Rio de Janeiro in 1981. There is a very useful version of this notion for pseudogroups.

Let $B(x, \delta) \subset \mathcal{T}$ denote the open δ -ball about $x \in \mathcal{T}$.

Definition: The *local entropy* of \mathcal{G} at x is

$$h_{loc}(\mathcal{G}, x) = \lim_{\delta \rightarrow 0} \left\{ \lim_{\epsilon \rightarrow 0} \left\{ \limsup_{n \rightarrow \infty} \frac{\ln h(B(x, \delta), n, \epsilon)}{n} \right\} \right\}$$

Example: \mathcal{G} generated by an expanding map $f: N \rightarrow N$ of a compact manifold N , then $h_{loc}(\mathcal{G}, x) = h_{loc}(f, x)$ is the usual local entropy of f .

Proposition: (Hurder [2005]) \mathcal{G} a finitely-generated pseudogroup:

$$h(\mathcal{G}) = \sup_{x \in \mathcal{T}} h_{loc}(\mathcal{G}, x)$$

Entropy and chaos and \mathcal{H}

Theorem: (Hurder [2005]) Let $\mathbf{K} \subset \mathcal{T}$ be a minimal set such that $h_{loc}(\mathcal{G}, x) > 0$ for some $x \in \mathbf{K}$. Then $\mathbf{K} \cap \mathcal{H} \neq \emptyset$.

Problem: If \mathcal{H} has positive Lebesgue measure and \mathcal{G} is $C^{1+\alpha}$ for some $\alpha > 0$, show that $h_{loc}(\mathcal{G}, x) > 0$ for almost every $x \in \mathcal{H}$.

Problem: Characterize the set $\{x \in \mathcal{T} \mid h_{loc}(\mathcal{G}, x) > 0\}$ – what are its properties? Is there an estimate for its Hausdorff dimension in terms of cohomology invariants of \mathcal{F} ?

Problem: Extend our understanding of the relation between cohomology invariants and properties of foliation dynamical systems to codimension greater than one.

No Problem: Happy Birthday, Paul!!