# The dynamical classification of arboreal actions

Steve Hurder, University of Illinois at Chicago
joint work with
Olga Lukina, University of Vienna

In this talk we consider:

$\star$ classification, up to return equivalence, minimal equicontinuous actions of a finitely generated group $\Gamma$ on a Cantor space $\mathfrak{X}$.

$\star$ a new approach, based on the Steinitz orders of profinite groups associated to the group action.

$\star$ two new classes of actions which are invariants of return equivalence - frothy and turbulent actions.

First, we discuss the motivation for the study of return equivalence.

Consider $\mathcal{P} = \{ q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$, where each $M_\ell$ is a compact connected manifold without boundary of dimension $n \geq 1$, and $q_\ell$ is a proper covering map. The inverse limit

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \; \{q_\ell \colon M_\ell \to M_{\ell-1}\} \; \subset \prod_{\ell \geq 0} \; M_\ell$$

is the *solenoidal manifold* associated to $\mathcal{P}$. It is compact and connected but not locally connected, has a fibration map $p_0 \colon \mathcal{S}_\mathcal{P} \to M_0$ and foliated by leaves which cover $M_0$.

For $x_0 \in M_0$, the fiber $\mathfrak{X}_0 = p_0^{-1}(x_0)$ is a Cantor set, and the monodromy along leaves gives an action of $\Gamma = \pi_1(M_0, x_0)$ on $\mathfrak{X}_0$.

**Problem:** Classify the solenoidal manifolds up to homeomorphism.

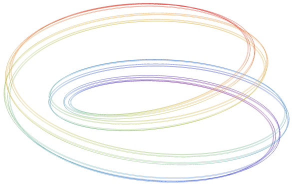
**Vietoris - van Dantzig Solenoid:**

$$\mathcal{S}(\vec{m}) = \varprojlim \, \{ \, q_\ell : \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 1 \}$$

where $q_\ell$ is a covering map of the circle $\mathbb{S}^1$ of degree $m_\ell > 1$. Let $\vec{m} = (m_1, m_2, \dots)$ be the collection of covering degrees, then the <u>Steinitz degree</u> of the covering map $q_0 \colon \mathcal{S}(\vec{m}) \to \mathbb{S}^1$ is the product

$$\Pi[\vec{m}] = m_1 \cdot m_2 \cdots m_i \cdots = \prod_{p \in \pi} p^{n(p)} \quad , \quad 0 \leq n(p) \leq \infty$$

When $m_i = 2$ for all $i \geq 1$ we get the Smale attractor:



Steinitz numbers are asymptotically equivalent, or $\Pi[\vec{m}] \overset{a}{\sim} \Pi[\vec{n}]$, if there exists $m, n \geq 1$ with $m \cdot \Pi[\vec{m}] = n \cdot \Pi[\vec{n}]$.

[Bing, 1960] and [McCord, 1965] showed the following:

**Theorem:** Solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{n})$ are homeomorphic if and only if $\Pi[\vec{m}] \overset{a}{\sim} \Pi[\vec{n}]$.

**Question:** Does such a result hold for the case of solenoidal manifolds with dimension greater than 1?

- If solenoidal manifolds $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ are homeomorphic, then their monodromy Cantor actions are *return equivalent*.

$\star$ Clark - Hurder - Lukina, "Classifying matchbox manifolds", Geom & Top, 23, 2019; arXiv:1311.0226.

**Minimal equicontinuous Cantor actions:**

- $\Gamma$ is a finitely generated group,
- $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is a topological action.
- $(\mathfrak{X}, \Gamma, \Phi)$ is minimal if every orbit $\mathcal{O}(x) = \{gx \mid g \in \Gamma\}$ is dense.
- $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\epsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$, we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(gx, gy) < \epsilon$.
- $\mathfrak{X}$ Cantor space then the *clopen* (closed and open) subsets $\mathrm{CO}(\mathfrak{X})$ form a basis for the topology.

**Fact:** $\mathfrak{X}$ a Cantor space, a minimal action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if and only if the $\Gamma$-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite for the induced action $\Phi_* \colon \Gamma \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$.

## Two Models

The tree model, or arboreal actions, where $\Gamma$ acts on a tree $\mathcal{T}$ preserving a root vertex $v$, then $\mathfrak{X}$ is identified with the ends of $\mathcal{T}$

The group chain model, where $\Gamma$ acts on the coset spaces $X_\ell = \Gamma/\Gamma_\ell$ where $\Gamma_1 \supset \Gamma_2 \supset \cdots$ is a descending chain of proper subgroups, so $\Gamma$ acts on the inverse limit space

$$\mathfrak{X} = \varprojlim \{X_1 \leftarrow X_2 \leftarrow \cdots\}$$

**Fact:** The two models are equivalent.

- Group chain model yields solenoidal manifolds directly.

**Definition:** $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, \Gamma, \Phi)$ if $U$ is a *non-empty clopen* subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$.

- Given $x \in \mathfrak{X}$ and clopen set $x \in W$, there is an adapted clopen set $U$ with $x \in U \subset W$.

- For $U$ adapted, the set of "return times" to $U$,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\}$$

is a subgroup of $\Gamma$, called the *stabilizer* of $U$.

- $\mathcal{H}_U = \Phi(\Gamma_U) \subset \mathbf{Homeo}(U)$ is the monodromy of $\Gamma_U$

**Definition:** Minimal equicontinuous Cantor actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are <u>return equivalent</u> if there exists an adapted set $U_1 \subset \mathfrak{X}_1$ for the action $\Phi_1$ and an adapted set $U_2 \subset \mathfrak{X}_2$ for the action $\Phi_2$, and a homeomoprhism $h \colon U_1 \to U_2$ which induces an isomorphism of the monodromy groups $\mathcal{H}_{U_1}$ with $\mathcal{H}_{U_2}$.

**Basic Problem:** Classify the minimal equicontinuous Cantor actions up to return equivalence.

$$\Uparrow$$

**Easier Problem:** Find properties of minimal equicontinuous Cantor actions which are return invariant.

$$\Uparrow$$

**Fun Problem:** Find interesting examples of minimal equicontinuous Cantor actions.

- **Properties**
  - ⋆ Steinitz orders
  - ⋆ Stable & Wild
  - ⋆ Frothy & Turbulent
- **Examples**
  - ⋆ Odometer Actions
  - ⋆ Heisenberg Actions

$(\mathfrak{X}, \Gamma, \Phi)$ minimal equicontinuous Cantor action

$\Longrightarrow$ $\Phi(\Gamma) \subset \mathbf{Homeo}(\mathfrak{X})$ is equicontinuous subgroup,

$\Longrightarrow$ closure $\mathfrak{G}(\Phi) = \overline{\Phi(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ is profinite group.

$\Longrightarrow$ $\mathfrak{G}(\Phi)$ acts transitively on $\mathfrak{X}$, so have $\widehat{\Phi} \colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$

$\Longrightarrow$ Isotropy subgroup $\mathfrak{D}(\Phi, x) = \{\widehat{g} \in \mathfrak{G}(\Phi) \mid \widehat{\Phi}(\widehat{g})(x) = x\}$

- $\star$ $\mathfrak{D}(\Phi, x)$ is finite, or Cantor group
- $\star$ $\mathfrak{D}(\Phi, x) \sim \mathfrak{D}(\Phi, y)$ for $x, y \in \mathfrak{X}$
- $\star$ $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{D}(\Phi, x)$

The closure $\overline{\Phi(\Gamma)}$ is also called the Ellis group of the action.

**Problem:** How does dynamics of action $(\mathfrak{X}, \Gamma, \Phi)$ depend on subgroup $\mathfrak{D}(\Phi, x)$? Or more precisely, on the left (adjoint) action of $\mathfrak{D}(\Phi, x)$ on $\mathfrak{G}(\Phi)/\mathfrak{D}(\Phi, x)$?

**Steinitz numbers:**

**Example:** Suppose $a$ and $b$ are Steinitz numbers, with

$$a = \prod_{p \in \pi} p^{n(p)} \quad , \quad b = \prod_{p \in \pi} p^{m(p)}$$

where $\pi$ is the set of distinct prime numbers.

$$LCM(a, b) = \prod_{p \in \pi} p^{\max\{n(p), m(p)\}}.$$

**Definition:** $\mathcal{N} = \{n_i \mid i \in \mathcal{I}\}$ collection of positive integers.

$$LCM(\mathcal{N}) = \prod_{p \in \pi} p^{n(p)} \quad , \quad 0 \leq n(p) \leq \infty$$

is *least common multiple* as Steinitz number

**Steinitz order:**

$\mathfrak{G}$ a profinite group

$\mathfrak{N} \subset \mathfrak{G}$ open normal subgroup then $\mathfrak{G}/\mathfrak{N}$ is finite group.

**Definition:** $\mathfrak{H} \subset \mathfrak{G}$ be a closed subgroup of the profinite group $\mathfrak{G}$.

$$\Pi[\mathfrak{G} : \mathfrak{H}] = LCM\{\#\{\mathfrak{G}/(\mathfrak{N}\cdot\mathfrak{H})\} \mid \mathfrak{N} \subset \mathfrak{G} \text{ clopen normal subgroup}\}$$

is the relative Steinitz order of $\mathfrak{H}$ in $\mathfrak{G}$.

- Steinitz order of $\mathfrak{G}$ is $\Pi[\mathfrak{G}] = \Pi[\mathfrak{G} : \{\widehat{e}\}]$.

- Steinitz numbers $\Pi_1 \overset{a}{\sim} \Pi_2$ (asymptotic equivalence)
  $\iff m \cdot \Pi_1 = n \cdot \Pi_2$ for integers $m, n \geq 1$

$\star$ J.S. Wilson, Chapter 2, **Profinite groups**, London Mathematical Society Monographs. New Series, Vol. 19, 1998.

**Theorem:** $(\mathfrak{X}, \Gamma, \Phi)$ minimal equicontinuous Cantor action, then the asymptotic relative Steinitz order $\Pi_a[\mathfrak{G}(\Phi) : \mathfrak{D}(\Phi)]$ is an invariant of return equivalence class of the action.

**Corollary:** Asymptotic Steintiz order of tower of coverings is an invariant of the homeomorphism class of solenoidal manifold.

**Definition:** Prime spectrum of $\mathfrak{G}$ is the collection

$$\pi(\Pi[\mathfrak{G}]) = \{p \text{ prime} \mid p \text{ divides } \Pi[\mathfrak{G}]\}$$

**Theorem:** $(\mathfrak{X}, \Gamma, \Phi)$ minimal equicontinuous Cantor action, then the prime spectra $\pi(\Pi[\mathfrak{G}(\Phi)])$ and $\pi(\Pi[\mathfrak{D}(\Phi)])$ are invariants of return equivalence of the action, modulo finite sets of primes.

**Remark:** Classification problem can be considered in terms of prime spectra of actions.

**Regularity properties of Cantor actions:**

Here are alternate versions of *topologically free* actions which are valid for Γ profinite group.

- $(\mathfrak{X}, \Gamma, \Phi)$ is quasi-analytic $\Longleftrightarrow$

for any clopen subset $U \subset \mathfrak{X}$ & any $g \in \Gamma$, if $\Phi(g)(U) = U$ and $\Phi(g)|U$ is the identity, then $\Phi(g)$ is the identity on all of $\mathfrak{X}$.

- $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic $\Longleftrightarrow$

if there exists $\epsilon > 0$ such that for any adapted subset $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$ & any $g \in \Gamma$ with $\Phi(g)(U) = U$, if there exists clopen $V \subset U$ with $\Phi(V) = V$ and the restriction $\Phi(g)|V$ is the identity, then $\Phi(g)|U$ is the identity on all of $U$.

**Definition:** $(\mathfrak{X}, \Gamma, \Phi)$ is <u>stable</u> if the *profinite* action $\widehat{\Phi} \colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic.

**Theorem:** Stable property is an invariant of return equivalence.

**Remark:** The classification problem for stable actions essentially reduces to a problem in algebra.


⋆ Cortez - Medynets, “Orbit equivalence rigidity of equicontinuous systems”, Journal Lond. Math. Soc. (2), 94, 2016.

⋆ Hurder - Lukina, “Orbit equivalence and classification of weak solenoids”, Indiana Univ. Math. Journal, Vol. 69, 2020; arXiv:1803.02098.

⋆ Hurder - Lukina, “Nilpotent Cantor actions”; arXiv:1905.07740.

**Definition:** $(\mathfrak{X}, \Gamma, \Phi)$ is <u>wild</u> if the *profinite* action $\widehat{\Phi}\colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ is not locally quasi-analytic.

Wild Cantor actions include:

- actions of weakly branch groups on their boundaries

⋆ Bartholdi - Grigorchuk - Šunić, "Branch groups", **Handbook of algebra, Vol. 3**, 2012.

- actions of higher rank arithmetic lattices on quotients of their profinite completions

⋆ Hurder- Lukina, "Wild solenoids", Transactions A.M.S., 371, 2019; arXiv:1702.03032.

- subgroups of wreath product groups acting on trees

⋆ Álvarez López - Barral Lijó - Lukina - Nozawa, "Wild Cantor actions", J. Math. Soc. Japan, to appear; arXiv:2010.00498.

**Classifying nilpotent Cantor actions:**

$(\mathfrak{X}, \Gamma, \Phi)$ is a nilpotent Cantor action $\Leftrightarrow$

- minimal & equicontinuous,
- $\Gamma$ contains a finitely-generated nilpotent subgroup of finite index.

**Question:** How do the dynamical properties of nilpotent Cantor actions differ from those of $\mathbb{Z}^n$-odometers?

**Theorem:** [Hurder - Lukina, 2021] Let $(\mathfrak{X}, \Gamma, \Phi)$ be a nilpotent Cantor action. Then

$$\text{prime spectrum } \pi(\Pi[\mathfrak{G}(\Phi)]) \text{ is finite } \implies \text{ action is stable}$$

**Problem:** Show there exist wild nilpotent Cantor actions.

**Two more properties:**

**Definition:** A wild Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is said to be frothy if $\mathfrak{D}(\Phi) \cong \prod_{i=1}^{\infty} H_i$ , where each $H_i$ is a finite group.

**Definition:** A wild Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is said to be turbulent if the set of points with non-trivial holonomy has full measure.

This notion has applications to the study of I.R.S.'s

$\star$ Gröger - Lukina, "Measures and regularity of group Cantor actions", Discrete Contin. Dynam. Sys.-A, 41(5) 2021; arXiv:1911.00680.

**Examples:**

- Toroidal Actions
- Heisenberg (nilpotent) Actions
  - ⋆ Stable
  - ⋆ Wild
  - ⋆ Frothy
  - ⋆ Turbulent

**Classic odometers:** Choose two disjoint sets of distinct primes,

$$\pi_f = \{q_1, q_2, \ldots\} \quad , \quad \pi_\infty = \{p_1, p_2, \ldots\}$$

where $\pi_f$ and $\pi_\infty$ can be chosen to be finite or infinite sets.

Choose multiplicities $n(q_i) \geq 1$ for the primes in $\pi_f$.

For each $\ell > 0$, define a subgroup of $\Gamma = \mathbb{Z}$ by

$$\Gamma_\ell = \{q_1^{n(q_1)} q_2^{n(q_2)} \cdots q_\ell^{n(q_\ell)} \cdot p_1^\ell p_2^\ell \cdots p_\ell^\ell \cdot n \mid n \in \mathbb{Z}\} \ ,$$

The completion $\widehat{\Gamma}$ of $\mathbb{Z}$ with respect to this group chain admits a product decomposition into its Sylow $p$-subgroups

$$\widehat{\Gamma} \ \cong \ \prod_{i=1}^{\infty} \mathbb{Z}/q_i^{n(q_i)}\mathbb{Z} \ \cdot \ \prod_{p \in \pi_\infty} \widehat{\mathbb{Z}}_{(p)} \quad , \ \pi(\Pi[\widehat{\Gamma}]) = \pi_f \cup \pi_\infty$$

$\mathbb{Z}$-action on $\mathfrak{X} = \widehat{\Gamma}$ is free, so certainly topologically free & stable.

**Heisenberg odometers:** $\mathcal{H} \subset \mathrm{GL}(\mathbb{Z}^3)$

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}. \tag{1}$$

The group operation $*$ in coordinates $(a, b, c), (a', b', c') \in \mathbb{Z}^3$,

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab')$$

The normal subgroups and representations of $\mathcal{H}$ are described in

⋆ Lightwood - Şahin - Ugarcovici, "The structure and spectrum of Heisenberg odometers", Proc. Amer. Math. Soc., 142(7), 2014.

⋆ Danilenko - Lemańczyk, "Odometer actions of the Heisenberg group", J. Anal. Math., 128, 2016.

Our interest is in group chains in $\mathcal{H}$ which are not normal.

Here is a very useful result:

**Theorem:** Let $\widehat{\Gamma}$ be a profinite completion of a finitely-generated nilpotent group $\Gamma$. Then there is a topological isomorphism

$$\widehat{\Gamma} \cong \prod_{p \in \pi(\Pi[\widehat{\Gamma}])} \widehat{\Gamma}_{(p)} ~ ,$$

where $\widehat{\Gamma}_{(p)} \subset \widehat{\Gamma}$ denotes the Sylow $p$-subgroup of $\widehat{\Gamma}$ for a prime $p$.

Thus the action of $\widehat{\Gamma}$ can be analyzed for each prime.

Conversely, actions of $\mathcal{H}$ can be constructed prime by prime.

**A model action of a finite $p$-group:**

Fix a prime $p \geq 2$.

For $n \geq 1$ and $0 \leq k < n$, we have the following finite groups:

$$G_{p,n} = \left\{ \begin{bmatrix} 1 & \overline{a} & \overline{c} \\ 0 & 1 & \overline{b} \\ 0 & 0 & 1 \end{bmatrix} \mid \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}/p^n\mathbb{Z} \right\}$$

$$H_{p,n,k} = \left\{ \begin{bmatrix} 1 & p^k\overline{a} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid \overline{a} \in \mathbb{Z}/p^n\mathbb{Z} \right\}$$

$$X_{p,n,k} = G_{p,n}/H_{p,n,k}$$

The isotropy group of the action of $G_{p,n}$ on $X_{p,n,k}$ at the coset $eH_{p,n,k}$ of the identity element is $H_{p,n,k}$.

**Construction of a wild example:**

Let $\pi_f$ and $\pi_\infty$ be two disjoint collections of primes, with $\pi_f$ an infinite set and $\pi_\infty$ arbitrary, possibly empty.

Enumerate $\pi_f = \{q_1, q_2, \ldots\}$ and choose integers $1 \leq r_i \leq n_i$ for $1 \leq i < \infty$.

Enumerate $\pi_\infty = \{p_1, p_2, \ldots\}$, again with the convention that if $\ell$ is greater than the number of primes in $\pi_\infty$ then we set $p_\ell = 1$.

For each $\ell \geq 1$, define the integers

$$\begin{aligned} M_\ell &= q_1^{r_1} q_2^{r_2} \cdots q_\ell^{r_\ell} \cdot p_1^\ell p_2^\ell \cdots p_\ell^\ell \ , \\ N_\ell &= q_1^{n_1} q_2^{n_2} \cdots q_\ell^{n_\ell} \cdot p_1^\ell p_2^\ell \cdots p_\ell^\ell \ . \end{aligned}$$

For $\ell \geq 1$, define a subgroup of $\mathcal{H}$, in the coordinates above,

$$\mathcal{H}_\ell = \{(aM_\ell, bN_\ell, cN_\ell) \mid a, b, c \in \mathbb{Z}\} \ ,$$

Its core subgroup is given by $C_\ell = \{(aN_\ell, bN_\ell, cN_\ell) \mid a, b, c \in \mathbb{Z}\}$.

For $k_i = n_i - r_i$ we then have

$$\widehat{\mathcal{H}}_\infty \ \cong \ \prod_{i=1}^{\infty} G_{q_i, n_i} \ \cdot \ \prod_{j=1}^{\infty} \widehat{\mathcal{H}}_{(p_j)} \quad , \quad D_\infty \ \cong \ \prod_{i=1}^{\infty} H_{q_i, n_i, k_i} \ .$$

The Cantor space $X_\infty = \widehat{\mathcal{H}}_\infty / D_\infty$ associated to the group chain $\{\mathcal{H}_\ell \mid \ell \geq 1\}$ is given by

$$X_\infty \ \cong \ \prod_{i=1}^{\infty} X_{q_i, n_i, k_i} \ \times \ \prod_{j=1}^{\infty} \widehat{\mathcal{H}}_{(p_j)} \ .$$

Let $x_i \in X_{q_i,n_i,k_i}$ denote the coset of the identity element.

For each $\ell \geq 1$, we define a clopen set in $X_\infty$

$$U_\ell = \prod_{i=1}^{\ell} \{x_i\} \times \prod_{i=\ell+1}^{\infty} X_{q_i,n_i,k_i} \times \prod_{j=1}^{\infty} \widehat{\mathcal{H}}_{(p_j)} .$$

- This action is wild.
- If the set $\pi_\infty$ is empty, then the action is frothy as well.
- With proper choices of integers $1 \leq r_i \leq n_i$ for $1 \leq i < \infty$, the action will be turbulent.

Details of calculations and more examples are in the paper

⋆ Hurder-Lukina, "The prime spectrum of solenoidal manifolds", 2021; arXiv:2103.06825