Variations of aperiodic flows

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Paul's first paper that I read (it was related to my PhD thesis work)

⋆ P. Schweitzer and A. Whitman. *Pontryagin polynomial residues of isolated foliation singularities*,

Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976), Paul A. Schweitzer, s.j. editor,

Lecture Notes in Mathematics, Vol. 652, 95–103 (1978)

Met Paul in person at the Institute for Advanced Study, Princeton in Fall 1981 - there was a year-long theme on *Foliations*.

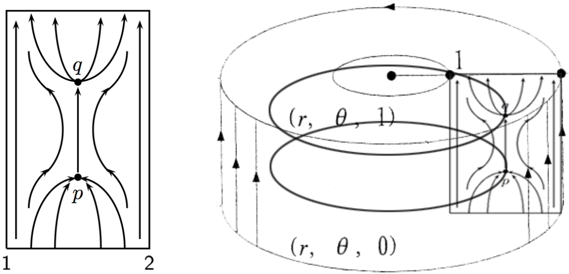
I encountered another side of Paul's work (almost 20 years late) at the *International Symposium/Workshop on Geometric Study of Foliations* in Tokyo, November 1993.

Conjecture [Seifert, 1950]: Every non-singular, continuous vector field on the 3-sphere has a periodic orbit.

Theorem [Schweitzer, 1974]: Every homotopy class of non-singular vector fields on a three-dimensional manifold M contains a C^1-vector field with no closed orbits.

$\star$ *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. 100(2), 386–400 (1974).

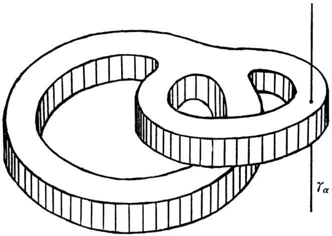
Recall the Wilson Plug [1966]: Wilson's fundamental idea was the construction of a plug which *trapped content*, and all trapped orbits have limit set a periodic orbit contained in the plug. Yields a flow with two periodic orbits by repeatedly inserting this plug.



The idea of the Schweitzer Plug:

• Each periodic circle in the Wilson flow is replaced by a *Denjoy minimal set* for a flow on a punctured 2-torus;

• The Denjoy minimal set is contained in a surface flow which embeds in $\mathbb{R}^3$.



The flow holonomy is a minimal rotation on a planar Cantor set, transverse to the minimal set for the flow.

The entertainment during the weeklong conference in Tokyo included an evening seminar over several days, on a preprint from October 1993 by Krystyna Kuperberg, where she proved.

Theorem [Kuperberg]: *Let M be an orientable three-dimensional manifold. Then M admits a C^∞ vector field with no closed orbits.*

This work appeared in the following year as

$\star$ Krystyna Kuperberg. *A smooth counterexample to the Seifert conjecture*, Ann. of Math. 140(2), 723–732 (1994).

Kuperberg's fundamental insight on the existence problem:

> *The problem is the Brouwer Fixed-Point Theorem: any map of a (transverse) disk to itself must have a fixed point, and so its suspension flow will have a periodic orbit.*
> *Thus, to build a flow without periodic orbits, the transverse holonomy for the flow must be a "translation" on an infinite line or region.*

Call this the Kuperberg Principle.

This led to an inspired geometric observation: such a translation is available in the Wilson plug (via the flow on the Reeb cylinder.)

⋆ *Kuperberg Dreams*,
https://celebratio.org/Kuperberg_KM/cover/960/ (2021)

The strategies

The first basic idea, due to Wesley Wilson, is to

> *"construct a plug which traps content, and all trapped orbits have limit set contained in the plug."*

Limit set is a union of circles for the Wilson flows.

The second basic idea, due to Paul Schweitzer, is to

> *"send the trapped orbits into a death spiral around a prescribed foliated continuum which has no closed orbits."*

Limit set is a Denjoy minimal set for the Schweitzer flows.

Krystyna Kuperberg modified the second basic idea:

> *"send the trapped orbits into a death spiral with no closed orbits, limiting onto some unknown continuum."*

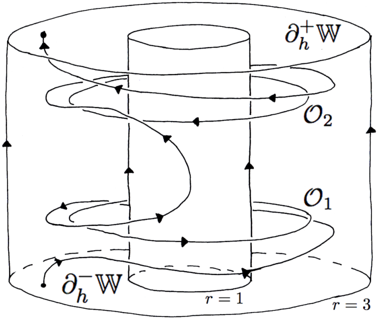
Limit set is *muito complicado* for the Kuperberg flows.

Kuperberg Plug

Begin with a modification of the Wilson Plug, as follows.

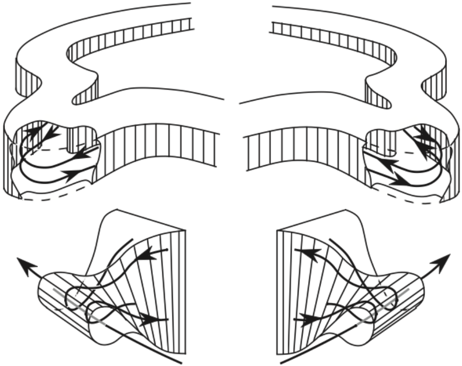
Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f \frac{f}{d\theta}$

f is asymmetric in z and $\vec{W}_1 = g \frac{f}{dz}$ is vertical.

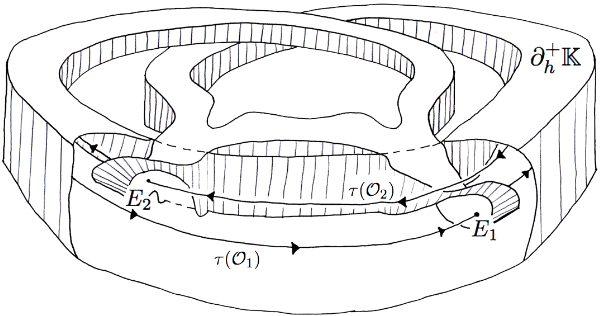


The two periodic orbits in $\mathbb{W}$ are now unstable.

Deform the modified Wilson Plug to have a pair of "horns"



Insert the horns with a twist and a bend, matching the flow lines on the boundaries, to obtain Kuperberg Plug $\mathbb{K}$



Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself. The periodic orbit for the Wilson flow is now captured by the ascending orbit back to the circle, so the resulting flow is aperiodic!

Question: Why study the dynamics of aperiodic flows?

Motivation #1. *Aperiodic* flows on 3-manifolds are exceptional, so their dynamics must also be exceptional.

Motivation #2. The minimal set of the known aperiodic flows on 3-manifolds have a "tree-like" structure, with a central axis along which the holonomy is contracting. It is analogous to the graph of a "self-similar group" in the sense of Nekrashevych.

Self-similar groups are *very* interesting.

Can the same be said for minimal sets of aperiodic flows?

Some dynamical properties of a Kuperberg flow (in a plug):

Theorem (A. Katok, 1980) *Let M be a closed, orientable 3-manifold. Then an aperiodic flow Φ_t on M has zero entropy.*

Theorem (Ghys, Matsumoto, 1995) *The Kuperberg flow has a unique minimal set $\mathcal{Z} \subset M$.*

Theorem (Matsumoto, 1995) *The Kuperberg flow has an open set of wandering points whose forward orbits limit to the unique minimal set.*

⋆ Étienne Ghys. *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, Astérisque 227, 283–307 (1995).

⋆ Shigenori Matsumoto. *K.M. Kuperberg's C^∞ counterexample to the Seifert conjecture*, Sūgaku, Mathematical Society of Japan, Vol. 47:38–45, (1995). Translation: Sugaku Expositions, A.M.S., Vol. 11, 39–49 (1998).

⋆ Greg & Krystyna Kuperberg. *Generalized counterexamples to the Seifert conjecture*, Ann. of Math. 144(2), 239–268 (1996).

Some puzzling questions

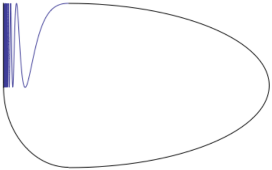
Problem: Give a geometric explanation for why a Kuperberg flow has zero entropy. For example, can one calculate the growth rates of ϵ-separated sets for the flow?

Problem: Describe the geometric properties of the unique minimal set $\mathcal{Z}$ in a Kuperberg flow. For example, when is the minimal set 1-dimensional? Or, is it always 2-dimensional?

Problem: Describe the topological shape of the unique minimal set $\mathcal{Z}$ in a Kuperberg flow. Does $\mathcal{Z}$ have *stable shape*? Is the shape of $\mathcal{Z}$ *movable*, a notion introduced by Karol Borsuk.

$\star$ *On movable compacta*, Fund. Math. 66, 137–146 (1969).

Shape, Definition 1: The circle and the Warsaw circle have the same shape:



Shape, Definition 2: ANR's X and Y have the same shape if they admit embeddings into Hilbert spaces, such that every open neighborhood of one, contains an open set homeomorphic to an open neighborhood of the other.

- The shape of the Denjoy minimal set in $\mathbb{T}^2$ is the wedge $\mathbb{S}^1 \vee \mathbb{S}^1$.

Ana Rechtman and I began our project in 2010, with the question:

Question [Kuperberg]: What is the *shape* of the minimal set for an aperiodic flow constructed using Kuperberg's construction?

⋆ *The dynamics of generic Kuperberg flows*, Astérisque 377 (2016).

⋆ *Perspectives on Kuperberg flows*, Topology Proc. 51, 197–244 (2018).

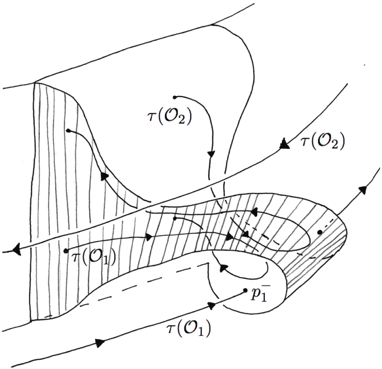
There is another natural question which was studied in the work

⋆ *Aperiodicity at the boundary of chaos*, Ergodic Theory Dynam. Systems 38, 2683–2728 (2018).

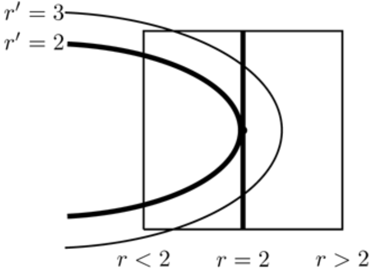
Problem: What are the dynamical properties of the flows which are smooth perturbations of a Kuperberg flow?

We first describe the *generic hypotheses* on a Kuperberg flow Φ_t.

Close up view of the lower embedding σ_1



The insertion map as it appears in a section $\mathbf{R}_0$:



A flow is *generic* if it is actually like its illustrations. That is to say, all choices are assumed to not be too pathological.

There are two types of *generic* conditions.

Generic 1: Consider the rectangle $\mathbf{R} = [1,3] \times [-2,2]$, with a vertical vector field $\vec{W}_1 = g\frac{f}{dz}$ where $g(r,z)$ vanishes at $(2,-1)$ and $(2,+1)$. We require that g *vanish to second order with positive definite Jacobian* at these two points.

Then the Wilson field on $\mathbb{W} = \mathbf{R} \times \mathbb{S}^1$ is $\vec{W} = \vec{W}_1 + f\frac{f}{d\theta}$ where f is asymmetric in z, and vanishes near the boundary.

Generic 2: These are conditions on the insertion maps $\sigma_i \colon D_i \to \mathcal{D}_i$. We require that the r-coordinate of the image *depends quadratically* on the θ-coordinate of the domain, for values of r near $r = 2$. Much stronger than the basic radius inequality.

Let Φ_t be a generic Kuperberg flow on a plug $\mathbb{K}$.

Theorem (H & R, 2015) *The unique minimal set $\mathcal{Z}$ for the flow is a 2-dimensional lamination "with dense boundary" and $\mathcal{Z}$ equals the non-wandering set of Φ_t.*

Theorem (H & R, 2015) *The flow Φ_t has positive "slow entropy", for exponent $\alpha = 1/2$.*

Thus, a generic Kuperberg flow *almost* has positive entropy.

Theorem (H & R, 2015) *The minimal set $\mathcal{Z}$ has unstable shape; but may be moveable.*

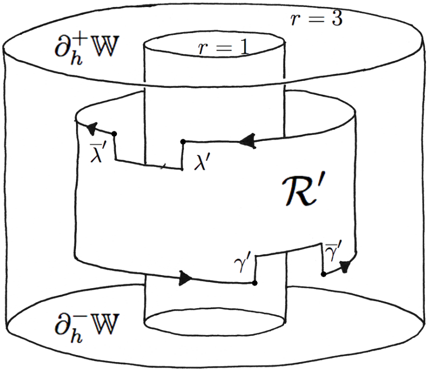
More precisely, for a descending chain of open neighborhoods $U_1 \supset U_2 \supset \cdots \supset U_\ell \supset \mathcal{Z}$ with $\cap U_\ell = \mathcal{Z}$, $\mathrm{rank}\ \pi_1(U_\ell, x_0) \to \infty$.

Theorem (Daniel Ingebretson, 2018) *The minimal set $\mathcal{Z}$ for the flow has Hausdorff dimension strictly between 2 and 3.*

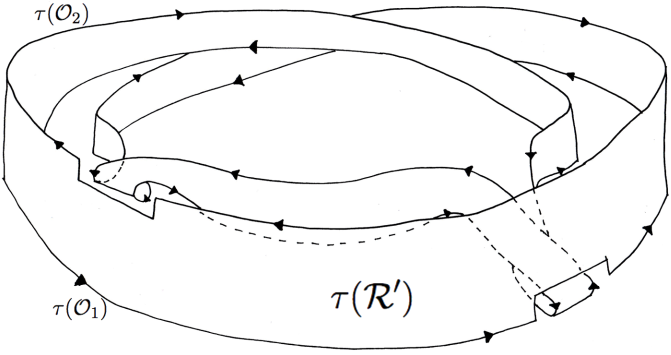
$\star$ *Hausdorff dimension of Kuperberg minimal sets*, arXiv:1801.04034

Problem: *How do the dynamical properties of the non-generic Kuperberg flows differ from those of the generic flows?*

We give a brief description of the *laminated structure* of the Kuperberg minimal set. Consider the punctured Reeb cylinder in the first step of the construction of Kuperberg Plug $\mathbb{K}$:



It gets twisted in the image $\tau(\mathcal{R}') \subset \mathbb{K}$



Consider the flow of the image $\tau(\mathcal{R}') \subset \mathbb{K}$ and its closure

$$\mathfrak{M}_0 \equiv \{\Phi_t(\tau(\mathcal{R}')) \mid -\infty < t < \infty\} \quad , \quad \mathfrak{M} \equiv \overline{\mathfrak{M}_0} \subset \mathbb{K} \, .$$

$\mathfrak{M}_0$ and $\mathfrak{M}$ are flow-saturated, so the minimal set $\mathcal{Z} \subset \mathfrak{M}$.

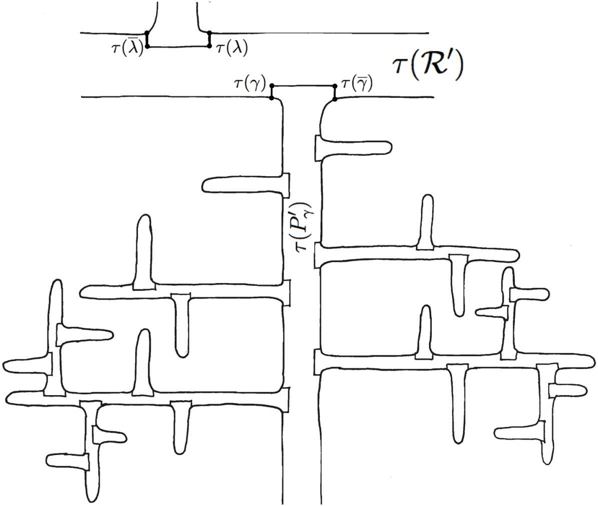
Theorem: *Let Φ_t be a generic Kuperberg flow, then $\mathcal{Z} = \mathfrak{M}$.*

Theorem: *For a generic Kuperberg flow, the space $\mathcal{Z} = \mathfrak{M}$ has the structure of a zippered lamination with 2-dimensional leaves.*

Proof: The closures of the boundary orbits of $\mathfrak{M}_0$ are dense in $\mathfrak{M}$. Hence, the boundary of each leaf of $\mathfrak{M}$ is dense in itself.

Surface $\mathfrak{M}_0$ is embedded in the Plug $\mathbb{K}$.

Each "propeller blade" wraps around the Reeb cylinder $\tau(\mathcal{R}')$.



The flow travels down this infinite surface, while contracting transversally. Thus, the *Kuperberg Principle* implies that tree structure gets copied repeatedly to itself as we flow down the tree.

This leads to a notion of a *scaling function* on the ends of the embedded surface $\mathfrak{M}_0$ - scaling functions are used to classify the dynamics of C^2-maps of Cantor sets:

$\star$ Tim Bedford and Albie Fisher. *Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets*, Ergodic Theory Dynam. Systems 17, 531–564 (1997).

- [Hurder & Rechtman] The “self-similar tree structure” is used to estimate the shape of the minimal set $\mathcal{Z}$.

- [Hurder & Rechtman] The subexponential area growth rate of the tree is used to estimate the entropy of the flow.

- [Ingebretson] The scaling function is used to estimate the Hausdorff dimension of the minimal set $\mathcal{Z}$.

Question: What happens to the tree structure for $\mathcal{Z}$ and the scaling function, when the flow is perturbed?

Derived from Kuperberg

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $-1 < \epsilon \leq 0$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ has two periodic orbits, and all orbits are properly embedded.*

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $0 \leq \epsilon < a$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ admits countably many families of "horseshoes" with dense periodic orbits, and so has positive entropy.*

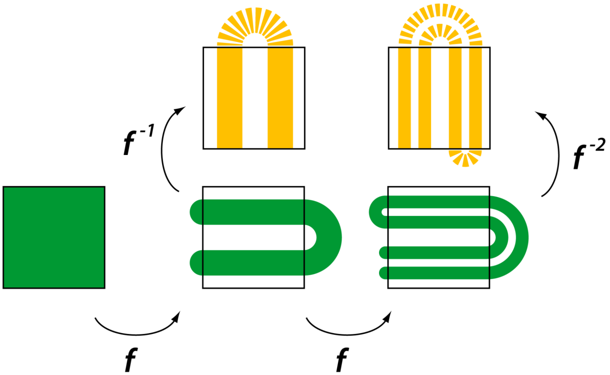
Conclusion: *The generic Kuperberg flows lie at the boundary of chaos (entropy > 0) and the boundary of tame dynamics.*

Idea of Proof:

$$\text{Reeb} + \text{Kuperberg} + \epsilon \implies \text{Horseshoes}$$

We give a brief indication of the idea of the proof.

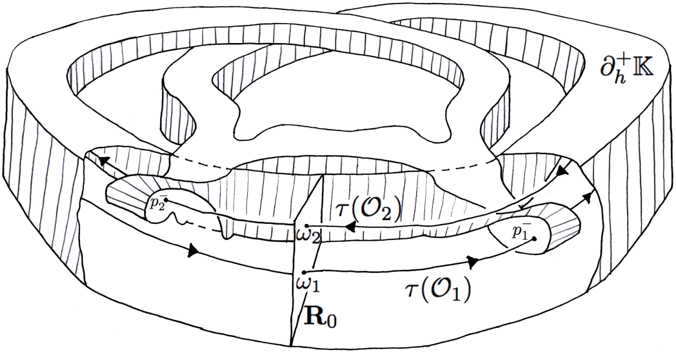
Recall the usual illustration of a horseshoe dynamical system.



The goal is to produce a variation of a Kuperberg flow, with these dynamics embedded in it.

The "Derived from Kuperberg" construction deforms a generic Kuperberg flow $\Phi_t = \Phi_t^0$ on a plug $\mathbb{K}$.

Introduce a section $\mathbf{R}_0 \subset \mathbb{K}$ to the flow Φ_t and the pseudogroup $\mathcal{G}_\Phi$ induced by the flow on $\mathbf{R}_0$.



The flow of Φ_t is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$, so the action of the pseudogroup has singularities along this line.

Critical difficulty: *There is not always a direct relation between the continuous dynamics of the flow Φ_t and the discrete dynamics of the action of the pseudogroup $\mathcal{G}_\Phi$.*

None the less, the introduction of the Kuperberg pseudogroup $\mathcal{G}_\Phi$ is a fundamental tool for the study of the dynamics of the flow Φ_t.

The semi-group formed by the generators of $\mathcal{G}_\Phi$ are used to give a complete description of the embedded space $\mathfrak{M}_0$.

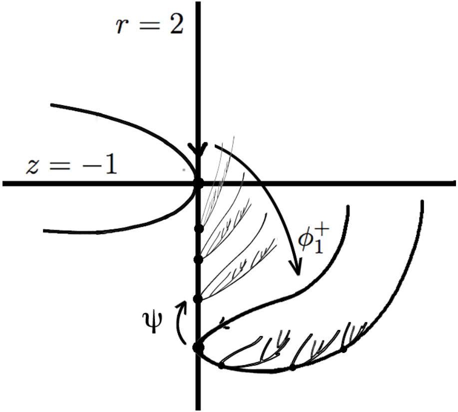
We consider two maps with domain in $\mathbf{R}_0$

- ψ which is the return map of the *Wilson flow* Ψ_t
- ϕ_1 which is the return map of the *Kuperberg flow* Φ_t

They generate a pseudogroup $\widehat{\mathcal{G}} = \langle \psi, \phi_1 \rangle$ acting on $\mathbf{R}_0$.

Proposition: *The restriction of* $\widehat{\mathcal{G}}$ *to the region* $\{r > 2\} \cap \mathbf{R}_0$ *is a sub-pseudogroup of* $\mathcal{G}_\Phi$ *The action of* $\widehat{\mathcal{G}}$ *on the line segment* $\mathcal{C} \cap \mathbf{R}_0$ *yields families of nested ellipses containing* $\mathbf{R}_0 \cap \mathfrak{M}_0$*.*

Corollary: *The elements of the pseudogroup* $\widehat{\mathcal{G}}$ *label the lower branches of the tree structure of* $\mathfrak{M}_0$*.*

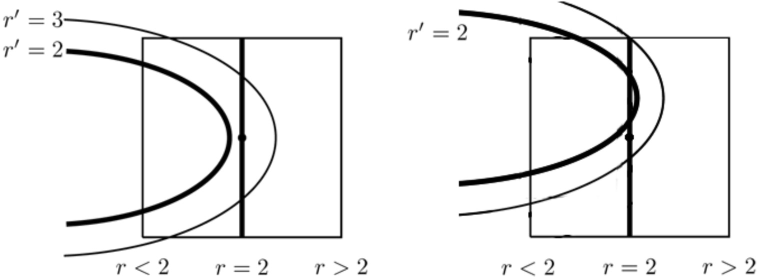


This looks like a ping-pong game, except that one player just barely keeps missing the ball. Besides, the play action is too slow to generate positive entropy.

Definition: A *Derived from Kuperberg* (DK) flow is obtained by choosing the embeddings so that we have:

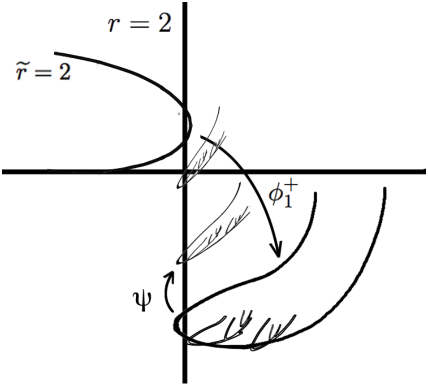
Parametrized Radius Inequality: For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^\epsilon(r', \theta', -2) \in \mathcal{L}_i$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

Modified radius inequality for the flow Φ_t^ϵ when $\epsilon < 0$ and $\epsilon > 0$:

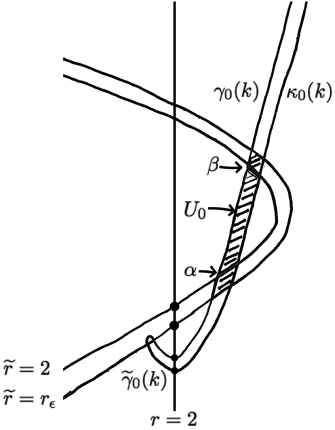


Meta-Principle: For $\epsilon > 0$ and "most" classes in the shape fundamental group, $[\gamma] \in \widehat{\pi}_1(\mathfrak{M}, \omega_0)$, there is a horseshoe subdynamics for the pseudogroup $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$ acting on $\mathbf{R}_0$ with a periodic orbit defining $[\gamma]$.

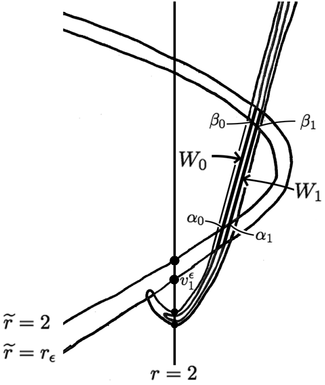
Action of $\widehat{\mathcal{G}}_\epsilon$ on the line $r = 2$ for $\epsilon > 0$.



Define a compact region $U_0 \subset \mathbf{R}_0$ which is mapped to itself by the map $\varphi = \psi^k \circ \phi_1^\epsilon$ for k sufficiently large. "k large" corresponds to translating the almost closed path far out in the surface $\mathfrak{M}_0$.



The images of the powers φ^ℓ form a δ-separated set for the action of the pseudogroup $\widehat{\mathcal{G}}_\epsilon$.



A horseshoe subsystem embedded in the *pseudogroup action*!

- For $\epsilon < 0$, the dynamics of the flow Φ^{ϵ} is tame, and completely predictable, except that as $\epsilon \to 0$ the dynamics approaches that of the Kuperberg flow.

- For $\epsilon > 0$, the dynamics of the flow Φ^{ϵ} is chaotic, as the flow generates horseshoes in the induced pseudogroup.

The calculation of the entropy and Hausdorff dimension for horseshoe dynamical systems is standard theory.

Question: How do the horseshoes for a perturbed flow Φ^{ϵ} vary with $\epsilon > 0$? Do they vary continuously? Same question for the Hausdorff dimensions of these subsystems?

• The map ψ above is the return map in the pseudogroup, generated by the flow on the Reeb cylinder. This illustrates the *Kuperberg Principle*.

Problem: Given a C^3-aperiodic flow on a 3-manifold, is there a way to identify a map in its induced pseudogroup that embodies the Kuperberg Principle?

• The map ϕ above is induced by the minimality of the aperiodic flow in its minimal set.

Question: Must the induced pseudogroup for a C^3-aperiodic flow on a 3-manifold always admit a nearby perturbation with horseshoe dynamics in its induced pseudogroup? Is this true if we assume that the minimal set has *unstable shape*?

feliz aniversário – parabéns Paul