Entropy for Y-like matchbox manifolds

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Motivation

van Dantzig – Vietoris solenoid, defined by tower of coverings:

$$\mathcal{P} \;\equiv\; \cdots \longrightarrow \mathbb{S}^1 \xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

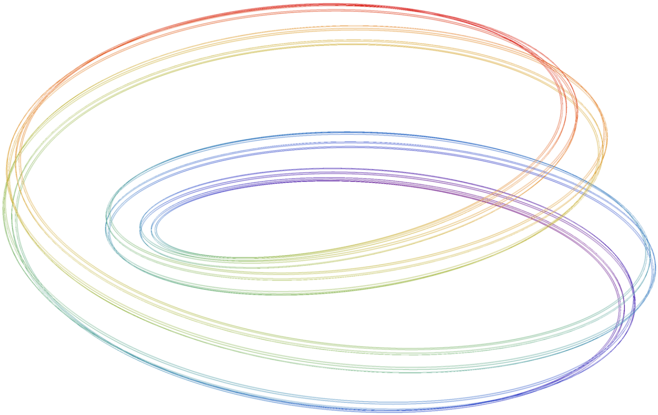
where each p_ℓ is a proper covering map of degree $n_\ell > 1$.

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1\} \;\subset \prod_{\ell \geq 0} \mathbb{S}^1$$

$\mathcal{S}_{\mathcal{P}}$ is given the (relative) product topology.

$\mathcal{P}$ is called a presentation for $\mathcal{S}_{\mathcal{P}}$. Set $n_{\mathcal{P}} = \{n_1, n_2, n_3, \ldots\}$.

Van Dantzig - Vietoris solenoid [1930]



Proposition: The homeomorphism type of $\mathcal{S}_{\mathcal{P}}$ depends only on the set of integers $n_{\mathcal{P}}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be sequences of positive integers, and let P be the infinite set of prime factors of the integers in the set $n_{\mathcal{P}}$, included with multiplicity, and Q the same of $n_{\mathcal{Q}}$.

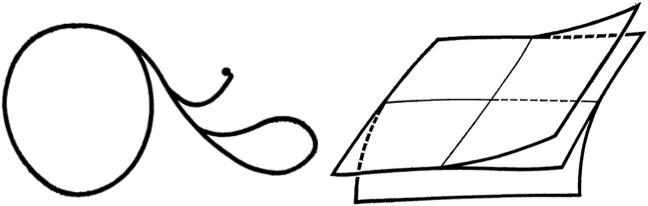
Theorem: [Bing, 1960; McCord, 1965; Aarts and Fokkink, 2004] The solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic if and only if there is bijection between a cofinite subset of P and a cofinite subset of Q.

Question: What are the invariants of homeomorphism type, for continua defined by inverse limits of manifold-like spaces?

Branched manifolds

Y is a *branched n-manifold* if it is a connected union of n-manifolds whose boundaries meet transversally.

Here are examples from [Williams, 1974] of how a branched 1-manifold and branched 2-manifold may look:



Y-like spaces

Y is a finite simplicial space.

$$\mathcal{P} \equiv \cdots \longrightarrow Y \xrightarrow{p_{\ell+1}} Y \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} Y \xrightarrow{p_1} Y$$

Each p_ℓ is a surjection. Set

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \{p_{\ell+1} \colon Y \to Y\} \subset \prod_{\ell \geq 0} Y$$

$\mathcal{S}_\mathcal{P}$ with the (relative) product topology is a *Y-like space*.

Problem: Give invariants of the homeomorphism type of $\mathcal{S}_\mathcal{P}$.

Smooth $\mathcal{Y}$-like spaces

Y_ℓ compact branched n-manifolds, $\ell \geq 0$.

$$\mathcal{P} \;\equiv\; \cdots \longrightarrow Y_{\ell+1} \xrightarrow{p_{\ell+1}} Y_\ell \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} Y_1 \xrightarrow{p_1} Y_0$$

Each p_ℓ is a proper covering map of branched manifolds. Set:

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \;\{p_{\ell+1}\colon Y \to Y\} \;\subset \prod_{\ell \geq 0} Y$$

$\mathcal{S}_{\mathcal{P}}$ with the (relative) product topology is a *smooth $\mathcal{Y}$-like space*.

Theorem: The *geometric entropy* is defined for $\mathcal{S}_{\mathcal{P}}$ and is an invariant of the homeomorphism type of $\mathcal{S}_{\mathcal{P}}$.

Matchboxes manifolds

Definition: A *matchbox manifold* is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that the transverse model space $\mathfrak{X}$ is totally disconnected, and for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{X}_x \subset \mathfrak{X}$ is a clopen subset, hence is homeomorphic to a Cantor set.

Figure: Blue tips are points in Cantor set $\mathfrak{X}_x$

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum $\equiv$ a compact, connected metric space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$;
- each $\mathfrak{X}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$;
- plaques $\mathcal{P}_i(z) = \varphi_i^{-1}([-1,1]^n \times \{z\})$ are connected, $z \in \mathfrak{X}_i$;
- for $U_i \cap U_j \neq \emptyset$, each plaque $\mathcal{P}_i(z)$ intersects at most one plaque $\mathcal{P}_j(z')$, and change of coordinates along intersection is smooth diffeomorphism;

$+$ some other technicalities.

Eaxamples of matchbox manifolds:

- *Exceptional minimal sets* for foliations of compact manifolds;
- *Inverse limit spaces* defined by a sequence of proper coverings of compact branched manifolds;
- *Expanding attractors* for Axiom A dynamical systems;
- *Tiling spaces* associated to aperiodic, locally-finite tilings of Euclidean space;
- *Suspensions* of minimal pseudogroup actions on a Cantor set, such as those obtained from the Ghys-Kenyon construction for infinite graphs.

Foliation pseudogroup

Let $\mathfrak{M}$ be a matchbox manifold, with a choice of a regular covering by foliated coordinate charts,

$$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$$

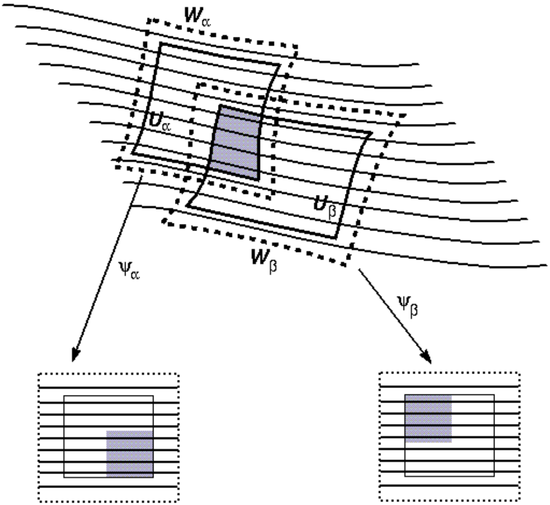
Identify $\mathfrak{T}_i = \varphi_i^{-1}(0 \times \mathfrak{X}_i) \subset \mathfrak{M}$ with $\mathfrak{X}_i$ and thus $\mathfrak{X}$ with $\mathfrak{T} \subset \mathfrak{M}$.

Let $\mathcal{G}_{\mathfrak{X}}$ be the pseudogroup for $\mathfrak{M}$ generated by the collection of transition maps

$$\mathcal{G}_{\mathfrak{X}}^0 \equiv \{h_{i,j} \mid U_i \cap U_j \neq \emptyset\}$$

The pseudogroup structure, or the pseudogroup action on a transverse Cantor set, is what distinguishes the study of inverse limits with a matchbox structure over other types of inverse limits.

The maps $h_{i,j}$ are pictured as this:



Let $\mathcal{G}_{\mathfrak{X}}^{*}$ be the collection of all compositions of elements of $\mathcal{G}_{\mathfrak{X}}^{0}$ on the open domains for which the composition is defined.

Definition: For $g \in \mathcal{G}^{*}$, the *word length* $\|g\| \leq m$ if g can be expressed as the composition of at most m elements of $\mathcal{G}_{\mathfrak{X}}^{0}$.

That is, $\|g\| \leq m$ implies that $g = h_{i_\ell} \circ \cdots h_{i_1}$ for $\ell \leq m$.

The inclusion $\mathfrak{T} \subset \mathfrak{M}$ induces a metric $d_{\mathfrak{T}}$ on the transversal, and hence a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$.

Geometric entropy

The *geometric entropy* for pseudogroup actions was introduced in [Ghys, Langevin & Walczak, 1988], to give a measure of the "exponential complexity" of the orbits of the holonomy pseudogroup for a foliation.

Let $\epsilon > 0$ and $\ell > 0$.

A subset $\mathcal{E} \subset \mathfrak{X}$ is said to be $(d_{\mathfrak{X}}, \epsilon, \ell)$-separated if for all $w, w' \in \mathcal{E} \cap \mathfrak{X}_i$ there exists $g \in \mathcal{G}^*_{\mathfrak{X}}$ with $w, w' \in \mathrm{Dom}(g) \subset \mathfrak{X}_i$, and $\|g\|_w \leq \ell$ so that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$.

If$w \in \mathfrak{X}_i$ and $w' \in \mathfrak{X}_j$ for $i \neq j$ then they are (ϵ, ℓ)-separated.

The "expansion growth function" is:

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathfrak{X} \text{ is } (d_{\mathfrak{X}}, \epsilon, \ell)\text{-separated}\}$$

The (geometric) entropy is the *asymptotic exponential growth type* of the expansion growth function:

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon) = \limsup_{\ell \to \infty} \ln \{h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell)\} / \ell$$

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = \lim_{\epsilon \to 0} h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon)$$

Then $0 \leq h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) \leq \infty$.

Properties of entropy – see [Ghys, Langevin & Walczak, 1988].

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup, acting on the compact space $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$. Then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}})$ is independent of the choice of metric $d_{\mathfrak{X}}$, so we can write $h(\mathcal{G}_{\mathfrak{X}})$.

Let $\mathfrak{X}_0 \subset \mathfrak{X}$ be a clopen subset which intersects every orbit of the action of $\mathcal{G}_{\mathfrak{X}}$, and let $h(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}_0)$ denote the restricted entropy.

Proposition:

- $h(\mathcal{G}_{\mathfrak{X}}) = 0 \iff h(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}_0) = 0$
- $h(\mathcal{G}_{\mathfrak{X}}) > 0 \iff h(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}_0) > 0$
- $h(\mathcal{G}_{\mathfrak{X}}) = \infty \iff h(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}_0) = \infty$

Theorem: Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be matchbox manifolds. If there exists a homeomorphism $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ then the entropies $h(\mathcal{G}_{\mathfrak{X}_1})$ and $h(\mathcal{G}_{\mathfrak{X}_2})$ have the same nature: they are both either zero, positive and finite, or infinite.

Proof: Uses ideas from

[*Classifying matchbox manifolds*, Clark, Hurder, Lukina, 2013].

[*Lipschitz mathchbox manifolds*, Hurder, 2013].

- *h* induces a Morita equivalence between actions of $\mathcal{G}_{\mathfrak{X}_1}$ and $\mathcal{G}_{\mathfrak{X}_2}$
- By a change of metric on $\mathfrak{X}_1$ the map *h* induces a Lipschitz Morita equivalence.
- Entropy is independent of metric.
- Result follows by relation of entropy for Lipschitz Morita equivalent pseudogroup actions.

Example: Let $\mathfrak{M}$ be homeomorphic to a Vietoris solenoid, then $h(\mathcal{G}_{\mathfrak{X}}) = 0$.

Example: Let $\mathfrak{M}$ be homeomorphic to a Denjoy generalized solenoid, then $h(\mathcal{G}_{\mathfrak{X}}) = 0$.

Using the results of the 2010 thesis of Aaron Brown, we have:

Example: Let $\mathfrak{M}$ be homeomorphic to an 2-dimensional attractor for an Axiom A diffeomorphism $f\colon M \to M$ for a compact 3-manifold, then $h(\mathcal{G}_{\mathfrak{X}}) > 0$.

There is a sharper form of the invariance of the entropy.

Let $\mathfrak{M}_i$ be a matchbox manifold with presentation $\mathcal{P}_i$ for $i = 1, 2$.

Assume that the base branched manifold for $\mathcal{P}_i$ is Y_0 for each.

Let $\Pi_i \colon \mathfrak{M}_i \cong \mathcal{S}_{\mathcal{P}_i} \to Y_0$ be the projection map.

A homeomorphism $h \colon \mathfrak{M}_1 \to \mathfrak{M}_2$ is base preserving if $\Pi_2 \circ h = \Pi_1$.

Theorem: Let $\mathfrak{M}_i$ be a matchbox manifold with $\mathfrak{M}_i \cong \mathcal{S}_{\mathcal{P}_i}$ for a presentation $\mathcal{P}_i$ for $i = 1, 2$. If there exists a base preserving homeomorphism $h \colon \mathfrak{M}_1 \to \mathfrak{M}_2$ then $h(\mathcal{G}_{\mathfrak{X}_1}) = h(\mathcal{G}_{\mathfrak{X}_2})$.

Question: Let $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$ for a presentation $\mathcal{P}$ by branched n-manifolds. How is $h(\mathcal{G}_{\mathfrak{X}})$ related to the properties of the presentation $\mathcal{P}$?

Definition: Let $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$ for a presentation $\mathcal{P}$ by branched n-manifolds. We say that $\mathcal{P}$ is *chaotic* if the typical leaf $L \subset \mathfrak{M}$ contains a quasi-isometrically embedded tree with exponential growth rate.

Theorem: Let $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$ for a presentation $\mathcal{P}$ by branched n-manifolds. Then $h(\mathcal{G}_{\mathfrak{X}}) > 0$ if and only if $\mathcal{P}$ is chaotic.

Problem: Give properties of the bonding maps of a Y-like presentation $\mathcal{P}$ by branched n-manifolds which suffices to imply that $\mathcal{P}$ is chaotic?

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Thank you for your attention.