A new perspective on Riemannian foliations

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New trends on Foliated and Stratified Spaces:
Topology, Geometry and Analysis

Commemorating the 70th birthday of Xosé Masa

Santiago do Compostela– June 18-19, 2018

In the beginning...

Bruce Reinhart, "*Foliated manifolds with bundle-like metrics*", **Ann. of Math**, 69:119–132, 1959

> *In addition to the foliation, there is postulated the existence of a "bundle-like"' Riemannian metric, which locally is very similar to a product metric. Such a metric always exists for the special case of a fibre space.*

Edmond Fedida, "*Sur les feuilletages de Lie*", **C. R. Acad. Sci. Paris Sér. A-B**, 272:A999–A1001, 1971.

Lawrence Conlon, "*Transversally parallelizable foliations of codimension two*", **Trans. Amer. Math. Soc.**, 194:79–102, 1974.

> *studied Riemannian foliations which are transversally Lie, with a dense leaf*

Pierre Molino, *Feuilletages transversalement parallèlisables et feuilletages de Lie. Applications*", **C. R. Acad. Sci. Paris Sér. A-B**, 282:A99–A101, 1976.

> *Si $\mathcal{F}$ admet un parallélisme au sens de L. Conlon, on obtient le théorème suivant: les adhérences des feuilles de $\mathcal{F}$ définissent une fibration $\pi \colon V \to W$ telle que $\mathcal{F}$ induit sur les fibres de π des feuilletages de Lie au sens de E. Fedida.*

This result includes the fundamental observation: the closures of the leaves of $\mathcal{F}$ are minimal sets of the foliation, and these closures form a continuous decomposition of the total space of the foliation.

Pierre Molino, **Feuilletages riemanniens**, Université des Sciences et Techniques du Languedoc, Institut de Mathématiques, Montpellier, 1983.

> *The principal technique in this study is to lift the foliation to a foliation of the same dimension on the orthonormal frame bundle associated to the normal bundle of the original foliation. The "lifted foliation"' is invariant under the action of the orthogonal group on the frame bundle and is transversely parallelizable, i.e., its normal bundle is parallelizable. The structure of the "lifted foliation" is studied as a special case of parallelizable foliations, and these results pushed down to the original foliation using the invariance under the orthogonal group.*

These lecture notes, and especially their 1988 translation from French by Grant Cairns, introduced the study of Riemannian foliations to a broad audience. This explosion of interest is evidenced by the Appendices of the English translation:

- A. *Variations on Riemannian flows*, by Y. Carrière;
- B. *Basic cohomology and tautness of Riemannian foliations*, by V. Sergiescu;
- C. *The duality between Riemannian foliations and geodesible foliations*, by G. Cairns;
- D. *Riemannian foliations and pseudogroups of isometries*, by E. Salem;
- E. *Riemannian foliations: examples and problems*, by É. Ghys.

The work of Xosé Masa solved a fundamental problem combining the topics of Appendices B. and C.

Xosé Masa, "*Duality and minimality in Riemannian foliations*", **Comment. Math. Helv.**, 67:17–27, 1992.

> *In this work, we prove that a Riemannian foliation $\mathcal{F}$ defined on a smooth closed manifold M is minimal, in the sense that there exists a Riemannian metric on M for which all the leaves are minimal submanifolds, if and only if $\mathcal{F}$ is unimodular, that is, the basic cohomology of $\mathcal{F}$ in maximal dimension is nonzero.*

This paper by Xosé Masa, and the related work by his student,

Jesús Álvarez López, "*The basic component of the mean curvature of Riemannian foliations*", **Ann. Global Anal. Geom.**, 10:179–194, 1992,

led to an explosion of further works studying the analytic and geometric properties of Riemannian foliations.

> *The study of Riemannian foliations is reaching mid-life. Perhaps, it is time for a Little Red Sports Car...*

In this talk, we consider a new model for further studies of "Riemannian foliations". It is sporty, and very algebraic.

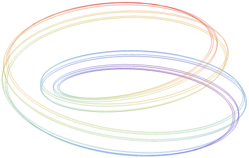
Motivated by a question posed by Ghys in Appendix E. to Molino's book, Alberto Candel and Jesús Álvarez López began the study of a topological version of Riemannian foliations, in their work

"*Equicontinuous foliated spaces*", **Math. Z.**, 263:725–774, 2009.

The starting point is the fact that a Riemannian foliation is a foliation whose holonomy pseudogroup is a pseudogroup of local isometries of a Riemannian manifold. More generally, for equicontinuous pseudogroups, one has the notion of equicontinuous foliated spaces. This leads to a version of the Molino structure theory, in the work

Álvarez López and Moreira Galicia, "*Topological Molino's theory*", **Pacific J. Math.**, 280:257–314, 2016.

We will consider foliated spaces which are transversally totally disconnected.



The objects of study are called various names in the literature:

- *Generalized laminations*, [Ghys, Lyubich & Minsky]
- *Matchbox manifolds*, [Aarts & Martens, Clark & Hurder]
- *Solenoidal manifolds*, [Sullivan]

All are foliated spaces as introduced in the book

- Moore & Schochet, **Global analysis on foliated spaces**, 1988.

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum $\equiv$ a compact, connected metric space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts
  $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$;
- each $\mathfrak{X}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$;
- plaques $\mathcal{P}_i(z) = \varphi_i^{-1}([-1,1]^n \times \{z\})$ are connected, $z \in \mathfrak{X}_i$;
- for $U_i \cap U_j \neq \emptyset$, each plaque $\mathcal{P}_i(z)$ intersects at most one plaque $\mathcal{P}_j(z')$, and changes of coordinates along intersection of plaques are smooth diffeomorphisms;

\+ some other technicalities.

The path connected components of $\mathfrak{M}$ are the leaves of the foliation $\mathcal{F}$. To the above list, we add the condition:

• there is a leafwise smooth Riemannian metric on the leaves of $\mathcal{F}$, so that each leaf of $\mathcal{F}$ is a complete Riemannian manifold with bounded geometry.

It follows that associated to a matchbox manifold are many of the traditional aspects of Riemannian manifolds, such as leafwise curvature, leafwise De Rham cohomology, leafwise operators, foliation-preserving transformation groups, and invariants constructed from these data.

Basic question: What are these spaces, and their properties?

The lamination analog of a Riemannian foliation is a matchbox manifold $\mathfrak{M}$ whose holonomy pseudogroup $\mathcal{G}$ is generated by minimal equicontinuous actions on a Cantor space $\mathfrak{X}$:

- A Cantor action $\varphi \colon \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in \mathcal{G}.$$

Here, $\mathcal{G}$ denotes either the pseudo$\star$group of local actions on the Cantor set $\mathfrak{X}$, or is an actual group action on $\mathfrak{X}$.

The simplest examples are given by the *Vietoris solenoids*.

Let $\mathbf{P} = (p_1, p_2, \ldots)$ be an infinite sequence of integers, $p_i > 1$.
Let $f^i_{i-1} \colon \mathbb{S}^1 \to \mathbb{S}^1$ be a p_i-to-1 self-covering map of a circle.

A Vietoris solenoid is the inverse limit space

$$\Sigma_{\mathbf{P}} = \{(y_i) = (y_0, y_1, y_2, \ldots) \mid f^i_{i-1}(y_i) = y_{i-1} \ \} \subset \prod_{i \geq 0} \mathbb{S}^1$$

with subspace topology from the Tychonoff topology on $\prod_{i \geq 0} \mathbb{S}^1$.
There is a projection map $\Pi \colon \Sigma_{\mathbf{P}} \to \mathbb{S}^1$, $\Pi(y_0, y_1, y_2, \ldots) = y_0$.

The fibre $\mathfrak{X}_b = \Pi^{-1}(b) = \{(b, y_1, y_2, \ldots)\} \subset \Sigma_{\mathbf{P}}$ is a Cantor set, for each $b \in \mathbb{S}^1$.

The fundamental group $\pi_1(\mathbb{S}^1, b) = \mathbb{Z}$ acts on $\mathfrak{X}_b$ via lifts of paths in $\mathbb{S}^1$, so the monodromy action on the fiber defines a group action $\Phi \colon \mathbb{Z} \times \mathfrak{X}_b \to \mathfrak{X}_b$ which is a classical odometer action.

Theorem (Bing 1961, McCord 1965): The homeomorphism class of a Vietoris solenoid is completely determined by the tail equivalence class of the prime divisors of the degrees of the coverings defining it.

McCord began the study of higher-dimensional generalizations of the Vietoris solenoid in 1965.

Let M_0 be a connected closed manifold, and let $\mathcal{P} \equiv \{f^i_{i-1} : M_i \to M_{i-1} \mid i \geq 1\}$ be a sequence of finite-to-one proper covering maps. $\mathcal{P}$ is called a *presentation*. Then

$$\mathfrak{M}_\infty = \varprojlim \{f^i_{i-1} : M_i \to M_{i-1} \mid i \geq 1\}$$

is a compact connected metrizable space, called a *(weak) solenoid*.

There is a fibration map $\Pi_0 \colon \mathfrak{M}_\infty \to M_0$, and for $b \in M_0$ the fiber $\mathfrak{X}_b = \Pi_0^{-1}(b)$ is a Cantor space. The monodromy of the fibration yields a group action $\varphi \colon G \times \mathfrak{X}_b \to \mathfrak{X}_b$ where $G = \pi_1(M_0, b)$.

Theorem [McCord, 1965]. A solenoid is a *matchbox manifold*, or *generalized lamination* (Ghys), or *solenoidal manifold* (Sullivan).

A topological space Z is homogeneous if for any pair of points $x, y \in Z$, there is a homeomorphism $h \colon Z \to Z$ with $h(x) = y$.

Proposition: A transversally parallelizable Riemannnian foliation of a compact connected manifold is homogeneous.

A homeomorphism $h \colon \mathfrak{M}_1 \to \mathfrak{M}_2$ of matchbox manifolds must map the leaves of $\mathcal{F}_1$ to the leaves of $\mathcal{F}_2$ since the leaves consist of the path components of the space. Thus, the following result can be considered as a matchbox version of Fedida's Theorem:

Theorem (Clark-H, 2013): Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid $\mathfrak{M}_\infty$ for some presentation $\mathcal{P}$. Moreover, if $\mathfrak{M}$ is a homogeneous space, then the fiber $\mathfrak{X}_b \subset \mathfrak{M}_\infty$ is a profinite group.

This is proved in the paper "*Homogeneous matchbox manifolds*", **Transactions A.M.S.**, 365:3151-3191, 2013.

However, if $\mathfrak{M}$ is not homogeneous, then the fiber $\mathfrak{X}_b$ is not a profinite group, but is a quotient of a profinite group by a non-trivial closed subgroup of some profinite group. It is the incredible variety of closed Cantor subgroups of profinite groups that is the source of troubles.

Topological Molino Theory deviates from the smooth theory in this non-homogeneous case, as there are issues with defining the paralellizable frame bundle associated to the foliation, which is a key step introduced by Molino for his general structure theory.

To understand this case, we digress into a discussion of the theory of equicontinuous minimal Cantor actions.

We recall some classical notions from Auslander, **Minimal flows and their extensions**, 1988. G denotes a countable group.

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ on a Cantor space $\mathfrak{X}$, if U is a non-empty clopen subset, and for any $g \in G$, $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. That is, the translates of U form a partition of the Cantor set $\mathfrak{X}$. It follows that

$$G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\}$$

is a subgroup of finite index in G, called the *stabilizer* of U.

For a Cantor space $\mathfrak{X}$, let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (compact open) subsets of $\mathfrak{X}$. Note that for $\phi \in \mathbf{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$.

Proposition (Glasner and Weiss, 1995): A minimal Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if and only if, for the induced action $\Phi_* \colon G \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$, the G-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite.

Corollary: Let $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous Cantor action. Then for all $x \in \mathfrak{X}$ and all $\delta > 0$, there exists an adapted clopen set U with $x \in U \subset B_{\mathfrak{X}}(x, \delta)$.

Definition: Let $\varphi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be minimal equicontinuous Cantor actions, for $i = 1, 2$. Say that φ_1 is return equivalent to φ_2 if there exist

- adapted clopen subsets $U_i \subset \mathfrak{X}_i$ for $i = 1, 2$
- a homeomorphism $h \colon U_1 \to U_2$

such that h induces an isomorphism $\alpha_h \colon G_1|U_1 \to G_2|U_2$ of the restricted groups, where $G_i|U_i \subset \mathbf{Homeo}(U_i)$.

Remark: When $U_i = \mathfrak{X}_i$ for $i = 1, 2$, and the actions are effective, this reduces to the notion of topological conjugacy of the actions, where $\alpha_h \colon G_1 \to G_2$ intertwines the actions.

Theorem (Clark, H, Lukina 2017): Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be weak solenoids. Suppose that there exists a homeomorphism $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$, then the fiber monodromy actions associated to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.

In Molino Theory, if two Riemannian foliations are diffeomorphic, then their Molino structure groups are isomorphic.

We next consider the structure theory for equicontinuous minimal Cantor actions, and how they behave under return equivalence.

We introduce the group chain (or odometer) model for an equicontinuous minimal Cantor action, and the related tree model for an action.

Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous Cantor action.

For a choice of basepoint $x \in \mathfrak{X}$ and scale $\epsilon > 0$, there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \epsilon$. Iterating this construction, for a given basepoint x, one can always construct the following:

Definition: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous action on a Cantor space $\mathfrak{X}$. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is said to be an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap\ U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.

For such a collection, setting $G_\ell = G_{U_\ell}$ we obtain a descending chain of finite index subgroups

$$\mathcal{G}_{\mathcal{U}} = \{ G = G_0 \supset G_1 \supset G_2 \supset \cdots \} \ .$$

Set $X_\ell = G/G_\ell$ and note that G acts transitively on the left on X_ℓ. The inclusion $G_{\ell+1} \subset G_\ell$ induces a natural G-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$\mathfrak{X}_\infty \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell > 0\}$$

which is a Cantor space with the Tychonoff topology, and the action on the factors X_ℓ induces a minimal equicontinuous action $\Phi_x \colon G \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$.

The action $\Phi_x \colon G \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$ is called the *generalized odometer* model, or also called a *subodometer*, by Cortez & Petite in their work "*G-odometers and their almost one-to-one extensions*", 2008.

We give some remarks on this construction.

- Each $X_i = G/G_i$ is a finite set with a left action of G. It is a group if G_i is normal in G, and then the Cantor space $\mathfrak{X}_\infty$ is a profinite group.
- The intersection $K(\mathcal{G}_\mathcal{U}) = \bigcap_{\ell \geq 0} G_\ell$ is called the kernel of $\mathcal{G}_\mathcal{U}$.
- For $g \in K(\mathcal{G}_\mathcal{U})$, the left action of g on X_ℓ fixes the coset $e_\ell \in X_\ell$ and hence fixes the limiting point $e_\infty \in \mathfrak{X}_\infty$.

We next recall the tree model for the action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$.

First, again choose an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action, $\mathcal{U} = \{U_i \subset \mathfrak{X} \mid i \geq 1\}$.

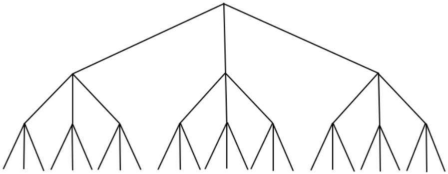
Note that by assumption we have $\bigcap U_i = \{x\}$.

Next, associate a vertex $v_{i,g}$ at level i to each $g \cdot U_i$.

Join $v_{i,g}$ and $v_{i+1,h}$ by an edge if and only if $h \cdot U_{i+1} \subset g \cdot U_i$.

A sequence of vertices $(v_{i,g_i})_{i \geq 0}$ is a path in the space $\mathcal{P}_T$ of paths in T, and $\mathfrak{X} \cong \mathcal{P}_T$.

The subgroup G_i of elements which stabilize U_i has finite index in G, and there is a group chain of stabilizers $\mathcal{G}_{\mathcal{U}} \equiv \{G_i\}_{i \geq 0}$ associated to the action.



We next introduce the Ellis group associated to an action.

An action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ induces a representation $\Phi\colon G \to \mathbf{Homeo}(\mathfrak{X})$ with image group

$$H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X})$$

Definition: The closure $E(\Phi)$ of H_Φ, in the topology of pointwise convergence on maps, is called the *Ellis (enveloping) semigroup*.

Proposition (Ellis, 1969): Let φ be an equicontinuous Cantor action. Then $E(\Phi) = \overline{H_\Phi} =$ closure of H_Φ in the *uniform topology on maps*. In particular, $\overline{H_\Phi}$ is a profinite group.

For $x \in \mathfrak{X}$ let $\overline{H_\Phi}_x = \{h \in \overline{H_\Phi} \mid h(x) = x\}$ be its isotropy group.

Lemma: The left action of $\overline{H_\Phi}$ on $\mathfrak{X}$ is transitive, hence $\mathfrak{X} \cong \overline{H_\Phi}/\overline{H_\Phi}_x$ and the closed subgroup $\overline{H_\Phi}_x \subset \overline{H_\Phi}$ is independent of the choice of basepoint x, up to topological isomorphism.

We give a representation for $\overline{H_\Phi}_x$ in terms of the odometer model for the action.

The normal core N of a subgroup $H \subset G$ is the largest subgroup $N \subset H$ which is normal in G.

Let $C_i \subset G_i$ be the normal core of G_i in G, then C_i has finite index in G. Define the profinite group

$$G_\infty \equiv \varprojlim \; \{q_i \colon G/C_{i+1} \to G/C_i \mid i > 0\} \ .$$

Each group G/C_i acts on the finite set $X_i = G/G_i$, so there is an induced action $\widehat{\Phi}_\infty \colon G_\infty \to \mathbf{Homeo}(\mathfrak{X}_\infty) \cong \mathbf{Homeo}(\mathfrak{X})$.

Theorem (Dyer-H-Lukina, 2016). $\overline{H_\Phi} \cong \widehat{\Phi}_\infty(G_\infty)$, and

$$\mathcal{D}_\infty \equiv \varprojlim \; \{\pi_i \colon G_{i+1}/C_{i+1} \to G_i/C_i \mid \ell \geq 0\} \; \cong \overline{H_{\Phi}}_x \; . \qquad (1)$$

The inverse limit group $\mathcal{D}_\infty$ is called the *discriminant group* for the action. Its *non-triviality* is the obstruction to the existence of a *transitive* right action on $\mathfrak{X}$ that commutes with the left action φ.

The approach to the study of weak solenoids via their monodromy Cantor actions obtained from group chains was initiated in the work of [Fokkink & Oversteegen, 2002].

Let $\Pi_0 \colon \mathfrak{M}_\infty \to M_0$ be a weak solenoid defined by the system of maps $\{f_0^i \colon M_i \to M_0 \mid i > 0\}$, where $f_0^i = f_0^1 \circ \cdots \circ f_{i-1}^i$.

Choose a basepoint $b \in M_0$ and basepoints $x_i \in M_i$ such that $f_0^i(x_i) = b$. Set $x = \lim x_i \in \mathfrak{X}_b \equiv \Pi_0^{-1}(b)$.

Define $G = G_0 = \pi_1(M_0, b)$, and let $G_i \subset G$ be the subgroup defined by $G_i = \mathrm{Image}\{(f_0^i)_\# \colon \pi_1(M_i, x_i) \to \pi_1(M_0, b)\}$.

$\{G_i \mid i \geq 0\}$ is a descending chain of subgroups of finite index in G.

The subgroups G_i are not assumed to be normal in G.

Example 1: Consider the Vietoris solenoid

$$\Sigma = \{f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1\}.$$

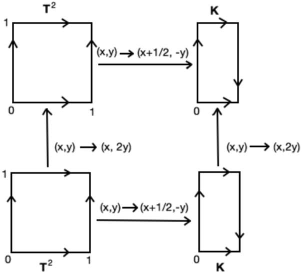
Then $G = \pi_1(\mathbb{Z}, 0) = \mathbb{Z}$, and $G_i = (p_1 \cdots p_i)\,\mathbb{Z}$, where p_i is the degree of f^i_{i-1}.

Then $G/G_i = \mathbb{Z}/p_1 \cdots p_i\mathbb{Z}$.

Since $\mathbb{Z}$ is abelian, $G_i = C_i$, and so G_i/C_i is a trivial group.

Thus $C_\infty \cong \mathfrak{X}_b$, where $\mathfrak{X}_b$ is a fibre of $\Sigma \to \mathbb{S}^1$, and so the discriminant group $\mathcal{D}_\infty$ of the Vietoris solenoid is trivial.

Example 2: Here is a more interesting example, with $\mathcal{D}_\infty$ non-trivial. It is due to [Rogers & Tollefson, 1971/72].



Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and consider an involution

$$r \times i(x, y) = (x + \tfrac{1}{2}, -y).$$

The quotient $K = \mathbb{T}^2/(x, y) \sim r \times i(x, y)$ is the Klein bottle.

The double cover $L\colon \mathbb{T}^2 \to \mathbb{T}^2\colon (x, y) \mapsto (x, 2y)$ induces a double cover $p : K \to K$.

Define K_∞ to be the inverse limit of the iterations of $p : K \to K$.

Since $i \circ L = p \circ i$, there is a double cover $i_\infty : \mathbb{T}_\infty \to K_\infty$.

The fundamental group of the Klein bottle is

$$G_0 = \pi_1(K, 0) = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

For the cover $p : K \to K$ we have

$$p_* \pi_1(K, 0) = \langle a^2, b \mid bab^{-1} = a^{-1} \rangle,$$

and for $p^n = p \circ \cdots \circ p : K \to K$ we have

$$G_n = (p^n)_* \pi_1(K, 0) = \langle a^{2^n}, b \mid bab^{-1} = a^{-1} \rangle.$$

The cosets of G/G_n are represented by $a^i G_i$, $i = 0, \ldots, n-1$,

$$C_n = \bigcap_{g \in G} g G_n g^{-1} = \langle a^{2^n} \mid bab^{-1} = a^{-1} \rangle.$$

Then $G_n / C_n = \{C_n, bC_n\}$, and so $\mathcal{D}_\infty \cong \mathbb{Z}/2\mathbb{Z}$.

The discriminant need not be an invariant of return equivalence for an equicontinuous Cantor action! We use the tree model for the action to analyze this, which leads to the notion of a wild action.

Recall that $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action, and $\mathcal{P}_T$ denotes the space of infinite paths starting at the root point corresponding to $\mathfrak{X}$.

Then we have a minimal action $\Phi\colon G \times \mathcal{P}_T \to \mathcal{P}_T$.

The group G_i stabilizes a branch of a tree, i.e fixes a vertex at level i. Then by minimality of the action, the set of vertices at level i is identified with G/G_i. There is a homeomorphism

$$\phi : \mathcal{P}_T \to \mathfrak{X}_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\},$$

equivariant with respect to the actions of G on $\mathcal{P}_T$ and $\mathfrak{X}_\infty$.

The core subgroup $C_i = \bigcap_{g \in G} g G_i g^{-1} \subset G$ fixes every vertex at level i, and the quotient group G/C_i acts transitively on the set of vertices at level i, which correspond to the set G/G_i. Then

$$C_\infty = \varprojlim \{G/C_i \to G/C_{i-1}\}$$

is a profinite group acting transitively on the path space $\mathcal{P}_T$.

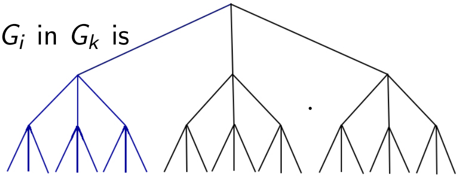
We use this model to consider the discriminant groups of the action φ restricted to an adapted clopen subset $U \subset \mathfrak{X}$.

The set of paths through vertex v_k at level k is a clopen set $U_k \subset \mathcal{P}_T$. Assume that $x \in U_k$.

The restricted action on U_k is given by $\Phi_k \colon G_k \to Homeo(U_k)$.

For each $k \geq i$, the normal core of G_i in G_k is

$$C_{k,i} = \bigcap_{g \in G_k} g G_i g.$$



Observe that $C_{k,i} \supset C_i$ as the action of C_i fixes all vertices at level i, while $C_{k,i}$ fixes just those vertices at level i in the branches of the tree through the vertex v_k.

The isotropy group of the action of $\overline{\Phi(G_k)}$ at x is represented by

$$\mathcal{D}_{x,k} = \varprojlim \{ G_i / C_{k,i} \to G_{i-1} / C_{k,i-1} \mid i \geq k \}$$

which is the *discriminant group* of the action $\Phi_k \colon G_k \times U_k \to U_k$.

Note that there are coset inclusions $G_i/C_i \to G_i/C_{k,i}$.

Theorem (Dyer-H-Lukina, 2017) Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action with group chain $\{G_i\}_{i\geq 0}$ associated to a basepoint $x \in \mathfrak{X}$. Then for any $k > j \geq 0$ there is a well-defined surjective homomorphism

$$\Lambda_{k,j} : \mathcal{D}_{x,j} \to \mathcal{D}_{x,k}$$

of discriminant groups.

Definition: The action φ is said to be *stable*, if there exists j_0, such that for all $k > j \geq j_0$ the homomorphism $\Lambda_{k,j}$ is an isomorphism. If no such j_0 exist, then the action is said to be *wild*.

Theorem (Dyer-H-Lukina 2017): If the global monodromy action of a weak solenoid $\mathfrak{M}_\infty$ is stable, then there is a well-defined (homogeneous) Molino space for $\mathfrak{M}_\infty$.

This is proved in the paper

Dyer, Hurder, Lukina, "*Molino theory for matchbox manifolds*"', **Pacific Journal Math.**, 289:91-151, 2017.

When an action is not stable, so it is wild, then we define a new invariant of the action as follows:

Definition (H-Lukina, 2017): The *asymptotic discriminant* of the action $(\mathfrak{X}, G, \Phi)$ is the equivalence class of the chain of surjective group homomorphisms

$$\mathcal{D}_{x,0} \to \mathcal{D}_{x,1} \to \mathcal{D}_{x,2} \to \cdots$$

with respect to the tail equivalence relation.

The notion of "tail equivalence" is precisely defined in the work with Lukina, *Wild solenoids*, **Transactions A.M.S.**, 2018.

In our work we also show the following key property:

Theorem (H-Lukina, 2017): The asymptotic discriminant of an equicontinuous minimal Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is invariant under the return equivalence of actions. In particular, the property of being stable or wild is an invariant of return equivalence.

Corollary: The asymptotic discriminant of the monodromy of a weak solenoid is an invariant of its homeomorphism type.

Our work constructs infinite families of Cantor actions which have distinct asymptotic discriminant invariants. This is completely unique to the topological Molino theory.

We next discuss the notion of an "analytic Cantor action", which was introduced in the works of Álvarez Lopez, and its relation to wildness and the Hausdorff property for the action.

Let $U, V \subset \mathfrak{X}$ be clopen subsets of a Cantor space $\mathfrak{X}$.

• A homeomorphism $h \colon U \to V$ is quasi-analytic (QA) if either $U = V$ and h is the identity map, or for every *clopen* subset $W \subset U$ the fixed-point set of the restriction $h|W \colon W \to h(W) \subset V$ has no interior.

• A homeomorphism $h \colon U \to V$ is locally quasi-analytic (LQA) if for each $x \in U$ there exists a clopen neighborhood $x \in U' \subset U$ such that the restriction $h_{U'} \colon U' \to V' = H(U')$ is QA.

• A group action $\varphi \colon G \colon \mathfrak{X} \to \mathfrak{X}$ is LQA if for each $x \in \mathfrak{X}$, there exists a clopen neighborhood $x \in U$, such that the restrictions of elements of G to U are quasi-analytic.

Remarks:

- A free action $G \times \mathfrak{X} \to \mathfrak{X}$ is quasi-analytic.

- The automorphism group of a spherically homogeneous rooted tree T_d, acting on the Cantor set of ends, is not LQA.

Proposition: Suppose that $\varphi\colon G\colon \mathfrak{X} \to \mathfrak{X}$ is the restriction of a C^ω action on $\mathbb{D}^k$ for some $k \geq 1$. Then the action of φ is LQA.

Theorem (H-Lukina, 2017). Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action, where G is finitely generated. Then the action φ is stable if and only if the action of the profinite group G_∞ on $\mathfrak{X}_\infty$ satisfies the LQA property.

These notions yield a "non-realizable" criteria:

Corollary. If an equicontinuous minimal Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is not LQA, then any weak solenoid whose monodromy action is return equivalent to this action cannot be realized as the minimal set for a C^ω-foliation.

Question: Is there a version of this result for C^2-foliations?

Corollary: A weak solenoid whose monodromy action is not LQA admits an infinitely increasing chain of closed groups in the fundamental group $G_0 = \pi_1(M_0, b_0)$ of the base manifold M_0.

Proposition: Let $\mathfrak{M}_\infty$ be a weak solenoid whose base manifold M_0 has nilpotent fundamental group G_0. Then the monodromy action of the solenoid is stable.

Problem: It is unknown if an equicontinuous minimal action of a finitely generated torsion free nilpotent group G must be stable.

There is a nice "geometric proof" that an LQA Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is stable using the tree model for the action.

Let U_k be the clopen set defined by choosing a vertex v_k of the tree model. Consider the restricted action of G_k on $U_k \subset \mathfrak{X}$ with group chain $\{G_i\}_{i \geq k}$.

The elements in $C_{k,i} \subset G_i$ stabilize all vertices at level i in a branch of T, while the elements in $C_i \subset C_{k,i}$ stabilize all vertices at level i. Then let

$$S_k = \varprojlim \{C_{k,i}/C_i \to C_{k,i-1}/C_i\} \cong \ker\{\mathcal{D}_x \to \mathcal{D}_{x,k}\}.$$

Suppose that $h \in S_k$, with $h \neq id$, then h acts trivially on U_k, but acts non-trivially on $\mathfrak{X}$. If the action φ is not LQA, then such an h exists for clopen sets U_k with arbitrarily small diameter, and hence the action is not stable.

Finally, we note that the LQA property for a group action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ can be related to properties of the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \varphi)$ associated to the action.

Recall that for $g_1, g_2 \in G$, we say that $\varphi(g_1)$ and $\varphi(g_2)$ are *germinally equivalent* at $x \in \mathfrak{X}$ if $\varphi(g_1)(x) = \varphi(g_2)(x)$, and there exists an open neighborhood $x \in U \subset \mathfrak{X}$ such that the restrictions agree, $\varphi(g_1)|U = \varphi(g_2)|U$. We then write $\varphi(g_1) \sim_x \varphi(g_2)$.

For $g \in G$, denote the equivalence class of $\varphi(g)$ at x by $[g]_x$. The collection of germs $\mathcal{G}(\mathfrak{X}, G, \varphi) = \{[g]_x \mid g \in G \ , \ x \in \mathfrak{X}\}$ is given the sheaf topology, and forms an *étale groupoid* modeled on $\mathfrak{X}$.

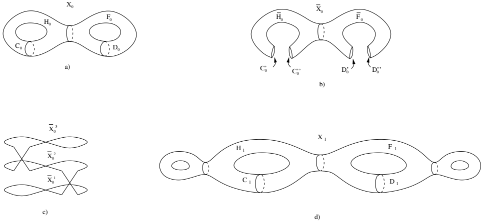
Theorem (H-Lukina, 2017). If an action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic, then $\mathcal{G}(\mathfrak{X}, G, \varphi)$ is Hausdorff.

The Hausdorff property for a germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \varphi)$ appears in the work of [Renault, 2008] on the C^*-algebra associated to the action, and has been studied in various works in C^*-algebras.

Problem: Find relations between the wild property for a group action, and the algebraic and topological invariants for the C^*-algebra associated to the action.

Some results on this problem are given in the work by Rui Excel, "*Non-Hausdorff étale groupoids*", **Proc. A.M.S.**, 2011.

Example 3: [Schori, 1966] gave the first example of a non-homogeneous weak solenoid. It is obtained by taking repeated 3-fold coverings starting with a closed surface Σ_2 of genus 2.



Proposition (Dyer-H-Lukina, 2017). The monodromy action of $G = \pi_1(\Sigma_2, b_0)$ on the fiber of the solenoid over Σ_2 is not LQA, and in particular is wild.

Example 4: Wild actions of arithmetic lattices. Lubotzky [1993] showed that the profinite completions of higher rank arithmetic lattices contain arbitrary products of finite torsion groups.

$\mathbf{SL}_N(\mathbb{Z}) = N \times N$ matrices with integer entries and determinant 1

$\widehat{\mathbf{SL}_N(\mathbb{Z})}$ profinite completion of $\mathbf{SL}_N(\mathbb{Z})$

$\mathcal{P}$ = set of all primes

$$\widehat{\mathbf{SL}_N(\mathbb{Z})} \equiv \varprojlim \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z}) \cong \mathbf{SL}_N(\widehat{\mathbb{Z}}) \cong \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \ . \qquad (2)$$

Let $G \subset \mathbf{SL}_N(\mathbb{Z})$ be a finite-index, torsion free subgroup.

Then G is finitely generated, and its profinite completion $\widehat{G}$ is a clopen subgroup of $\widehat{\mathbf{SL}_N(\mathbb{Z})}$, hence there is a cofinite $\mathcal{P}' \subset \mathcal{P}$, with

$$\prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \subset \widehat{G} \subset \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p)$$

Set $\widehat{H} = \prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$. Then there is a homomorphism with dense image $\alpha\colon G \to \widehat{H}$. For each $p \in \mathcal{P}'$, choose $D_p \subset \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$ with trivial normal core. Set $\mathcal{D} = \prod_{p \in \mathcal{P}'} D_p$.

Theorem [Hurder & Lukina, 2017]. For a closed subgroup $\mathcal{D} \subset \widehat{H}$ as above, the induced action $\varphi_{\alpha,\mathcal{D}}$ of G on $\widehat{H}/\mathcal{D}$ by α satisfies:

- The action $\varphi_{\alpha,\mathcal{D}}$ is minimal and equicontinuous;
- The action $\varphi_{\alpha,\mathcal{D}}$ is wild for suitable choices of $\mathcal{D}$;
- The actions $\varphi_{\alpha,\mathcal{D}}$ for uncountably many such choices of $\mathcal{D}$ yield non-homeomorphic weak solenoids.

Example 5: Arboreal actions of Galois groups.

The analogy between theory of finite coverings and Galois theory of finite field extensions suggests looking for examples of minimal Cantor actions arising from purely arithmetic constructions.

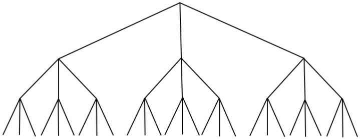
• [R.W.K. Odoni, 1985] began the study of arboreal representations of absolute Galois groups on the rooted trees formed by the solutions of iterated polynomial equations.

• [Jones, 2013] gives a nice introduction and survey of this program, from the point of view of arithmetic dynamical systems and number theory.

The following discussion concerns results of [Lukina, 2018].

Let $X = \mathcal{P}_d$ be the space of paths
in a spherically homogeneous rooted tree T_d.

Let G be any discrete group, acting on T_d
by permuting edges at each level
so that the paths are preserved.
The space of paths with the
cylinder topology is a Cantor set

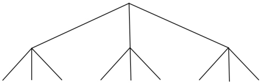


This action is equicontinuous.

Let $f(x)$ be an irreducible polynomial of degree d over a number field K. Let $\alpha \in K$, and suppose $f(x) = \alpha$ has d distinct solutions.

Identify α with the root of a d-ary tree T_d, and identify every solution $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1d}$ of $f(x) = \alpha$ with a vertex at level 1 in the tree.

$\mathrm{Gal}(K(f^{-1}(\alpha))/K)$ is identified with a subgroup of the symmetric group S_d.



For every α_{1i}, consider the equation

$$f(x) = \alpha_{1i}, \text{ so } f \circ f(x) = f(\alpha_{1i}) = \alpha.$$

Suppose there are d^2 distinct roots. Identify the solutions of $f(x) = \alpha_{1i}$ with the d vertices at level 2 connected with α_{1i} at level 1.

The action of $\mathrm{Gal}(K(f^{-2}(\alpha))/K)$ preserves the structure of the tree, so

$$\mathrm{Gal}(K(f^{-2}(\alpha))/K) \subseteq [S_d]^2,$$

where $[S_d]^2$ denotes the two-fold wreath product of symmetric groups S_d.

Continue by induction, assuming that for each $i > 0$ the polynomial $f^i(x)$ has d^i distinct roots.

In the limit, we get a d-ary infinite tree T_d of preimages of α under the iterations of $f(x)$, and the profinite group

$$\mathrm{Gal}_\infty(f) = \varprojlim \{\mathrm{Gal}(K(f^{-i}(\alpha))/K) \to \mathrm{Gal}(K(f^{-(i-1)}(\alpha))/K)\},$$

a subgroup of the infinite wreath product $Aut(T_d) = [S_d]^\infty$.

The group $\mathrm{Gal}_\infty(f)$ is called an *arboreal representation* of the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$.

The representation depends on the polynomial f and on α.

Thus $\mathrm{Gal}_\infty(f)$ is a *profinite* group acting on the Cantor set of paths in the tree T_d.

Example [Odoni, 1985]. If $K = \mathbb{Q}$, $\alpha = 2$, $f(x) = x^2 - x + 1$, then

$$Gal_\infty(f) \cong Aut(T_2) \cong [S_2]^\infty \ .$$

Theorem [Lukina, 2018]. Let $f(x)$ be a polynomial of degree $d \geq 2$ over a field K, suppose all roots of $f^i(x)$ are distinct and $f^i(x) - \alpha$ is irreducible for all $i \geq 0$.

Let $\mathbf{v}$ be a path in the space of paths $\mathcal{P}_d$ of the tree T_d.

Then there exists a countably generated group G_0, a homomorphism $\Phi : G_0 \to Homeo(\mathcal{P}_d)$ and a chain $\{G_i\}_{i \geq 0}$ of subgroups in G_0 such that

(1) There is an isomorphism $\widetilde{\phi} : \overline{\Phi(G_0)} \to \mathrm{Gal}_\infty(f)$,

(2) There is a homeomorphism $\phi : \varprojlim\{G_0/G_i\} \to \mathcal{P}_d$ with $\phi(eG_i) = \mathbf{v}$,

(3) For all $\mathbf{u} \in \mathcal{P}_d$ and $\mathbf{g} \in \overline{\Phi(G_0)}$ we have

$$\widetilde{\phi}(\mathbf{g}) \cdot \phi(\mathbf{u}) = \phi(\mathbf{g}(\mathbf{u})).$$

Theorem [Lukina, 2018]. Suppose the image of an arboreal representation $\mathrm{Gal}_\infty(f)$ is a subgroup of finite index in $Aut(T_d)$. Then the action of the dense subgroup G_0 on the path space $\mathcal{P}_d$ is not LQA, and in particular is wild.

Remark: The proof of this result is geometric, it uses the fact that the action of $Aut(T_d)$ is not locally quasi-analytic.

Remark: There are many techniques, in the literature and developing, for calculating arboreal representations.

Into the future...

Problem: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous action, where G is a finitely generated, torsion free nilpotent group. Show that the action is tame.

Problem: Characterize the algebraic number fields and polynomials whose arboreal representations are wild.

Problem: Develop a Molino Theory for equicontinuous foliated spaces which are not minimal.

That is, adapt the Topological Molino Theory of Álvarez López and Moreira Galicia, to the case where the actions are wild.

There are hints at what such a theory may be like in the work on iterated monodromy groups associated to the absolute Galois groups of function fields.

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Happy
Birthday