Lipschitz matchbox manifolds

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$\mathcal{F}$ is a C^1-foliation of a compact manifold M.

Problem: Let L be a complete Riemannian smooth manifold without boundary. When is L *quasi-isometric* to a leaf of a C^1-foliation $\mathcal{F}$ of a compact smooth manifold M?

Definition: $\mathcal{Z} \subset M$ is minimal if it is closed, a union of leaves, and contains no proper subset with these two properties.

Theorem: [Cass, 1985] A leaf in a minimal set must be "quasi-homogeneous", and that this property is an invariant of the quasi-isometry class of a Riemannian metric on L.

The point of this talk is to discuss an analog of this result for the *geometry of Cantor minimal systems*.

Problem: Given a minimal pseudogroup $\mathcal{G}_{\mathfrak{X}}$ action on a Cantor set, can it be realized as the holonomy pseudogroup of a minimal set in a C^r-foliation, $r \geq 1$?

A minimal requirement is that:

Definition: A pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$ is *compactly generated*, if there exists two collections of *clopen* subsets $\{U_1, \ldots, U_k\}$ and $\{V_1, \ldots, V_k\}$ of $\mathfrak{X}$ and homeomorphisms $\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ which generate all elements of $\mathcal{G}_{\mathfrak{X}}$.

$\mathcal{G}^*_{\mathfrak{X}}$ is defined to be all compositions of the generators on the maximal domains for which the composition is defined.

Definition: The action of a compactly generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on a Cantor set $\mathfrak{X}$ is *Lipschitz* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if there exists $C \geq 1$ such that for each $1 \leq i \leq k$ then for all $w, w' \in U_i = \mathrm{Dom}(h_i)$ we have

$$C^{-1} \cdot d_{\mathfrak{X}}(w, w') \leq d_{\mathfrak{X}}(h_i(w), h_i(w')) \leq C \cdot d_{\mathfrak{X}}(w, w') \ .$$

We then say that $\mathcal{G}_{\mathfrak{X}}^*$ is C-Lipschitz with respect to $d_{\mathfrak{X}}$.

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly-generated pseudogroup acting on a Cantor set $\mathfrak{X}$. If $\mathcal{G}_{\mathfrak{X}}$ is the defined by the restriction of the holonomy for a minimal set $\mathcal{Z}$ of a C^1-foliation to a transversal $\mathfrak{X} = \mathcal{Z} \cap \mathcal{T}$, then $\mathfrak{X}$ has a metric $d_{\mathfrak{X}}$ with Lipschitz action.

There is a only one "Cantor set", but the Cantor set has many metrics, and need not be "locally homogeneous".

Two metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are *Lipschitz equivalent*, if they satisfy a Lipschitz condition for some $C \geq 1$,

$$C^{-1} \cdot d_{\mathfrak{X}}(x, y) \ \leq \ d'_{\mathfrak{X}}(x, y) \ \leq \ C \cdot d_{\mathfrak{X}}(x, y) \quad \text{for all } x, y \in \mathfrak{X} \quad (1)$$

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The study of the *Lipschitz geometry* of the pair $(\mathfrak{X}, d_{\mathfrak{X}})$ investigates the geometric properties common to all metrics in the Lipschitz class of the given metric $d_{\mathfrak{X}}$.

Hausdorff dimension is an invariant of Lipschitz geometry.

Theorem: There exist compactly-generated pseudogroups $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, such that there is no metric on $\mathfrak{X}$ for which the generators of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipschitz condition.

Associated to such each point of such an action is a Cayley graph of its orbits, and these graphs have properties analogous to non-embeddable manifolds in C^1-foliations. That is, they are highly irregular, and not "quasi-homogeneous" as defined by Cass.

Sketch of the proof, details are in:
Lipschitz matchbox manifolds, arXiv:1309.1512.

We begin by constructing a standard model for a shift space.

First, introduce the Cantor set $\mathfrak{X}$, with metric $d_{\mathfrak{X}}$.

Let $G_\ell = \mathbb{Z}/(2^\ell\,\mathbb{Z})$ be the cyclic group of order 2^ℓ.

Let $p_{\ell+1} \colon G_{\ell+1} \to G_\ell$ be the natural quotient map. Set:

$$\mathfrak{X} \ = \ \varprojlim \ \{p_{\ell+1} \colon G_{\ell+1} \to G_\ell\} \ \subset \prod_{\ell \geq 1} \mathbb{Z}/(2^\ell\,\mathbb{Z}) \ .$$

Metric on $\mathfrak{X}$: $\overline{x} = (x_1, x_2, x_3, \ldots)$ and $\overline{y} = (y_1, y_2, y_3, \ldots)$, then

$$d_{\mathfrak{X}}\,(\overline{x}, \overline{y}) = \sum_{\ell=1}^{\infty} \ 3^{-\ell}\delta(x_\ell, y_\ell),$$

where $\delta(x_\ell, y_\ell) = 0$ if $x_\ell = y_\ell$, and is equal to 1 otherwise.

Define action $A\colon \mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$, where $\mathbb{Z}$ acts on each factor $\mathbb{Z}/(2^{\ell}\,\mathbb{Z})$ by translation.

Action of A on $\mathbb{Z}$ on $\mathfrak{X}$ is minimal.

Let $\sigma\colon \mathfrak{X} \to \mathfrak{X}$ be the shift map, $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.

σ is a $2-1$ map, and so is not invertible.

σ is a 3-times expanding map.

Partition $\mathfrak{X}$ into clopen subsets, for $i = 0, 1$,

$$U_1(i) = \{(i, x_2, x_3, \ldots) \mid 0 \leq x_j < 2^j \ , \ p_{j+1}(x_{j+1}) = x_j \ , \ j > 1\}.$$

$\mathrm{diam}_{\mathfrak{X}}(U_1(0)) = \mathrm{diam}_{\mathfrak{X}}(U_1(1)) = d_{\mathfrak{X}}(U_1(0), U_1(1)) = 1/3.$

Inverse map $\tau_i = \sigma_i^{-1}\colon \mathfrak{X} \to U_1(i)$ given by the usual formula for the section, $\tau_i(x_1, x_2, x_3, \ldots) = (i, x_1, x_2, x_3, \ldots)$.

For $\overline{x} \in \mathfrak{X}$, set $\overline{x}_\ell = (x_1, \ldots, x_\ell)$.

For $\ell \geq 1$, define the clopen neighborhood of $\overline{x}$,

$$\begin{array}{rcl} U_\ell(\overline{x}) & = & \{(x_1, \ldots, x_\ell, \xi_{\ell+1}, \xi_{\ell+2}, \ldots) \\ & & \mid 0 \leq \xi_j < 2^j \;,\; p_{j+1}(\xi_{j+1}) = \xi_j \;,\; j > \ell\}. \end{array}$$

The restriction $\sigma^\ell \colon U_\ell(\overline{x}) \to \mathfrak{X}$ is $1-1$ and onto, 3^ℓ-expansive.

$\mathrm{diam}_{\mathfrak{X}}(U_\ell(\overline{x})) = 3^{-\ell}/2$.

The key point: add a hypercontraction $\varphi\colon \mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$.

Choose two distinct points $\overline{y}, \overline{z} \in \mathfrak{X}$, and choose a sequence $\{\overline{x}_k \mid -\infty < k < \infty\} \subset \mathfrak{X} - \{\overline{y}, \overline{z}\}$ of distinct points with $\lim_{k\to\infty} \overline{x}_k = \overline{y}$ and $\lim_{k\to-\infty} \overline{x}_k = \overline{z}$.

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Choose disjoint clopen neighborhoods $V_k \subset \mathfrak{X}$ of the points $\overline{x}_k$ recursively.

$\mathrm{diam}_{\mathfrak{X}}(V_k) = \mathrm{diam}_{\mathfrak{X}}(V_{-k}) < \rho_k/(3\,\ell_k!)$

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Homeomorphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}$ such that for all $-\infty < k < \infty$,

the restriction $\varphi_k\colon V_k \to V_{k+1}$ is a homeomorphism onto, and

φ is defined to be the identity on the complement of the union $V = \cup\{V_k \mid -\infty < k < \infty\}$.

The map φ is a homeomorphism.

Let $\mathcal{G}_{\mathfrak{X}} = \langle A, \tau_1, \tau_2, \varphi \rangle$ be pseudogroup they generate.

Claim: There does not exists a metric $d'_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators $\{A, \tau_1, \tau_2, \varphi\}$ of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipschitz condition.

Proof: If such a metric $d'_{\mathfrak{X}}$ exists, then some power of the contractions τ_i are contractions for the new metric $d'_{\mathfrak{X}}$.

But then the Lipschitz condition on φ becomes impossible, as the metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are uniformly related on a fixed ϵ-tube around the diagonal.

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What is going on?

The above result is equivalent to constructing a graph which cannot be embedded quasi-isometrically in a leaf of a foliation, because the hyper-contracting map φ corresponds to adding segments to the graph of the orbit of the affine model which violate the "quasi-homogeneous" property.

Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup acting on a Cantor space $\mathfrak{X}$, and let $V \subset \mathfrak{X}$ be a clopen subset. $\mathcal{G}_{\mathfrak{X}}|V$ is defined as the restrictions of all maps in $\mathcal{G}_{\mathfrak{X}}$ with domain and range in V.

Definition: Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{X}$ via Lipschitz homeomorphisms with respect to the metric $d_{\mathfrak{X}}$. Likewise, let $\mathcal{G}_{\mathfrak{Y}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{Y}$ via Lipschitz homeomorphisms with respect to the metric $d_{\mathfrak{Y}}$. Then

- $(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is *Morita equivalent* to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if there exist clopen subsets $V \subset \mathfrak{X}$ and $W \subset \mathfrak{Y}$, and a homeomorphism $h\colon V \to W$ which conjugates $\mathcal{G}_{\mathfrak{X}}|V$ to $\mathcal{G}_{\mathfrak{Y}}|W$.
- $(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is *Lipschitz equivalent* to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if the conjugation h is Lipschitz.

Problem: Given a compactly-generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, and suppose there exists a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators are Lipschitz, when is there a Lipschitz equivalence to the pseudogroup defined by an exceptional minimal set of a C^1 foliation?

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Problem: Classify the compactly-generated pseudogroups acting minimally on a Cantor set $\mathfrak{X}$, up to Lipschitz equivalence.

Definition: A *matchbox manifold* is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that the transverse model space $\mathfrak{X}$ is totally disconnected, and for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{X}_x \subset \mathfrak{X}$ is a clopen subset, hence is homeomorphic to a Cantor set.

Figure: Blue tips are points in Cantor set $\mathfrak{X}_x$

Matchbox dynamics converts the geometry of minimal Cantor pseudogroups into foliation geometry and dynamics.

Weak solenoids

Presentation is a collection $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$,

• each M_ℓ is a connected compact simplicial complex, dimension n,

• each "bonding map" $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers.

The *generalized solenoid*

$$\mathcal{S}_{\mathcal{P}} \equiv \lim_{\longleftarrow} \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \, \subset \prod_{\ell \geq 0} M_\ell$$

$\mathcal{S}_{\mathcal{P}}$ is given the product topology.

Presentation is *stationary* if $M_\ell = M_0$ for all $\ell \geq 0$, and the bonding maps $p_\ell = p_1$ for all $\ell \geq 1$.

Definition: $\mathcal{S}_{\mathcal{P}}$ is a *weak solenoid* if for each $\ell \geq 0$, M_ℓ is a compact manifold without boundary, and the map $p_{\ell+1}$ is a proper covering map of degree $m_{\ell+1} > 1$.

Classic example: Vietoris solenoid, defined by tower of coverings:

$$\longrightarrow \mathbb{S}^1 \xrightarrow{n_{\ell+1}} \mathbb{S}^1 \xrightarrow{n_\ell} \cdots \xrightarrow{n_2} \mathbb{S}^1 \xrightarrow{n_1} \mathbb{S}^1$$

where all covering degrees $n_\ell > 1$.

Weak solenoids are the most general form of this construction.

Proposition: A weak solenoid is a *matchbox manifold*.

Remark: A generalized solenoid may be a matchbox manifold, such as for Williams solenoids, and Anderson-Putnam construction of finite approximations to tiling spaces.

Associated to a presentation: sequence of proper surjective maps

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0.$$

and a fibration map $\Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$ obtained by projection onto the ℓ-th factor. $\Pi_0 = \Pi_\ell \circ q_\ell \colon \mathcal{S}_\mathcal{P} \to M_0$ for all $\ell \geq 1$.

Choice of a basepoint $x \in \mathcal{S}_\mathcal{P}$ gives basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$.

$$\mathcal{H}_\ell = image\{q_\ell \colon \pi_1(M_\ell, x_\ell) \to \pi_1(M_0, x_0)\} \subset \mathcal{H}_0$$

Definition: $\mathcal{S}_\mathcal{P}$ is a *McCord (or normal) solenoid* if for each $\ell \geq 1$, $\mathcal{H}_\ell$ is a normal subgroup of $\mathcal{H}_0$.

$\mathcal{P}$ normal presentation $\Longrightarrow$ fiber $\mathfrak{X}_x = (\Pi_0)^{-1}(x)$ of $\Pi_0 \colon \mathcal{S}_\mathcal{P} \to M_0$ is a Cantor group, and monodromy action of $\mathcal{H}_0$ on $\mathfrak{X}_x$ is minimal.

A continuum Ω is *homogeneous* if its group of homeomorphisms is point-transitive. Alex Clark and I proved the following in 2010.

Theorem: Let $\mathfrak{M}$ be a matchbox manifold.

• If $\mathfrak{M}$ has equicontinuous pseudogroup, then $\mathfrak{M}$ is homeomorphic to a weak solenoid as foliated spaces.

• If $\mathfrak{M}$ is homogeneous, then $\mathfrak{M}$ is homeomorphic to a McCord solenoid as foliated spaces.

This last result is a higher-dimensional version of the *Bing Conjecture* for 1-dimensional matchbox manifolds.

Solenoids have many possible metrics. For a weak solenoid:

Choose a metric d_ℓ on each X_ℓ.

Choose a series $\{a_\ell \mid a_\ell > 0\}$ with total sum $< \infty$.

Define a metric on $\mathfrak{X}_x$ by setting, for $u, v \in \mathfrak{X}_x$ so

$u = (x_0, u_1, u_2, \ldots)$ and $v = (x_0, v_1, v_2, \ldots)$,

$$d_{\mathfrak{X}}(u, v) = a_1 d_1(u_1, v_1) + a_2 d_1(u_2, v_2) + \cdots$$

Simple example: Vietoris solenoids.

Let m_ℓ be the covering degrees for a presentation $\mathcal{P}$ with base $M_0 = \mathbb{S}^1$, given by $m_\ell = 2$ for ℓ odd, and $m_\ell = 3$ for ℓ even.

Let n_ℓ be the covering degrees for a presentation $\mathcal{Q}$ with base $M_0 = \mathbb{S}^1$, given by $\{n_1, n_2, n_3, \ldots\} = \{2, 3, 2, 2, 3, 2, 2, 2, 2, 3, \ldots\}$. The ℓ-th cover of degree 3 is followed by 2^ℓ covers of degree 2.

Sequences are equivalent for Baer classification of solenoids.

But for the metrics they define, the solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ are not Lipschitz equivalent as matchbox manifolds.

Theorem: [Clark & Hurder, 2009] For all $r \geq 1$, there exists irreducible solenoids over $\mathbb{T}^n$ for all $n \geq 1$ which can be realized as minimal sets of C^r-foliations.

Problem: Classify the McCord solenoids which arise as foliation minimal sets, up to Lipschitz equivalence.

- For subshifts of finite type, it is one of the standard equivalences.

- For more genera invariant sets in dynamical systems, such as solenoids, this appears to be a completely open question.

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Thank you for your attention.