LS category of foliations and Fölner properties[1]

Joint work with Carlos Meniño-Cotón

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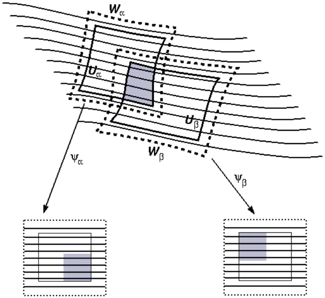
Foliation charts

Let M be a smooth manifold of dimension n.

Definition: M a smooth manifold of dimension n is *foliated* if there is a covering of M by coordinate charts whose change of coordinate functions preserve leaves

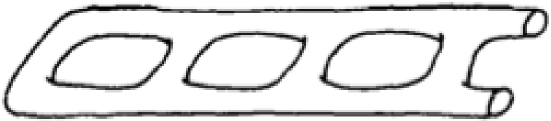


p is leaf dimension, $q = n - p$ is codimension.

Average Euler characteristic

A. Phillips and D. Sullivan
Geometry of leaves, [Topology, 1981]

Suppose L is leaf of dimension $p = 2$ and there is a sequence of connected submanifolds $K_\ell \subset L$ with $length(\partial K_\ell)/area(K_\ell) \longrightarrow 0$.



The sets $\{K_\ell\}$ define an averaging sequence for $\mathcal{F}$.

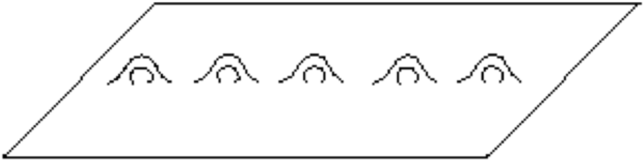
Pass to a subsequence to obtain convergence, then define:

$$\begin{aligned} E_\mu &= \text{Average Euler } (\{K_\ell \mid \ell = 1, 2, \ldots\}) \\ &= \lim_{\ell \to \infty} \frac{\chi(K_\ell)}{Area(K_\ell)} \\ &= \lim_{\ell \to \infty} \frac{1}{Area(K_\ell)} \cdot \int_{K_\ell} e(T\mathcal{F}) \\ &= \langle C_\mu, e(T\mathcal{F}) \rangle \end{aligned}$$

where $e(T\mathcal{F})$ is the closed Euler 2-form for $T\mathcal{F} \to M$, and

- μ is the invariant measure defined by the averaging sequence $\{K_\ell\}$
- C_μ is the Ruelle-Sullivan current on 2-forms associated to it.

It is easy to get zero for an answer:



T. Januszkiewicz
Characteristic invariants of noncompact Riemannian manifolds, [Topology, 1984].

The Phillips-Sullivan ideas extend to *average Pontrjagin invariants* of a foliation.

Average leaf topology

The Phillips-Sullivan and Januszkiewicz results suggest:

Question: How do you measure the "average topology" of the leaves of foliations, not just their average characteristic invariants?

First, you need an invariant measure to average with.

Second, you need a way to "count" the topology of a space, like a leaf.

Covering number

Here is a question that Reeb must have wondered at some point:

Question: Let $\mathcal{F}$ be a foliation of closed manifold M with $p, q > 0$. What is the least number of foliation charts required to cover M?

This can be asked for topological and smooth foliations. Let $Cov(M, \mathcal{F})$ be the minimum number of foliation charts required.

Theorem: [Foulon, 1994] $Cov(M, \mathcal{F}) > 2$.

There does not seem to be much more known about $Cov(\mathcal{F})$, and its relation to the topology of M or dynamics of $\mathcal{F}$. One obvious relation:

- $Cov(M, \mathcal{F}) \geq cat(M) + 1$, the Lusternik-Schnirelmann category of M.

Lusternik-Schnirelmann category

L. Lusternik and L. Schnirelmann,
Méthodes topologiques dans les problèmes variationnels, [Hermann, Paris, 1934].

X a connected topological space, $x_0 \in X$ a basepoint.

$U \subset X$ is *categorical* (in X) if there exists a homotopy $H_t \colon U \to X$ with $H_0 = Id$ and $H_1(U) = x_0$.

Definition: $cat(X) \leq k$ if there is an open covering $\{U_0, U_1, \ldots, U_k\}$ of X where each U_i is an open set which is categorical in X.

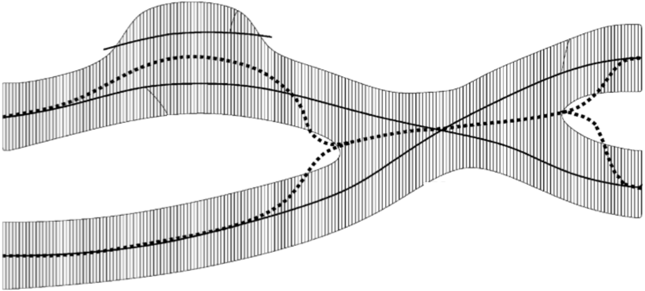
- $cat(\mathbb{S}^n) = 1$.
- $cat(\mathbb{T}^n) = n$.
- $cat(M^n) \leq n$.

Tangential Lusternik-Schnirelmann category

H. Colman and E. Macias-Virgós
Tangential Lusternik-Schnirelmann category of foliations, [Journal L.M.S., 2002]

H_t is *foliated homotopy* map if the curves $t \mapsto H_t(x)$ remains in the leaf L_x containing x, for all $x \in M$. In particular, $H_t(L_x) \subset L_x$ for all $x \in M$, $0 \leq t \leq 1$.

$U \subset M$ is $\mathcal{F}$*-categorical* if there exists a foliated homotopy $H_t \colon U \to M$ with $H_0 = Id$ and $H_1(L'_y|U) = x_y$ where L'_y is the connected leaf of the restricted foliation $\mathcal{F}|U$ which contains $y \in U$.



$cat_{\mathcal{F}}(M)$ calculations

Definition: $cat_{\mathcal{F}}(M) \leq k$ if there is an open covering $\{U_0, U_1, \ldots, U_k\}$ of M where each U_i is an open set which is $\mathcal{F}$-categorical.

Calculating $cat_{\mathcal{F}}(M)$ is surprisingly subtle. A basic tool remains the observation:

Remark: $cat_{\mathcal{F}}(M) \geq cat(L_x)$ for all leaves L_x of $\mathcal{F}$.

- $cat_{\mathcal{F}}(\mathbb{T}^2) = 1$ where $\mathcal{F}$ is the Reeb foliation of $\mathbb{T}^2$.
- $cat_{\mathcal{F}}(\mathbb{S}^3) = 2$ where $\mathcal{F}$ is the Reeb foliation of $\mathbb{S}^3$.
- $cat_{\mathcal{F}}(M) = p$ where $\mathcal{F}$ is a linear foliation of $\mathbb{T}^n$ with dense leaves.

Relation to topology

W. Singhof and E. Vogt
Tangential category of foliations, [Topology, 2003]

Theorem: $\mathcal{F}$ is C^2-foliation implies that $cat_{\mathcal{F}}(M) \leq p$.

Corollary: $\mathcal{F}$ is C^2, $L \subset M$ with $cat(L) = p$, then $cat_{\mathcal{F}}(M) = p$.

Corollary: $\mathcal{F}$ is C^2, $q = 1$, $p \geq 2$, $cat_{\mathcal{F}}(M) = 1 \quad \Rightarrow \quad \mathcal{F}$ is foliation by spheres.

There has been recent work relating it to more standard notions of topology.

J.-P. Doeraene, E. Macias-Virgós, D. Tanré, *Ganea and Whitehead definitions for the tangential Lusternik-Schnirelmann category of foliations*, [Top. Apps., 2010].

Remark: $cat_{\mathcal{F}}(M)$ does not depend (much) on the dynamical properties of $\mathcal{F}$.

Foliated cohomology

H. Colman and S. Hurder, *Tangential LS category and cohomology for foliations*, [Contemp. Math. Vol. 316, 2002].

$\Omega^r(\mathcal{F})$ the space of smooth r-forms along the leaves.

$d_{\mathcal{F}} \colon \Omega^r(\mathcal{F}) \to \Omega^{r+1}(\mathcal{F})$ leafwise differential.

The *foliated cohomology* $H^r_{\mathcal{F}}(M)$ is the cohomology of the complex $(\Omega^r(\mathcal{F}), d_{\mathcal{F}})$.

Product of forms yields product of foliated cohomology

$$\wedge \colon H^r_{\mathcal{F}}(M) \otimes H^s_{\mathcal{F}}(M) \longrightarrow H^{r+s}_{\mathcal{F}}(M)$$

Theorem: $cat_{\mathcal{F}}(M) \geq 1 + \mathrm{nil}\ H^+_{\mathcal{F}}(M)$.

Theorem: $\mathcal{F}$ a C^2-foliation: $0 \neq GV(\mathcal{F}) \in H^{2q+1}(M;\mathbb{R}) \ \Rightarrow \ cat_{\mathcal{F}}(M) \geq q+2$.

Leafwise cohomology yields tangential categorical invariants.

Holonomy pseudogroup of $\mathcal{F}$

Idea: Use the tangential LS category to define an "average topology of leaves". Need some preliminary notions.

Let $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ be a complete transversal for $\mathcal{F}$ associated to an open covering of M by foliation charts $\{\varphi_i \colon U_i \to (-1,1)^p \times (-1,1)^q \mid 1 \leq i \leq k\}$.

Let $\gamma \colon [0,1] \to M$ be a leafwise path with $\gamma(0) = x \in \mathcal{T}$ and $\gamma(1) = y \in \mathcal{T}$.

Let $h_\gamma \colon U_x \to V_y$ denote the holonomy map defined by γ, where $U_x, V_y \subset \mathcal{T}$.

The holonomy maps between points in $\mathcal{T}$ defines pseudogroup $\mathcal{G}_\mathcal{F}$ acting on $\mathcal{T}$.

Transverse invariant measures

A Borel measure μ on $\mathcal{T}$ is *holonomy invariant* if $\mu(E) = \mu(h_\gamma(E))$ for every leafwise path γ and Borel set $E \subset \mathrm{domain}(h_\gamma)$.

Remark: A holonomy invariant Borel measure μ extends to a measure on Borel transversals $f \colon X \to M$. μ is *probability* measure means that $\mu(\mathcal{T}) = 1$.

Theorem: [Plante 1975] An averaging sequence $\{K_\ell\}$ determines a Borel probability measure μ on $\mathcal{T}$ which is holonomy invariant.

Definition: For X compact topological space, a Borel map $f \colon X \to M$ is a transversal for $\mathcal{F}$ if the intersection $f(X) \cap L$ is discrete for all leaves L.

Measured LS category

Carlos Meniño-Cotón:
LS category, foliated spaces and transverse invariant measure, [Thesis, 2012].

Suppose the foliation $\mathcal{F}$ admits a transverse invariant measure μ.

Question: Is there a way to average the leafwise LS category?

Can the idea of LS category be used to define numerical invariants of $(\mathcal{F}, \mu)$? What would they measure?

Let $U \subset M$ be categorical set and $H_t \colon U \to M$ be foliated homotopy.

$\Rightarrow$ $T_U = H_1(U)$ intersects each leaf in a discrete set of points,

$\Rightarrow$ $\mu(T_U) < \infty$.

Let μ be a holonomy invariant, Borel probability measure μ on $\mathcal{T}$.

Definition: (μ-category of $\mathcal{F}$)

$$cat_{\mathcal{F},\mu}(M) = \inf \left\{ \sum_{i=0}^{k} \mu(T_{U_i}) \mid \{U_0, U_1, \ldots, U_k\} \text{ is an } \mathcal{F} \text{ categorical cover} \right\}$$

- $cat_{\mathcal{F},\mu}(M)$ measures how "efficiently" the space M can be decomposed and squeezed into transversals.

- If all leaves of $\mathcal{F}$ are compact and bounded, then $cat_{\mathcal{F},\mu}(M) > 0$ depends on the orbifold quotient $M/\mathcal{F}$ and LS-category of fibers.

- If all leaves of $\mathcal{F}$ are dense, then $cat_{\mathcal{F},\mu}(M) = 0$.

This last result seems surprising, but follows because there is no "price to pay" for moving parts of leaves long distances. The "price" should be a measure of how far a point in U_i has to travel to the basepoint x_0.

- The highest "prices" should be along the boundary ∂U_i of each U_i.

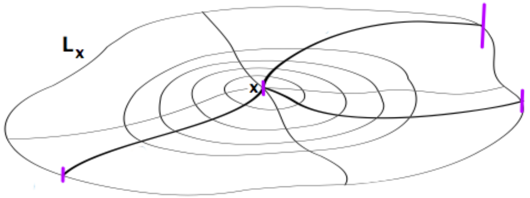
Isoperimetric measured LS category

Definition: (Iso-μ-category of $\mathcal{F}$)

$$cat^{\partial}_{\mathcal{F},\mu}(M) = \inf \left\{ \sum_{i=0}^{k} \mu(\partial U_i) \mid \{U_0, U_1, \ldots, U_k\} \text{ is an } \mathcal{F} \text{ categorical cover} \right\}$$

The notion of $\mu(\partial U_i)$ requires some explanation. Cover U_i by flow charts, then take the measure of the transversals corresponding to the boundary plaques of U_i.

Each U_i defines a Borel subset $E_i \subset \mathcal{T}$ for the plaques in U_i. Let E_i^{∂} denote the transversal points for the boundary plaques. Then $\mu(\partial U_i) = \mu(E_i^{\partial})$.



$cat^{\partial}_{\mathcal{F},\mu}(M)$ is well-defined, for:

- fixed covering of M by foliation charts
- fixed transverse invariant measure μ.

Finally, take the infimum over all open coverings of M by foliation charts.

Remarks:

- $cat^{\partial}_{\mathcal{F},\mu}(M)$ is always finite: take finite cover of M by foliation flow boxes.
- Suppose that $L = \mathbb{S}^p$ is compact leaf supporting μ. Then $cat^{\partial}_{\mathcal{F},\mu}(M) = 0$.
- Suppose that $L = \mathbb{T}^p$ is compact leaf supporting μ. Then $cat^{\partial}_{\mathcal{F},\mu}(M) > 0$.
- $cat^{\partial}_{\mathcal{F},\mu}(\mathfrak{M})$ defined for foliated spaces $\mathfrak{M}$ and *matchbox manifolds* (transversally Cantor foliated spaces).

Suspensions

Let $\Gamma = \pi_1(B, b_0)$ be fundamental group of closed manifold B, and $\phi\colon \Gamma \times X \to X$ an action on compact space X preserving a Borel probability measure μ on X. The *suspension* of ϕ is the foliated space

$$\mathfrak{M}_\phi = \widetilde{B} \times X/(b, x) \sim (b \cdot \gamma^{-1}, \phi(\gamma)(x))$$

Problem: How is $cat^{\partial}_{\mathcal{F},\mu}(\mathfrak{M})$ related to:

- $cat(B)$?
- properties of Γ?
- dynamics of ϕ?

Suppose that $\mathfrak{M}_\phi$ is foliated space defined by suspension of minimal action $\phi\colon \Gamma \times X \to X$ on Cantor set X, preserving measure μ.

For free, minimal Cantor actions by $\Gamma = \mathbb{Z}^p$, all such actions are *topologically hyperfinite*, or *affable*, a topological form of the Fölner condition.

Theorem: [Forrest, 1999] Let $\phi\colon \mathbb{Z}^p \times X \to X$ be a free, minimal action, then ϕ is *topologically Fölner* on a subset $X_0 \subset X$, whose complement $Z = X - X_0$ has measure zero for any transverse invariant measure μ on X.

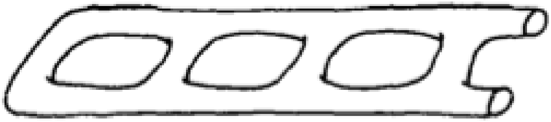
Corollary: Let $\mathfrak{M}_\phi$ be the suspension for $B = \mathbb{T}^p$ of a minimal action of $\mathbb{Z}^p$ on a Cantor set X, preserving a measure μ. Then $cat^{\partial}_{\mathcal{F},\mu}(\mathfrak{M}) = 0$.

In the case where the leaves of $\mathcal{F}$ are contractible, $cat^{\partial}_{\mathcal{F},\mu}(\mathfrak{M})$ is a form of average *Cheeger isoperimetric constant* $\lambda(\Gamma)$ for Γ.

Theorem: Let $\mathfrak{M}_\phi$ be the suspension of a free, minimal action $\phi\colon \Gamma \times X \to X$ which preserves a probability measure μ. Then $cat^{\partial}_{\mathcal{F},\mu}(M) \geq \lambda(\Gamma)$.

Average Euler class, revisited

Consider again the example of “Jacob’s Ladder”



A decomposition of this leaf into categorical sets for a foliation $\mathcal{F}$ must have “very long edges”, which have uniform fraction of total mass.

Let $U \subset M$ be $\mathcal{F}$-categorical set, and $\mathcal{P} \subset U$ a leaf for $\mathcal{F}|U$. Then the Euler form $e(T\mathcal{F})|\mathcal{P} = d_{\mathcal{F}}(T(e|\mathcal{P}))$ for a *uniformly bounded* 1-form on U.

$$\int_{\mathcal{P}} e(T\mathcal{F}) = \int_{\mathcal{P}} d_{\mathcal{F}}(T(e|\mathcal{P})) = \int_{\partial\mathcal{P}} T(e|\mathcal{P}) \leq C_e \cdot length(\partial\mathcal{P})$$

Semi-locality

DeRham cohomology invariants of bundles $\mathcal{E} \to M$ have a semi-local property:

Proposition: [Folklore] Let L have bounded geometry and let $U \subset L$ be a "nice" categorical set. Then for any geometric class $P(\mathcal{E})$ for $\mathcal{E}$ of degree p, formed from products of the Euler, Chern and Pontrjagin classes, there exists a bounded $(p-1)$-form $TP(\mathcal{E})$ on U such that $d_{\mathcal{F}}TP(\mathcal{E}) = P(\mathcal{E})$, where the bound depends on the geometry of M and $\mathcal{E}$ but not on U.

Theorem: Let $(M, \mathcal{F})$ have invariant measure μ, and let $P(T\mathcal{F})$ be a geometric form of degree p. If $\langle C_\mu, P(\mathcal{E})\rangle \neq 0$ then $cat^{\partial}_{\mathcal{F},\mu}(M) > 0$.

Corollary: If $\mathcal{F}$ contains a leaf L which is quasi-isometric to the Jacobs Ladder, and μ is a transverse invariant measured associated to an averaging sequence defined by L, then $cat^{\partial}_{\mathcal{F},\mu}(M) > 0$.

Average topology of leaves

The invariant $cat^{\partial}_{\mathcal{F},\mu}(M) > 0$ depends on:

- The isoperimetric constant of leaves of $\mathcal{F}$.
- The topology of leaves in the support of μ.

Remark: $cat^{\partial}_{\mathcal{F},\mu}(M)$ is analogous to the invariants that Robert Brooks studied:

The spectral geometry of foliations, [Amer. Journal Math., 1984]

Problem: Relate $cat^{\partial}_{\mathcal{F},\mu}(M)$ to the spectrum of leafwise elliptic operators for $\mathcal{F}$.

Thank you for your attention!