Foliation dynamics, shape and classification
I. Introduction

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These lectures are about the study of continua which are locally the product of an open subset of $\mathbb{R}^n$ with a totally disconnected space.

The work was originally motivated by questions from foliation theory, but is just as much about applying ideas and techniques from foliation theory to the study of this special class of continua.

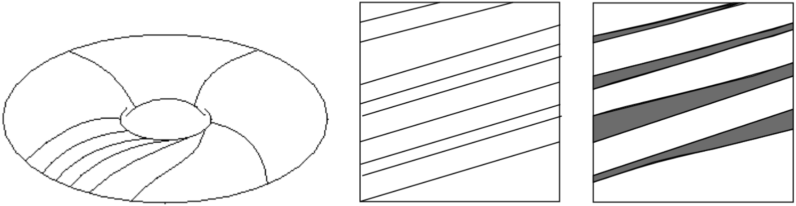
We state some of the problems, and show how to prove very general forms of results from the literature, shown there for flow spaces, here for spaces with $n \geq 1$.

The ideas and results reported on in these lectures are based on various joint works with Alex Clark and Olga Lukina.

Problem 1: How do you classify the (exceptional) minimal sets for C^r-foliations of compact manifolds, $r \geq 1$?

For for codimension-one foliations, this was asked by Gilbert Hector, in his thesis from 1972.

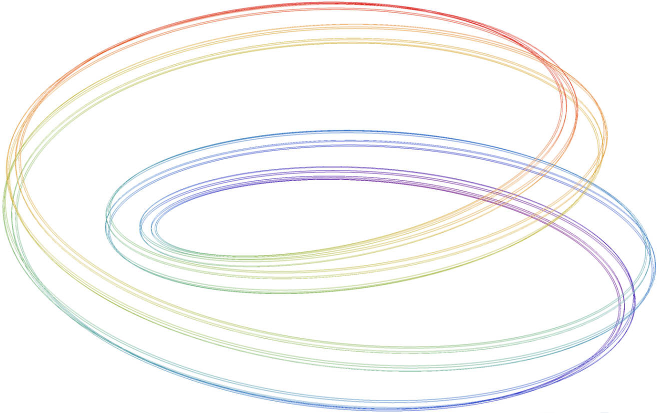
Denjoy Example [1932]



Problem 2: Which solenoids embed as minimal sets of foliations?

This has ben studied extensively for solenoids defined by coverings of the circle. Alex Clark posed the same question for foliations defined by coverings of the torus $\mathbb{T}^2$.

Van Dantzig - Vietoris solenoid [1930]

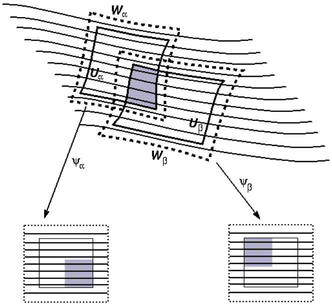


In both examples, the spaces to be understood have a local product structure, of a product of intervals with a Cantor set $\mathfrak{X}$.

In other words, they admit a covering by "matchboxes", terminology introduced by [Aarts & Martens, 1988], [Aarts & Oversteegen, 1995] and [Fokkink, 1991].

Figure: Blue tips are points in Cantor set $\mathfrak{X}$

Definition: A C^r-foliation $\mathcal{F}$ of a manifold M is a "uniform partition" of M into submanifolds of constant dimension p and codimension q, such that there is a covering of M by C^r-coordinate charts whose change of coordinate functions preserve the leaves, for $r \geq 1$.



Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum $\equiv$ a compact, connected metric space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts
  $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$;
- each $\mathfrak{X}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$;
- plaques $\mathcal{P}_i(z) = \varphi_i^{-1}([-1,1]^n \times \{z\})$ are connected, $z \in \mathfrak{X}_i$;
- for $U_i \cap U_j \neq \emptyset$, each plaque $\mathcal{P}_i(z)$ intersects at most one plaque $\mathcal{P}_j(z')$, and change of coordinates along intersection is smooth diffeomorphism;

$+$ some other technicalities.

The path connected components of $\mathfrak{M}$ are the leaves of the foliation $\mathcal{F}$. To the above list, we add the condition:

• there is a leafwise smooth Riemannian metric on the leaves of $\mathcal{F}$, which is continuous in each foliation chart.

Proposition: Each leaf of $\mathcal{F}$ is a complete Riemannian manifold with bounded geometry.

Problem 3: Let L be a complete Riemannian smooth manifold without boundary. When is L *quasi-isometric* to a leaf of a foliation $\mathcal{F}$ of a matchbox manifold $\mathfrak{M}$?

The study of Problem 3 is one of the themes of foliation theory, but it turns out to be even more suggestive question in the topological setting of matchbox manifolds.

Types of matchbox manifolds:

- *Exceptional minimal sets* for C^r-foliations of compact manifolds;
- *Inverse limit spaces* defined by a sequence of proper covers of a compact manifold, or more generally compact branched manifolds;
- *Expanding attractors* for Axiom A dynamical systems;
- *Tiling spaces* associated to aperiodic locally-finite tilings of Euclidean space;
- *Suspensions* of minimal pseudogroup actions on a Cantor set, such as those obtained from the Ghys-Kenyon construction for infinite graphs.

Solenoids: generalized, weak, and normal

Classic example: Vietoris solenoid, defined by tower of coverings:

$$\mathcal{P} \equiv \cdots \longrightarrow \mathbb{S}^1 \xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

where each p_ℓ is a covering map of degree $n_\ell > 1$.

Set $n_{\mathcal{P}} = \{n_1, n_2, n_3, \ldots\}$.

$\mathcal{P}$ is called a presentation.

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$

$\mathcal{S}_{\mathcal{P}}$ is given the (relative) product topology.

Proposition: The space $\mathcal{S}_{\mathcal{P}}$ is a matchbox manifold, for which every leaf is diffeomorphic to $\mathbb{R}$, and dense in $\mathcal{S}_{\mathcal{P}}$.

Some "classical" results on Vietoris solenoids:

Proposition: The homeomorphism type of $\mathcal{S}_{\mathcal{P}}$ depends only on the set of integers $n_{\mathcal{P}}$.

More is true. Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations, and let P be the infinite set of prime factors of the integers in the set $n_{\mathcal{P}}$, included with multiplicity, and Q the same of $n_{\mathcal{Q}}$.

Theorem: [Bing, 1960; McCord, 1965] The solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic, if and only if there is a bijection between a cofinite subset of P with a cofinite subset of Q.

Proposition: The space $\mathcal{S}_{\mathcal{P}}$ is circle-like.

That is, given a metric on $\mathcal{S}_{\mathcal{P}}$, for all $\epsilon > 0$ there exists a continuous map $f_\epsilon \colon \mathcal{S}_{\mathcal{P}} \to \mathbb{S}^1$ such that for each $x \in \mathbb{S}^1$, the inverse image $f_\epsilon^{-1}(x)$ is a subset of with diameter at most ϵ.

Associated to $\mathcal{P}$ is a sequence of proper covering maps

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1 \ .$$

Projection onto the ℓ-th factor in the product $\prod_{\ell \geq 0} \mathbb{S}^1$ yields a map $\Pi_\ell \colon \mathcal{S}_{\mathcal{P}} \to \mathbb{S}^1$, for which $\Pi_0 = \Pi_\ell \circ q_\ell \colon \mathcal{S}_{\mathcal{P}} \to \mathbb{S}^1$.

The diameters of the fibers of Π_ℓ tend to 0 uniformly, as $\ell \to \infty$.

Theorem:[Bing, 1960] If $\mathfrak{M}$ is a 1-dimensional matchbox manifold which is circle-like, then $\mathfrak{M}$ is homeomorphic to a Vietoris solenoid.

The space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h \colon X \to X$ such that $h(x) = y$.

Theorem:[Hagopian, 1977] If $\mathfrak{M}$ is a 1-dimensional homogeneous matchbox manifold, then $\mathfrak{M}$ is circle-like, hence homeomorphic to a Vietoris solenoid.

Theorem:[folklore] Let $\mathcal{S}_{\mathcal{P}}$ be a Vietoris solenoid, then there exists a C^1-flow on $\mathbb{S}^3$ which has a minimal set homeomorphic to $\mathcal{S}_{\mathcal{P}}$.

Generalized solenoids.

Presentation is a collection $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$,

- M_ℓ is a connected compact branched manifold of dimension n;
- each *bonding map* $p_{\ell+1}$ is a proper (smooth) submersion;

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \; \subset \prod_{\ell \geq 0} M_\ell$$

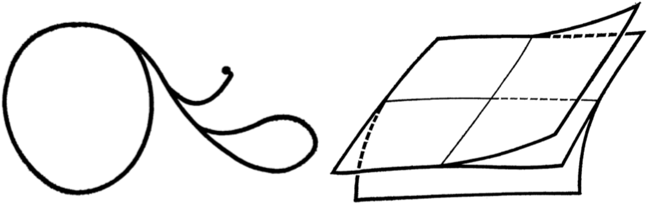
is a *generalized solenoid*, where $\mathcal{S}_{\mathcal{P}}$ is given the product topology.

Presentation is *stationary* if $M_\ell = M_0$ for all $\ell \geq 0$, and the bonding maps $p_\ell = p_1$ for all $\ell \geq 1$.

If each M_ℓ is a compact manifold, and each bonding map p_ℓ is a *proper covering map*, then we call $\mathcal{S}_{\mathcal{P}}$ a *weak solenoid*.

Proposition:[McCord, 1967] A weak solenoid is a matchbox manifold.

The definition of branched manifolds is too technical to properly define here. Here are examples from [Williams, 1974] of how a branched 1-manifold and branched 2-manifold may look:



Examples of 2-dimensional matchbox manifolds.

Suppose that M_0 is the 2-torus, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Suppose that $A \in GL(2,\mathbb{Z}) \subset GL(2,\mathbb{R})$ is a 2×2 invertible integer matrix, then $\Gamma_A = A \cdot \mathbb{Z}^2$ is a subgroup of finite index in $\Gamma_0 = \mathbb{Z}^2$.

Then there is an induced proper covering map $\phi_A \colon \mathbb{T}^2 \to \mathbb{T}^2$, where the degree of the covering is the index of Γ_A in Γ_0, which equals the determinant $\det(A) \in \mathbb{Z}$.

Given an infinite collection $\mathcal{A} \equiv \{A_\ell \in GL(2,\mathbb{Z}) \mid \ell = 1, 2, \ldots\}$ set $p_\ell = \phi_{A_\ell}$ and we obtain a presentation

$$\mathcal{P}_{\mathcal{A}} = \{p_\ell \colon \mathbb{T}^2 \to \mathbb{T}^2 \mid \ell = 1, 2, \ldots\}$$

which defines the solenoid $\mathcal{S}_{\mathcal{A}}$.

Examples of 2-dimensional matchbox manifolds.

Let $M_0 = \Sigma_g$ be a Riemann surface of genus $g \geq 2$, pick a basepoint $x_0 \in M_0$ and let $\Gamma_0 = \pi_1(M_0, x_0)$.

Γ_0 contains a free subgroup of rank g, and is residually finite, and each subgroup $\Gamma \subset \Gamma_0$ of finite index defines a proper covering $\pi \colon \Sigma_\Gamma \to \Sigma_g$. Given an infinite descending chain

$$\mathcal{G} \equiv \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

let $M_\ell = \Sigma_{\Gamma_\ell}$ and $p_{\ell+1} \colon M_{\ell+1} \to M_\ell$ be the induced proper covering map. Then this defines a presentation $\mathcal{P}_\mathcal{G}$ and a corresponding solenoid $\mathcal{S}_\mathcal{G}$.

The universal Riemann surface introduced in [Sullivan, 1988] is a matchbox manifold.

The above constructions of 2-dimensional weak solenoids have natural extensions to solenoids of dimension $n > 2$.

Not necessary to increase the dimension before getting into trouble, and considering generalized solenoids is even more probematic.

For each of the results stated above about 1-dimensional solenoids, the corresponding result for 2-dimensional weak solenoids requires substantial effort and new techniques to prove, if possible.

Or possibly, the generalized form of the result is not known yet.

Problem: What are the homogeneous matchbox manifolds?

Problem: What are the invariants of the homeomorphism type of matchbox manifolds?

Problem: What are the invariants of the orbit-equivalence type of matchbox manifolds?

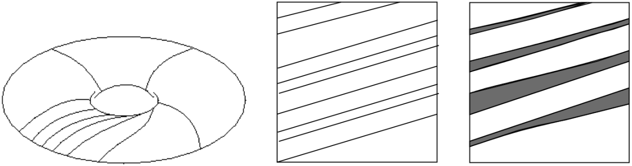
Problem: What does shape classify?

Problem: Which matchbox manifolds embed as minimal sets for C^r-foliations, where $r \geq 1$?

Good questions, all.

The first step in almost all studies of foliations, is to create a discrete model for the geometry and dynamics - the *pseudogroup* associated to the foliation.

Consider the Denjoy example again, for foliation $\mathcal{F}$ of $\mathbb{T}^2$:

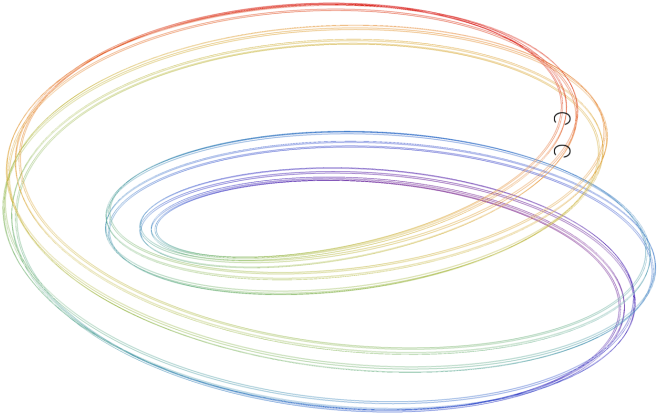


Choose an embedded circle $\eta\colon \mathbb{S}^1 \to \mathbb{T}^2$ which is transverse to the leaves of the minimal set $\mathfrak{M} \subset \mathbb{T}^2$.

We obtain a transversal $\mathfrak{T} = \eta(\mathbb{S}^1) \cap \mathfrak{M}$ which is a Cantor set.

The flow on the leaves of $\mathcal{F}$ defines a homeomorphism $h_{\mathcal{F}}\colon \mathfrak{T} \to \mathfrak{T}$.

The Vietoris solenoid example again



Choose local sections $\eta_i \colon \mathbb{D}^2 \to \mathbb{R}^3$ for $1 \leq i \leq k$, which are transverse to the flow with $\mathcal{S}_{\mathcal{P}}$ as minimal set. We obtain a transversal

$$\mathfrak{T} = \bigcup_{1 \leq i \leq k} \eta_i(\mathbb{D}^2) \cap \mathcal{S}_{\mathcal{P}}$$

which is a Cantor set. The return map of the flow defines a homeomorphism $h_{\mathcal{F}} \colon \mathfrak{T} \to \mathfrak{T}$.

These are simple examples of general construction of the *holonomy pseudogroup* for a matchbox manifold $\mathfrak{M}$.

Definition: A pseudogroup $\mathcal{G}$ modeled on a topological space X is a collection of homeomorphisms between open subsets of X satisfying the following properties:

- For every open set $U \subset X$, the identity $Id_U \colon U \to U$ is in $\mathcal{G}$.

- For every $\varphi \in \mathcal{G}$ with $\varphi \colon U_\varphi \to V_\varphi$ where $U_\varphi, V_\varphi \subset X$ are open subsets of X, then also $\varphi^{-1} \colon V_\varphi \to U_\varphi$ is in $\mathcal{G}$.

- For every $\varphi \in \mathcal{G}$ with $\varphi \colon U_\varphi \to V_\varphi$, and each open subset $U' \subset U_\varphi$, then the restriction $\varphi \mid U'$ is in $\mathcal{G}$.

- For every $\varphi \in \mathcal{G}$ with $\varphi \colon U_\varphi \to V_\varphi$, and every $\varphi' \in \mathcal{G}$ with $\varphi' \colon U_{\varphi'} \to V_{\varphi'}$, if $V_\varphi \subset U_{\varphi'}$ then the composition $\varphi' \circ \varphi$ is in $\mathcal{G}$.

- If $U \subset X$ is an open set, $\{U_\alpha \subset X \mid \alpha \in \mathcal{A}\}$ are open sets whose union is U, $\varphi \colon U \to V$ is a homeomorphism to an open set $V \subset X$, and for each $\alpha \in \mathcal{A}$ we have $\varphi_\alpha = \varphi \mid U_\alpha \colon U_\alpha \to V_\alpha$ is in $\mathcal{G}$, then φ is in $\mathcal{G}$.

For the case where $X = \mathfrak{X}$ is totally disconnected, we have

Definition: The pseudogroup $\mathcal{G}_{\mathfrak{X}}$ modeled on $\mathfrak{X}$ is *compactly generated*, if there exists two collections of *clopen* subsets $\{U_1, \ldots, U_k\}$ and $\{V_1, \ldots, V_k\}$ of $\mathfrak{X}$, and homeomorphisms $\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ which generate all elements of $\mathcal{G}_{\mathfrak{X}}$.

The collection $\mathcal{G}_{\mathfrak{X}}^0 \equiv \{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ is called a generating set for $\mathcal{G}_{\mathfrak{X}}$.

$\mathcal{G}_{\mathfrak{X}}^*$ is defined to be all compositions of elements of $\mathcal{G}_{\mathfrak{X}}^0$ on the maximal domains for which the composition is defined.

Definition: For $g \in \mathcal{G}^*$, the *word length* $\|g\| \leq m$ if g can be expressed as the composition of at most m elements of $\mathcal{G}_{\mathfrak{X}}^0$.

That is, $\|g\| \leq m$ implies that $g = h_{i_\ell} \circ \cdots h_{i_1}$ for $\ell \leq m$.

Let $\mathfrak{M}$ be a matchbox manifold, with a regular covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$. Then there is a compactly generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ associated to this covering, where:

- $\mathfrak{X} = \mathfrak{X}_1 \cup \cdots \cup \mathfrak{X}_k$
- For $U_i \cap U_j \neq \emptyset$ we have $h_{i,j}(x) = y$ if $x \in \mathfrak{X}_i$ and $y \in \mathfrak{X}_j$ so that the plaques they define satisfy $\mathcal{P}_i(x) \cap \mathcal{P}_j(y) \neq \emptyset$.

The map $h_{i,j}$ is well-defined by the hypotheses on the covering, and is a homeomorphism with inverse $h_{i,j}^{-1} = h_{j,i}$.

The collection $\mathcal{G}_{\mathfrak{X}}^0 \equiv \{h_{i,j} \mid U_i \cap U_j \neq \emptyset\}$ generates $\mathcal{G}_{\mathfrak{X}}$.

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts such that that action of the pseudogroup $\mathcal{G}_{\mathfrak{X}}$ is equicontinuous. That is, for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h \in \mathcal{G}^*_{\mathcal{F}}$ we have

$$x, y \in D(h) \text{ with } d_{\mathfrak{X}}(x, y) < \delta \quad \Longrightarrow \quad d_{\mathfrak{X}}(h(x), h(y)) < \epsilon$$

$\mathfrak{X}$ is minimal if for each leaf $L \subset \mathfrak{X}$, the closure $\overline{L} = \mathfrak{X}$.

Theorem: $\mathfrak{M}$ equicontinuous implies $\mathfrak{M}$ is minimal.

This is folklore for flows; the case of group actions is in book "Minimal flows and their extensions," by Joe Auslander.

The case of pseudogroups is a technical adaptation.

Theorem: [McCord 1965] A weak solenoid $\mathcal{S}$ is equicontinuous.

Definition: $\mathfrak{M}$ is an *expansive matchbox manifold* if it admits some covering by foliation charts such that action of the pseudogroup $\mathcal{G}_{\mathfrak{X}}$ is expansive. That is, there exists $\epsilon > 0$, so that for all $x \neq y \in \mathfrak{X}$, there exists $h \in \mathcal{G}^*_{\mathcal{F}}$ so that

$$x, y \in D(h) \text{ and } d_{\mathfrak{X}}(h(x), h(y)) \geq \epsilon$$

Examples:

- The pseudogroup for a Denjoy minimal set.
- The pseudogroup for a tiling space associated to an aperiodic tiling of finite local complexity on $\mathbb{R}^n$.

Remarks:

• In smooth dynamics, the study of group or pseudogroup actions which act via isometries is a well-understood subject. In Cantor dynamics, there are many more possibilities.

• In these lectures, we are interested in the interplay between the structure of the pseudogroup actions on Cantor sets, and the topological properties of the matchbox manifolds that induce them.

• A discussion of problems for the dynamics of smooth foliations can be found in the survey *Lectures on Foliation Dynamics: Barcelona 2010*, [Hurder, 2014].

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