Foliation dynamics, shape and classification
II. Shape and dynamics

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Recall from yesterday's lecture:

Theorem:[Bing, 1960] If $\mathfrak{M}$ is a 1-dimensional matchbox manifold which is circle-like, then $\mathfrak{M}$ is homeomorphic to a Vietoris solenoid.

Suppose that $\mathfrak{M}$ is a matchbox manifold with leaf dimension $n \geq 2$. What is the analog of "circle-like"?

There are two extensions of this idea – the obvious extension, and the other based on the observation that $\mathbb{S}^1$ is the unique closed 1-manifold.

Definition: Let Y be a closed connected n-manifold. Then $\mathfrak{M}$ is Y-like, if for all $\epsilon > 0$ there exists a continuous map $f_\epsilon \colon \mathfrak{M} \to Y$ such that for each $x \in Y$, the inverse image $f_\epsilon^{-1}(x) \subset \mathfrak{M}$ has diameter at most ϵ.

Let $\mathcal{Y}$ be a collection of closed connected manifolds.

Definition: $\mathfrak{M}$ is $\mathcal{Y}$-like, if for all $\epsilon > 0$ there exists $Y_\epsilon \in \mathcal{Y}$ and a continuous map $f_\epsilon \colon \mathfrak{M} \to Y_\epsilon$ such that for each $x \in Y_\epsilon$, the inverse image $f_\epsilon^{-1}(x) \subset \mathfrak{M}$ has diameter at most ϵ.

There are then two versions of Bing's Theorem above to consider:

- Classify all of the matchbox manifolds which are Y-like.
- Classify all of the matchbox manifolds which are $\mathcal{Y}$-like.

Here are our two main results, which we discuss today.

Theorem: Let $\mathfrak{M}$ be a $\mathcal{Y}$-like matchbox manifold, where $\mathcal{Y}$ is the collection of all closed manifolds. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid.

Recall that $\mathfrak{M}$ is *homogeneous* if for every $x, y \in \mathfrak{M}$ there exists a homeomorphism $h \colon \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$.

Theorem: Let $\mathfrak{M}$ be a homogeneous matchbox manifold, then $\mathfrak{M}$ is homeomorphic to a normal solenoid.

The 1-dimensional case was shown by [Hagopian,1977], [Mislove & Rogers, 1989], [Aarts, Hagopian & Oversteegen, 1991]

The case for Cantor bundles over $\mathbb{T}^n$ was shown by [Clark, 2002].

The proofs of these results takes us into a deeper analysis of the structure of matchbox manifolds.

There are two main techniques we discuss:

• Coding the orbits of the holonomy pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on the transverse Cantor set $\mathfrak{X}$.

• The extension of the "Long Box Lemma" for flows by Fokkink & Oversteegen, 2002, to the existence of "Reeb slabs" for arbitrary matchbox manifolds.

We first discuss the coding of the orbits of a pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$.

Consider first the special case where $\mathcal{G}_{\mathfrak{X}}$ is generated by an equicontinuous homeomorphism $h\colon \mathfrak{X} \to \mathfrak{X}$ of a Cantor set.

The topology of a Cantor set is generated by the clopen subsets of $\mathfrak{X}$. It follows that for $\epsilon_1' > 0$ there exists a finite partition into clopen subsets

$$\mathfrak{X} = W_1^1 \cup \cdots \cup W_{k_1}^1$$

where $diam(W_i^1) < \epsilon_1'$ for each $1 \leq i \leq k_1$.

The labeling of this partition is thought of as an alphabet,

$$\mathcal{B}_1 = \{1, 2, \ldots, k_1\}$$

The coding function for the action of $h\colon \mathfrak{X} \to \mathfrak{X}$ is defined as, for $x \in \mathfrak{X}$ and $\ell \in \mathbb{Z}$,

$$C_x^1(\ell) = i \quad \Longleftrightarrow \quad h^\ell(x) \in W_i^1$$

Proposition: Let $h\colon \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous action, and $\mathcal{W}_1 = \{W_1^1, \ldots, W_{k_1}^1\}$ a clopen partition of $\mathfrak{X}$.

Then there exists $\delta_1 > 0$ so that for all $x, y \in \mathfrak{X}$ with $d_{\mathfrak{X}}(x, y) < \delta_1$ then $C_x^1(\ell) = C_y^1(\ell)$ for all $\ell \in \mathbb{Z}$.

That is, the coding function $x \mapsto C_x^1$ is locally constant.

This is *local stability* of the orbit.

Proof: Let $\epsilon_1 = \min\{\epsilon_1', \min\{dist(W_i^1, W_j^1) \mid i \neq j\}\}$.

If $x \in W_i^1$ and $d_{\mathfrak{X}}(x, y) < \epsilon_1$ then this implies $y \in W_i^1$ as well.

Let δ_1 be the equicontinuous constant for ϵ_1. So

$$d_{\mathfrak{X}}(x, y) < \delta_1 \Longrightarrow d_{\mathfrak{X}}(h^\ell(x), h^\ell(y)) < \epsilon_1 \quad \text{for all } \ell \in \mathbb{Z}$$

Combine these two facts to obtain that if $d_{\mathfrak{X}}(x, y) < \delta_1$ then $C_x^1(\ell) = C_y^1(\ell)$ for all $\ell \in \mathbb{Z}$.

Define a refinement of the partition $\mathcal{W}_1$ into the *coding partition*

$$\mathcal{V}_1 = \{V_1^1, V_2^1, \ldots, V_{n_1}^1\}$$

where the coding function $x \mapsto C_x^1$ is constant on each $V^1(i)$.
It follows that each $V^1(i)$ is clopen. Also, observe that:

Lemma: The action of $\mathbb{Z}$ on $\mathfrak{X}$ permutes the sets in $\mathcal{V}_1$.

We then repeat this procedure recursively.

Set $\epsilon_2' = \epsilon_1/2$

Choose a clopen partition of $\mathfrak{X}$ which is a refinement of $\mathcal{V}_1$

$\mathcal{W}_2 = \{W_1^2, \ldots, W_{k_2}^2\}$, $diam(W_i^2) < \epsilon_2'$ for each $1 \leq i \leq k_2$.

Define the coding function $x \mapsto C_x^2$ as before, and we obtain the refined coding partition of $\mathcal{V}_1$

$$\mathcal{V}_2 = \{V_1^2, V_2^2, \ldots, V_{n_2}^2\}$$

and an alphabet $\mathcal{B}_2 = \{1, 2, \ldots, n_2\}$.

Repeat to obtain refined coding partitions $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \mathcal{V}_3 \supset \cdots$ where each $V_i^\ell \in \mathcal{V}^\ell$ satisfies $diam(V_i^\ell) < \epsilon_\ell \leq \epsilon_0/2^\ell$.

We have shown:

Proposition: Let $h\colon \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous homeomorphism. Then for every $\epsilon > 0$, there exists a *periodic* homeomorphism $h_\epsilon\colon \mathfrak{X} \to \mathfrak{X}$ so that

$$d_{\mathfrak{X}}(h^\ell(x), h_\epsilon^\ell(x)) < \epsilon \quad \text{for all } x \in \mathfrak{X}, \ell \in \mathbb{Z}$$

For the case where $\mathfrak{X}$ is the fiber of a Vietoris solenoid $\Pi\colon \mathcal{S}_{\mathcal{P}} \to \mathbb{S}^1$, the conclusion of that the holonomy map h is approximated by periodic maps, is equivalent to the conclusion that $\mathcal{S}_{\mathcal{P}}$ is ϵ-approximated by circles.

The above construction easily extends to the case of group actions:

Theorem: Let $\varphi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous action, where Γ is a finitely generated group. Then there exists, for $\ell \geq 1$,

- finite set $\mathcal{B}_\ell$ with cardinality $|\mathcal{B}_\ell| = n_\ell \to \infty$;
- continuous surjection $\psi_\ell \colon \mathfrak{X} \to \mathcal{B}_\ell$, whose fibers $\mathfrak{X}_b^\ell \equiv \psi_\ell^{-1}(b)$ satisfy $diam(\mathfrak{X}_b^\ell) \leq \epsilon_\ell$ where $\epsilon_\ell \to 0$ as $\ell \to \infty$;
- representation $\rho_\ell \colon \Gamma \to \mathrm{Perm}(\mathcal{B}_\ell)$ with image a finite group G_ϵ

so that for all $\gamma \in \Gamma$ and $x \in \mathfrak{X}$,

$$\rho_\ell(\gamma)(\psi_\ell(x)) = \psi_\ell(\varphi(\gamma)(x))$$

But we do not know if there exists a lift of the representations $\rho_\ell \colon \Gamma \to \mathrm{Perm}(\mathcal{B}_\ell)$ to $\varphi_\ell \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ which are intertwined by the map ψ_ℓ.

Now consider the general case of the holonomy pseudogroup of a matchbox manifold. Assume there is a fixed a regular covering by foliated coordinate charts

$$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$$

- Set $\mathfrak{X} = \mathfrak{X}_1 \cup \cdots \cup \mathfrak{X}_k$

The local transition maps define a compactly generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$, where:

- For $U_i \cap U_j \neq \emptyset$ we have $h_{i,j}(x) = y$ if $x \in \mathfrak{X}_i$ and $y \in \mathfrak{X}_j$ so that the plaques they define satisfy $\mathcal{P}_i(x) \cap \mathcal{P}_j(y) \neq \emptyset$.

The collection $\mathcal{G}^0_{\mathfrak{X}} \equiv \{h_{i,j} \mid U_i \cap U_j \neq \emptyset\}$ generates $\mathcal{G}_{\mathfrak{X}}$.

The plan is to extend the orbit coding technique to the action of the pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on the Cantor set $\mathfrak{X}$.

One problem is that for $h \in \mathcal{G}_{\mathfrak{X}}$ the domain $\mathrm{Dom}(h)$ of h is only assumed to be a clopen subset of $\mathfrak{X}$, and need not be $\mathfrak{X}$.

The following result follows from the definition of equicontinuity:

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be an equicontinuous finitely generated pseudogroup. Then for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $h \in \mathcal{G}_{\mathfrak{X}}$ and $x \in \mathrm{Dom}(h)$, we have

$$B(x, \delta_\epsilon) \subset \mathrm{Dom}(h) \quad \text{and} \quad diam(h(B(x, \delta_\epsilon)) \leq \epsilon$$

Theorem: Let $\mathcal{G}_{\mathfrak{X}}$ be an equicontinuous finitely generated pseudogroup which acts minimally on $\mathfrak{X}$. Then there exists

- finite set $\mathcal{B}_\ell$ with cardinality $|\mathcal{B}_\ell| = n_\ell \to \infty$;
- continuous surjection $\psi_\ell \colon \mathfrak{X} \to \mathcal{B}_\ell$, whose fibers $\mathfrak{X}_b^\ell \equiv \psi_\ell^{-1}(b)$ satisfy $diam(\mathfrak{X}_b^\ell) \leq \epsilon_\ell$ where $\epsilon_\ell \to 0$ as $\ell \to \infty$;
- a "pseudogroup representation" $\rho_\ell \colon \mathcal{G}_{\mathfrak{X}} \to \mathrm{Perm}'(\mathcal{B}_\ell)$ such that ψ_ℓ intertwines the action of $\mathcal{G}_{\mathfrak{X}}$.

By the notation $\mathrm{Perm}'(\mathcal{B}_\ell)$, we mean that, if $\psi \in \mathrm{Perm}'(\mathcal{B}_\ell)$, then there are subsets $X_\psi, Y_\psi \subset \mathcal{B}_\ell$ for which $\psi \colon X_\psi \to Y_\psi$ is a bijection. That is, ψ is a *partial isometry* for the set $\mathcal{B}_\ell$.

For $h \in \mathcal{G}_{\mathfrak{X}}$ and $x \in \mathrm{Dom}(h)$ with $x \in V_i^\ell$, then claim of the proof is then that $\rho_\ell(h)(i) = j$ is well-defined, where $h(x) \in V_j^\ell$.

Well-defined means, for all $x' \in \mathrm{Dom}(h)$, we have $h(x') \in V_j^\ell$.

We next discuss the analog of the "Long Box Lemma", for the case of n-dimensional matchbox manifolds, where $n \geq 1$.

Identify $\mathfrak{T}_i = \varphi_i^{-1}(0 \times \mathfrak{X}_i) \subset \mathfrak{M}$ with $\mathfrak{X}_i$ and thus $\mathfrak{X}$ with $\mathfrak{T} \subset \mathfrak{M}$.

For $x \in \mathfrak{M}$, let L_x be the leaf of $\mathcal{F}$ containing x.

Proposition: Let $\mathfrak{M}$ be a minimal matchbox manifold, $U \subset \mathfrak{T}$ a clopen subset, and $x \in \mathfrak{M}$. Then $N(L_x, U) = L_x \cap U$ is a net in L_x.

That is, there exists constants $0 < A(x, U) < B(x, U)$ so that

- for every $y \neq z \in N(L_x, U)$ we have $d_{L_x}(y, z) \geq A(x, U)$;
- for every $y \in L_x$ there exists $z \in N(L_x, U)$ so that $d_{L_x}(y, z) \leq B(x, U)$.

Example: Let $\mathfrak{M}$ be a 1-dimensional minimal matchbox manifold defined by a flow, then for an open section $U \subset \mathfrak{T} \subset \mathfrak{M}$, the set $N(L_x, U) \subset \mathbb{R}$ is the set of return times for the orbit starting at x.

Example: Let $\mathfrak{M}$ be a 2-dimensional minimal matchbox manifold defined by an action of $\mathbb{R}^2$, then for an open section $U \subset \mathfrak{T} \subset \mathfrak{M}$, the set $N(L_x, U) \subset \mathbb{R}^2$ is a net in $\mathbb{R}^2$.

Proposition: Let $\mathfrak{M}$ be a minimal matchbox manifold whose leaves are without holonomy, then the minimum spacing constant $A(x, U)$ tends to ∞ as $diam(U) \to 0$.

Remark: If $L_x \subset \mathfrak{M}$ is a leaf *with holonomy*, then $A(x, U)$ is bounded above by half the length of the shortest leafwise path with non-trivial germinal holonomy.

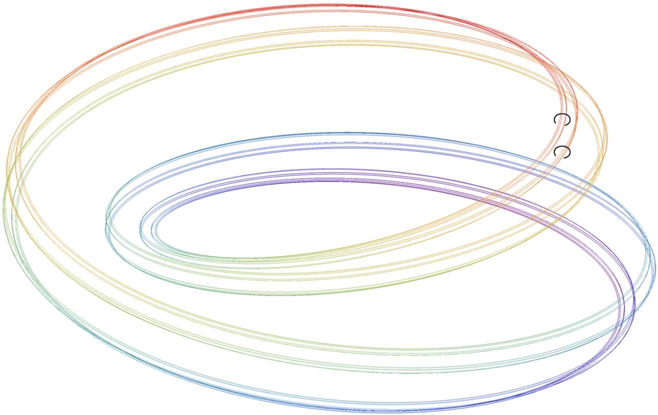
Associated to a net $N \subset L$ in a complete Riemannian manifold with metric d_L there is a *Voronoi decomposition* into cells defined by

$$V(L, N, y) = \{z \in L \mid d_L(y, z) \leq d_L(y, w) \text{ for all } w \in N, w \neq y\}$$

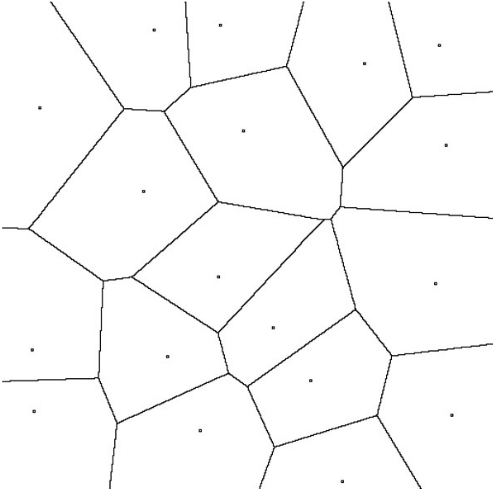
Proposition:

- If $y \neq y' \in N$ then $int(V(L, N, y)) \cap int(V(L, N, y')) = \emptyset$
- $L = \bigcup_{y \in N} V(L, N, y)$

Cantor sections to a solenoid



Voronoi cells in $\mathbb{R}^2$



Definition: A *Reeb slab* for $\mathfrak{M}$ is a foliated inclusion,

$$\phi \colon K \times \mathfrak{C} \to \mathfrak{M}$$

- where $K \subset L$ is a connected subset of a leaf
- $\mathfrak{C}$ is a Cantor set
- $K \times \mathfrak{C}$ is given the product foliation.
- the images $\psi(x \times \mathfrak{C}) \subset \mathfrak{M}$ are *Cantor transversals* to $\mathcal{F}$

For flows, these are sometimes called "long boxes".

The key property is that the long box are bi-foliated, where the transverse foliation to the leaves of $\mathcal{F}$ are Cantor foliations.

We next state the most technical result of this construction, which is a form of "Reeb Stability Theorem" for matchbox manifolds.

Theorem: $\mathfrak{M}$ an equicontinuous matchbox manifold.

Let $\psi_\ell \colon \mathfrak{X} \to \mathcal{B}_\ell$ be the surjection whose fibers $\mathfrak{X}_b^\ell \equiv \psi_\ell^{-1}(b)$ satisfy $diam(\mathfrak{X}_b^\ell) \leq \epsilon_\ell$. We assume that ϵ_ℓ is sufficiently small.

Fix basepoint $x_0 \in V_1^\ell$. For each coding partition $V_i^\ell = \psi_\ell^{-1}(i)$ where $1 \leq i \leq n_i$ choose $x_i \in L_{x_0} \cap V_i^\ell$

Let $N_\ell \subset L_{x_0}$ be the net this defines. For each $y \in N_\ell$ let $V(L_{x_0}, N_\ell, y) \subset L_{x_0}$ be the associated Voronoi cell.

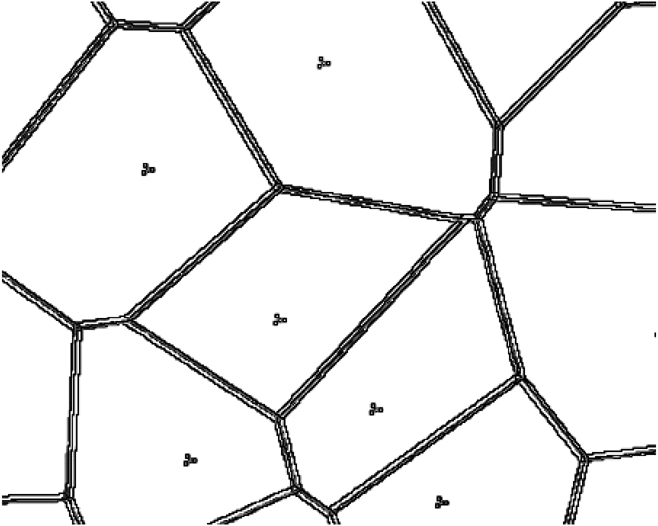
Then there is a collection of Reeb slabs

$$\phi^i \colon V(L_{x_0}, N_\ell, y) \times V_i^\ell \to \mathcal{R}(\ell, i) \subset \mathfrak{M}$$

whose intersections are "neat" along boundaries: boundary of $\mathcal{R}(\ell, i)$ is contained in the union of boundaries of $\mathcal{R}(\ell, j)$, and the transverse Cantor foliations on $\mathcal{R}(\ell, i)$ agree on overlaps.

These conditions are compatible for refinements $\mathcal{V}_{\ell'} \subset \mathcal{V}_\ell$.

Reeb slabs fitting together along boundaries



The details of the above constructions are extremely technical, and appear in the authors' three papers. They yield:

Theorem: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid.

We sketch the idea of the proof.

By the above constructions, for $\epsilon_\ell > 0$ sufficiently small, there is a decomposition of $\mathfrak{M}$ into Reeb slabs, for which the Cantor foliations are compatible on overlaps.

Use the Cantor foliations to project onto the core Voronoi cells, whose union defines a manifold M_ℓ. We obtain a map $\Pi_\ell \colon \mathfrak{M} \to M_\ell$ which restricts to a covering map on leaves of $\mathcal{F}$.

The nesting condition for the clopen sets in the coding partitions V^ℓ implies that these projections are compatible, so they define a weak solenoid structure on $\mathfrak{M}$.

Definition: A metric space X has the *Effros property* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d_X(x, y) < \delta$, there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$, and $d_X(z, h(z)) < \epsilon$ for all $z \in X$.

This is otherwise know as *micro-transitivity*.

Theorem: [Effros,1965] The group $\mathrm{Homeo}(\mathfrak{M})$ is micro-transitive.

This seems to be a very mysterious property - almost magic! - though perhaps the most clear explanation for it is the proof by [Ancel, 1987] which reduces it to a consequence of the *Open Mapping Theorem* in Functional Analysis.

A continuous map $h\colon \mathfrak{M} \to \mathfrak{M}$ maps path components into path components, thus:

$$h\colon \mathfrak{M} \to \mathfrak{M} \text{ homeomorphism } \implies h \text{ is a foliated homeomorphism}$$

Thus, the conclusion of the Effros Theorem is about the stability for foliated homeomorphisms.

The Effros property was used by [Aarts, Hagopian, Oversteegen, 1991] to prove equicontinuity for flows.

It is a technical exercise in foliation holonomy maps to conclude the analogous result for holonomy pseudogroups:

Proposition: $\mathfrak{M}$ homogeneous implies $\mathfrak{M}$ is equicontinuous.

Corollary: If $\mathfrak{M}$ is homogeneous, then it is homeomorphic to a weak solenoid.

How much of the above remains valid when $\mathfrak{M}$ is just a minimal matchbox manifold?

First, there is a bit of omission in the above, as the case where the foliation has a leaf with non-trivial germinal holonomy requires additional treatment. It is necessary to pass to the holonomy cover of the leaf L_{x_0}, then form the Reeb slabs for the covering, and project down again. These projections of Reeb slabs only match up on boundaries, if $\mathcal{G}_{\mathfrak{X}}$ acts equicontinuously.

There is a more fundamental issue though with the construction of the coding functions for expansive actions. The coding partitions need not be clopen sets, even in the case of group actions, and not just for pseudogroups.

The solution is to code only for restricted words in the pseudogroup, where we code for the orbits of a ball of radius 2ℓ

$$\mathcal{G}_{\mathfrak{X}}(2\ell) = \{h \in \mathcal{G}^{*}_{\mathfrak{X}} \mid \|h\| \leq 2\ell\}$$

then use the Reeb slabs for a Voronoi decomposition associated to a finer leafwise net.

The technical details of this occupy our most recently published paper.

The following gives a generalization of the results of [Anderson and Putnam, 1998] and [Sadun, 2003], that the tiling space for an aperiodic, locally finite tiling defined by a substitution is homeomorphic to a generalized solenoid.

Theorem: Let $\mathfrak{M}$ be a minimal matchbox manifold, whose foliation $\mathcal{F}$ is without germinal holonomy. Then $\mathfrak{M}$ is homeomorphic to a generalized solenoid

$$\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}} \equiv \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \, \subset \prod_{\ell \geq 0} M_\ell$$

where the spaces M_ℓ are branched manifolds, obtained from the quotients of the Reeb slabs defined by the coding.

Theorem: Let $\mathfrak{M}$ be a $\mathcal{Y}$-like matchbox manifold. Then $\mathfrak{M}$ is equicontinuous, hence is homeomorphic to a weak solenoid.

By assumption, for any $\epsilon > 0$ there exists a closed manifold M_ϵ and surjection $f_\epsilon \colon \mathfrak{M} \to M_\epsilon$ whose fibers have diameters less than ϵ. It follows from the construction of the Reeb slabs, that the holonomy group cannot expand by more than ϵ.

Two mysteries:

Mystery 1: What can be said for the case of a minimal matchbox manifold $\mathfrak{M}$, where there are leaves of $\mathcal{F}$ which have non-trivial holonomy, possibly of infinite order.

It is a puzzle whether the inverse limit results can be obtained even for the case of an action $\varphi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ where there are points $x \in \mathfrak{X}$ for which the isotropy group Γ_x is infinite.

Mystery 2: Suppose that Y is a branched n-manifold, for $n \geq 1$. Let $\mathfrak{M}$ be a Y-like matchbox manifold. Is $\mathfrak{M}$ homeomorphic to the inverse limit of surjective maps $f\colon Y \to Y$?

This would give a generalization of the result by [Williams, 1974] that an attractor of an Axiom A diffeomorphism has this structure.

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Thank you for your attention.