Foliation dynamics, shape and classification
IV. Embeddings

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Fix some $r > 0$, usually $r = 1$ or $r = 2$.

Problem: Which matchbox manifolds embed in C^r-foliations?

Problem: Given a finitely generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ modeled on a Cantor set $\mathfrak{X}$, with generating set

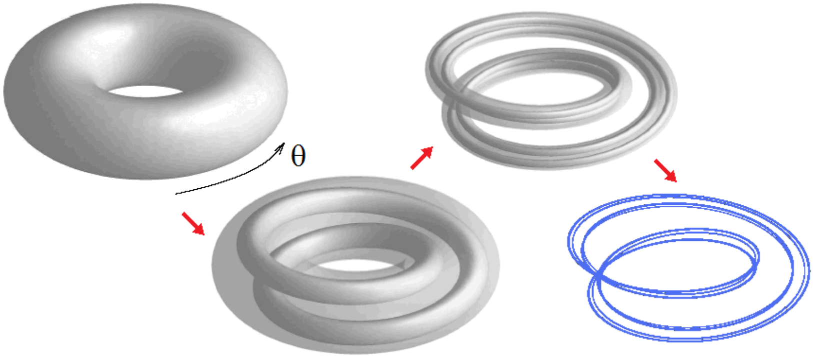
$$\mathcal{G}^0_{\mathfrak{X}} \equiv \{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\},$$

when is there an embedding $\tau \colon \mathfrak{X} \to M$ into a manifold M, and smooth C^r-diffeomorphisms $f_i \colon N_i \to M_i$ defined on open neighborhoods of the images of the domains and ranges of the generators h_i, for $1 \leq i \leq k$, whose restrictions agree with the h_i?

Recall that a Vietoris solenoid $\mathcal{S}_{\mathcal{P}}$ is defined up to homeomorphism by the collection of exponents $P = \{m_1, m_2, \ldots\}$ with $m_\ell \geq 2$, the covering multiplicities.

Theorem:[Markus & Meyer, 1980] Let M be a compact symplectic manifold of dimension $2m \geq 4$, and let an infinite collection P be fixed. Then there exists a generic set of flows in the space of Hamiltonian dynamical systems on M, so that each flow in this set contains a minimal set homeomorphic to $\mathcal{S}_{\mathcal{P}}$.

The proof is based on the Smale embedding trick in [Smale, 1967]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The illustration above is for the case where all covering degrees are the same, $n_\ell = p^\ell$ for integer $p > 1$, and the resulting solenoid is an attractor for an Axiom A diffeomorphism which is expanding on the leaves of $\mathcal{S}_\mathcal{P}$ by construction.

A variation on this idea was used in [Williams, 1967], [Williams, 1974] to show that a generalized solenoid defined by a stable presentation,

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \, \{\sigma \colon M \to M\}$$

where M is a branched n manifold, and the bonding map $\sigma \colon M \to M$ is a smooth uniformly expanding surjection, admits an embedding into a foliation $\mathcal{F}$ of an open set $U \subset \mathbb{R}^m$ for some $m > n$. Moreover, $\mathcal{S}_\mathcal{P}$ is again an attractor for an Axiom A diffeomorphism, which is expanding on the leaves of $\mathcal{F}$.

Question: Is there a similar embedding result for all generalized solenoids?

The works by [Gambaudo, Tresser, 1994], [Gambaudo, Sullivan, Tresser, 1994] in the 1990's studied the dynamical properties of C^r-diffeomorphisms with solenoidal attractors.

The thesis of [Brown, 2010] studied solenoidal attractors for partially hyperbolic diffeomorphisms of compact 3-manifolds.

Extensions of the results above, from minimal sets for flows to minimal sets for foliations, is not so immediate.

Every non-singular 1-dimensional C^1-distribution $F \subset TM$ can be integrated to give a foliation $\mathcal{F}$ of M, so a perturbation of a flow is always integrable. This is very useful!

For $n \geq 2$, a smooth distribution $F \subset TM$ of dimension n can only be integrated to a foliation $\mathcal{F}$ if the *Frobenius integrability conditions* are satisfied. A celebrated result of Bott shows that there are distributions for which no perturbation can be integrated.

In [Clark-Hurder, 2011] we replaced the technique of taking nested perturbations of flows near a periodic orbit as above, with the techniques of taking variations of flat bundles over a manifold. This guarantees that integrability issues do not arise.

It is necessary to show that an infinite sequence of variations of flat bundles yields a C^r-foliation in the limit, for $r > 0$.

For $M_0 = \mathbb{T}^n$, and a presentation $\mathcal{P} = \{p_{\ell+1} \colon \mathbb{T}^n \to \mathbb{T}^n \mid \ell \geq 0\}$, recall that the homeomorphism type of $\mathcal{S}_{\mathcal{P}}$ is determined by a sequence of integer matrices A_ℓ with $\det(A_\ell) > 1$ which induce the proper non-trivial covering maps p_ℓ.

Let $\mathcal{A} = \{A_1, A_2, \ldots\}$ denote such a sequence, and $\mathcal{S}_{\mathcal{A}}$ the corresponding normal solenoid. In [Clark-Hurder, 2011] we showed:

Theorem: For each $r \geq 1$, there are recursive conditions on the sequence $\mathcal{A}$ which guarantee that $\mathcal{S}_{\mathcal{A}}$ embeds as a minimal set for a C^r-foliation of codimension $q \geq 2n$. Moreover, for fixed r there are uncountably many such $\mathcal{A}$ for which the spaces $\mathcal{S}_{\mathcal{A}}$ are pairwise non-homeomorphic.

The recursive conditions are very technical to state, as they are based on giving uniform estimates to show the convergence of the flat bundle perturbations in the C^r-topology on foliations.

An alternate approach to proving that a given matchbox manifold does not embed into any foliation, is to look for criteria showing that it does not embed.

The simple observation is that a foliated embedding $h\colon \mathfrak{M} \to M$ where $\mathcal{F}_M$ is a C^r-foliation on M, induces a type of C^r-geometry on the transversal space $\mathfrak{T}$ for $\mathcal{F}$, and the holonomy of the induced pseudogroup $\mathcal{G}_{\mathfrak{X}}$ must preserve this "geometry". So, we can look for invariants of these geometries which serve as obstructions to the existence of an embedding.

There are two notions in the literature related to this: Lipschitz geometry, and quasi-conformal structures. We consider the Lipschitz geometries.

Definition: Metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ on $\mathfrak{X}$ are *Lipschitz equivalent*, if for some $C \geq 1$, they satisfy the condition:

$$C^{-1} \cdot d_{\mathfrak{X}}(x,y) \;\leq\; d'_{\mathfrak{X}}(x,y) \;\leq\; C \cdot d_{\mathfrak{X}}(x,y) \quad \text{for all } x, y \in \mathfrak{X}$$

Definition: The action of a pseudogroup $\mathcal{G}_{\mathfrak{X}}$, which is compactly generated by the collection

$$\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\} \;,\; U_i, V_i \subset \mathfrak{X}$$

is *C-Lipschitz* with respect to $d_{\mathfrak{X}}$, if there exists $C \geq 1$, such that for each $1 \leq i \leq k$ and for all $w, w' \in U_i$ we have

$$C^{-1} \cdot d_{\mathfrak{X}}(w, w') \leq d_{\mathfrak{X}}(h_i(w), h_i(w')) \leq C \cdot d_{\mathfrak{X}}(w, w')$$

That is, $\mathcal{G}_{\mathfrak{X}}$ is generated by *bi-Lipschitz homeomorphisms*.

Let $\mathfrak{M}$ be a matchbox manifold, with a choice of a regular covering by foliated coordinate charts,

$$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$$

Identify $\mathfrak{T}_i = \varphi_i^{-1}(0 \times \mathfrak{X}_i) \subset \mathfrak{M}$ with $\mathfrak{X}_i$ and thus $\mathfrak{X}$ with $\mathfrak{T} \subset \mathfrak{M}$.

Let $\mathcal{G}_{\mathfrak{X}}$ be the pseudogroup for $\mathfrak{M}$ generated by the collection of transition maps

$$\mathcal{G}_{\mathfrak{X}}^0 \equiv \{h_{i,j} \mid U_i \cap U_j \neq \emptyset\}$$

The inclusion $\mathfrak{T} \subset \mathfrak{M}$ induces a metric $d_{\mathfrak{T}}$ on the transversal, and hence a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$.

Proposition: Let $\mathfrak{M}$ be a minimal matchbox manifold, and M a smooth Riemannian manifold with a C^1-foliation $\mathcal{F}_M$. If there exists a homeomorphism $f\colon \mathfrak{M} \to \mathcal{Z} \subset M$, then there exists a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the action of the holonomy pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on $\mathfrak{X}$ is Lipschitz.

It is immediate that the holonomy maps associated to a C^1-foliation of a compact manifold satisfy a bi-Lipschitz condition.

This implies the induced transition maps $h_{i,j}$ for the clopen subsets of $\mathfrak{X}$ are Lipschitz, though with some technical considerations.

The *geometric entropy* for pseudogroup actions was introduced in [Ghys, Langevin & Walczak, 1988], to give a measure of the "exponential complexity" of the orbits of the holonomy pseudogroup for a C^1-foliation. Conveniently, their work formulates the main ideas and results for pseudogroups of local homeomorphisms of topological spaces. We recall some details.

Let $\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ be a generating set.

$\mathcal{G}^*_{\mathfrak{X}}$ is defined to be all compositions of elements of $\mathcal{G}^0_{\mathfrak{X}}$ on the maximal domains for which the composition is defined.

Recall that the *word length* of $g \in \mathcal{G}^*$ satisfies $\|g\| \leq m$ if g can be expressed as the composition of at most m elements of $\mathcal{G}^0_{\mathfrak{X}}$.

That is, $\|g\| \leq m$ implies that $g = h_{i_\ell} \circ \cdots h_{i_1}$ for $\ell \leq m$.

Let $\epsilon > 0$ and $\ell > 0$.

A subset $\mathcal{E} \subset \mathfrak{X}$ is said to be $(d_{\mathfrak{X}}, \epsilon, \ell)$-separated if for all $w, w' \in \mathcal{E} \cap \mathfrak{X}_i$ there exists $g \in \mathcal{G}^*_{\mathfrak{X}}$ with $w, w' \in \mathrm{Dom}(g) \subset \mathfrak{X}_i$, and $\|g\|_w \leq \ell$ so that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$.

If$w \in \mathfrak{X}_i$ and $w' \in \mathfrak{X}_j$ for $i \neq j$ then they are (ϵ, ℓ)-separated.

The "expansion growth function" is:

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathfrak{X} \text{ is } (d_{\mathfrak{X}}, \epsilon, \ell)\text{-separated}\}$$

The (geometric) entropy is the *asymptotic exponential growth type* of the expansion growth function:

$$\begin{aligned} h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon) &= \limsup_{\ell \to \infty} \ln \{h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell)\} / \ell \\ h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) &= \lim_{\epsilon \to 0} h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon) \end{aligned}$$

Then $0 \leq h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) \leq \infty$.

We recall some properties of the entropy which were shown in [Ghys, Langevin & Walczak, 1988].

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup, acting on the compact space $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$. Then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}})$ is independent of the choice of metric $d_{\mathfrak{X}}$.

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup, defined as the induced pseudogroup for a compact leafwise saturated subset $\mathcal{Z} \subset M$ for a C^1-foliation $\mathcal{F}_M$ on a compact manifold M. Then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$.

Definition: The action of the pseudogroup $\mathcal{G}_{\mathfrak{X}}$ is equicontinuous, if for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, y \in D(h) \text{ with } d_{\mathfrak{X}}(x, y) < \delta \quad \Longrightarrow \quad d_{\mathfrak{X}}(h(x), h(y)) < \epsilon$$

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup, acting on the compact space $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$. If the action of $\mathcal{G}_{\mathfrak{X}}$ is equicontinuous, then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = 0$.

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup associated to a regular covering of $\mathfrak{M}$, acting on the compact space $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$. Then the property that $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = 0$, $0 < h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$, $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = \infty$ is independent of the choice of the regular covering of $\mathfrak{M}$.

We can thus speak of a matchbox manifold $\mathfrak{M}$ as having zero, positive, or infinite entropy.

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly generated pseudogroup associated to a regular covering of $\mathfrak{M}$, acting on the compact space $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$. If $\mathfrak{M}$ admits a foliated embedding into a C^1-foliation of a compact manifold M, then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$.

Question: Which matchbox manifolds have infinite entropy?

Proposition: Let $\mathfrak{M}$ be a matchbox manifold which is homeomorphic to a weak solenoid $\mathcal{S}_{\mathcal{P}}$. Then for any regular covering of $\mathfrak{M}$, and metric $d_{\mathfrak{X}}$ on the transversal space $\mathfrak{X}$, the associated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ satisfies $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = 0$.

Thus, it is reasonable to ask which weak solenoids admit embeddings into C^1-foliations.

Proposition: Let $\mathfrak{M}$ be a matchbox manifold which is homeomorphic to a generalized solenoid $\mathcal{S}_{\mathcal{P}}$, associated to the pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$. If $\mathfrak{X}$ has finite Hausdorff dimension, hence finite box dimension, then $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$.

The proof uses that for any $\epsilon > 0$, there is a finite stage M_ℓ of the presentation for $\mathfrak{M}$ for which the fibers of the projection $\Pi \colon \mathfrak{M} \to M_\ell$ have diameters less than ϵ. Then for any $\ell > 0$ the holonomy of words of length at most ℓ in $\mathcal{G}_{\mathfrak{X}}$ is defined by a leafwise path in $\mathfrak{M}$ whose length is bounded above by $k \cdot \ell$ for some constant k. These paths project to paths in the branched manifold M_ℓ with length at most $k' \cdot \ell$, for some k'. Such a path crosses branch submanifolds in M_ℓ at most an exponential function of ℓ. As $\mathfrak{X}$ has finite box dimension, this implies that $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$.

Recall the space $\mathfrak{M}_2$ obtained from the Ghys-Kenyon construction, using the Cayley tree for the free group $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ on 2-generators.

From the talk yesterday by Lukina, and in [Lukina, 2014], one can show that :

Theorem: The matchbox manifold $\mathfrak{M}_2$ is the suspension of a Lipschitz pseudogroup action on a Cantor set $\mathfrak{X}_2$ with Hausdorff dimension infinity, and with $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = \infty$.

Corollary: $\mathfrak{M}_2$ does not embed into any C^1-foliation of a manifold.

Note: there is no requirement in the corollary, that the embedding is Lipschitz. This big space just doesn't embed!

Recall the definition:

Definition: Matchbox manifolds $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are *return equivalent* if there exists

- regular coverings of $\mathfrak{M}^1$ and $\mathfrak{M}^2$ by foliated coordinate charts, with transversals $\mathfrak{T}^1$ and $\mathfrak{T}^2$,
- pseudogroups $\mathcal{G}_{\mathfrak{X}^1}$ and $\mathcal{G}_{\mathfrak{X}^2}$, respectively,
- sections $C^1 \subset \mathfrak{T}^1$ and $C^2 \subset \mathfrak{T}^2$,
- a homeomorphism $h\colon C^1 \to C^2$ which conjugates the restricted pseudogroups $\mathcal{G}_{C^1}$ to $\mathcal{G}_{C^2}$.

Definition: Lipschitz pseudogroups $(\mathcal{G}_{\mathfrak{X}^1}, d_{\mathfrak{X}^1})$ and $(\mathcal{G}_{\mathfrak{X}^2}, d_{\mathfrak{X}^2})$ are *Lipshitz return equivalent* if there exists

- sections $C^1 \subset \mathfrak{T}^1$ and $C^2 \subset \mathfrak{T}^2$,
- a bi-Lipschitz homeomorphism $h\colon C^1 \to C^2$ which conjugates $\mathcal{G}_{C^1}$ to $\mathcal{G}_{C^2}$.

Proposition: Lipschitz return equivalence is an equivalence relation on minimal Lipschitz pseudogroups.

Meta-problems:

- Find invariants of Lipschitz return equivalence.
- For a group action $\Gamma \times \mathfrak{X} \to \mathfrak{X}$, does Lipschitz return equivalence become tractable for particular classes of groups Γ? Examples to consider include $\Gamma = \mathbb{Z}^n$, and Γ a higher rank lattice for example.

Thank you for your continued attention and interest, throughout these four lectures!

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