Coarse entropy and transverse dimension

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Let (X, d_X) and (Y, d_Y) be metric spaces.

A homeomorphism $f : X \to Y$ is said to be a *quasi-isometry* if there exists a constant $C > 0$ so that for all $x, x' \in X$

$$\lambda^{-1} \cdot d_X(x, x') \leq d_Y(f(y), f(y')) \leq \lambda \cdot d_X(x, x')$$

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In his 1974 ICM address, Dennis Sullivan asked:

Question: Let L be a complete Riemannian smooth manifold without boundary. When is L *quasi-isometric* to a leaf of a C^r-foliation $\mathcal{F}_M$ of a compact smooth manifold M, for $r \geq 1$?

To solve this, you need some property of complete Riemannian manifolds which is an invariant of quasi-isometry, which distinguishes when the manifold is a leaf.

There have been two types of obstructions, to date:

- The *average Euler class* of leaves with subexponential growth, in [Phillips & Sullivan, 1981] and [Cantwell & Conlon, 1982], and the average or Pontrjagin classes, introduced in [Januszkiewicz, 1984]
- The *coarse entropy* of a complete Riemannian manifold, introduced in [Attie & Hurder, 1996]

In this talk, we recall the definition of coarse entropy, and give the relation between this invariant and the work of Lukina in the previous talk.

We begin with a discussion of graph spaces.

Let $\mathcal{G}$ be a metric graph of finite type k. That is, there is a countable set of vertices $V(\mathcal{G})$ and edges $E(\mathcal{G})$ such that:

- each edge $e \in E(\mathcal{G})$ connects to two vertices, $\partial^+ e, \partial^- e \in V(\mathcal{G})$;
- each vertex $v \in V(\mathcal{G})$ is connected to at least one edge;
- each vertex $v \in V(\mathcal{G})$ is connected to no more than k edges;
- each edge has length 1.

The space $\mathcal{G}$ is given the path length metric, denoted $d_{\mathcal{G}}$.

Denote the closed ball by $B_{\mathcal{G}}(v, R) = \{x \in \mathcal{G} \mid d_{\mathcal{G}}(v, x) \leq R\}$.

Fix $R > 0$. An *R-quasi-tiling* of $\mathcal{G}$ is a collection $\{K_1, \ldots, K_\mu\}$ of compact metric spaces with diameters at most $2R$, the *tiles*, and a countable set of homeomorphisms into $\{f_i \colon K_{\alpha_i} \to \mathcal{G} \mid i \in \mathcal{I}\}$ with:

- Each f_i is an isometry onto its image.
- For any $v \in V(\mathcal{G})$, there exist $i \in \mathcal{I}$ so that $B_{\mathcal{G}}(v, R) \subset f_i(K_{\alpha_i})$.

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Let $H(\mathcal{G}, d_{\mathcal{G}}, R)$ denote the least number of sets in an R-quasi-tiling for $\mathcal{G}$. If no R-quasi-tiling exists, then set $H(\mathcal{G}, d_{\mathcal{G}}, R) = \infty$.

Example: Let $\mathcal{G}$ be the Cayley graph of a finitely presented group Γ. Then $H(\mathcal{G}, d_{\mathcal{G}}, R) = 1$ for R sufficiently large.

Fix a base vertex $v_0 \in V(\mathcal{G})$.

Let $\#_e(S)$ be the number of vertices in a subgraph $S \subset \mathcal{G}$.

Example: Let $\mathcal{G}(\mathbb{F}^2)$ be the tree for the free group $\mathbb{F}^2 = \mathbb{Z} * \mathbb{Z}$.
Then $\#_e(B_{\mathcal{G}}(v_0, R)) \sim e^{3R}$ is an exponential function of R.

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Define the *entropy* of the graph $(\mathcal{G}, d_{\mathcal{G}})$

$$h(\mathcal{G}, d_{\mathcal{G}}) = \limsup_{R \to \infty} \frac{\ln\{H(\mathcal{G}, d_{\mathcal{G}}, R)\}}{\#_e(B_{\mathcal{G}}(v_0, R))}$$

Thus if $h(\mathcal{G}, d_{\mathcal{G}}) > 0$, then the number of quasi-tile classes in the collection of balls, $\{B_{\mathcal{G}}(v, 2R) \mid v \in V(\mathcal{G})\}$ is a super-exponential function.

Given $v \in V(\mathcal{G})$ observe that each connected tree in the set with $v \in \mathcal{T} \subset B_{\mathcal{G}}(v, R)$ is a "tile" in the sense above. In the above definition, we identify these tree-tiles if they are isometric, so the number of tiles needed to cover may be less.

In the previous talk, the tree-tiles are identified if they agree up to translation by the (isometric) group action. We call these equivalence classes the "patterns".

Proposition: The number of tree patterns in the collection of balls, $\{B_{\mathcal{G}}(v, R) \mid v \in V(\mathcal{G})\}$, is bounded below by $H(\mathcal{G}, d_{\mathcal{G}}, R)$.

Hence by the calculations in [Lukina, 2014], the Hausdorff dimension of the Ghys-Kenyon graph space for $\mathcal{G}$ is infinite.

Let M be a compact Riemannian manifold, and $\mathcal{F}$ a foliation, with a complete transversal $\mathcal{T} \subset M$. For example, we can take $\mathcal{T}$ to be the union of the transversals in a covering of M by foliation charts.

Give each leaf $L \subset M$ the leafwise Riemannian metric,

Let $\mathcal{N}_L = L \cap \mathcal{T}$ be the net defined by the choice of transversal, and give $\mathcal{N}_L$ the restricted metric from d_L.

Definition: The Cayley graph of a leaf $\mathcal{G}(L)$ has vertices $V(L) = \mathcal{N}_L$, and an edge between two vertices if their corresponding plaques in the coordinate charts overlap. See [Lozano-Rojo, 2006] for example. We declare all edges to have length 1 as before, and give $\mathcal{G}(L)$ the path length metric $d_{\mathcal{G}(L)}$.

The number $h(\mathcal{G}(L), d_{\mathcal{G}(L)})$ may depend upon the choices made to define it, so it is not an invariant of the leaf itself.

If there is a naturally given net $\mathcal{N}_L \subset L$ and metric on L, then $h(\mathcal{G}(L), d_{\mathcal{G}(L)})$ may be an invariant for this data. This is the case for tiling spaces for aperiodic, locally-finite tilings of $\mathbb{R}^n$. Then $h(\mathcal{G}(L), d_{\mathcal{G}(L)})$ is well-defined, and related to the *pattern complexity function* in the work [Julien, 2014], which also has references to other works for complexity of tiling spaces.

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However, for leaves of foliations, there is no reason why this invariant should be independent of the choices of Riemannian metric on M and covering of M by foliation charts. The solution is to "coarsify" the pattern entropy (terminology due to John Roe).

That is, we allow a controlled amount of distortion in our tiling patterns, and then let this coarsening tend to infinity.

Let (X, d_X) and (Y, d_Y) be metric spaces.

For $\epsilon > 0$ we say that a subset $Z \subset Y$ is ϵ-dense, if for every $y \in Y$ there exists $z \in Z$ with $d_Y(y, z) < \epsilon$.

A map $f : X \to Y$ is said to be a λ-*coarse isometry* if, for all $x, x' \in X$,

$$\lambda^{-1} \cdot d_X(x, x') - \lambda \leq d_Y(f(y), f(y')) \leq \lambda \cdot d_X(x, x') + \lambda$$

and the image $f(X) \subset Y$ is $\lambda > 0$ dense.

A map $f : X \to Y$ is said to be a *coarse isometry*, if it is λ-coarse isometry for some $\lambda \geq 1$.

Note that a quasi-isometry is a coarse isometry, and a composition of coarse isometries is again a coarse isometry.

Example: The inclusion $\mathbb{Z}^n \subset \mathbb{R}^n$ is a λ-coarse isometry, for $\lambda \geq 1$.

Example: Let M be a compact Riemannian manifold, and $\mathcal{F}$ a foliation, with a complete transversal $\mathcal{T} \subset M$. Each leaf $L \subset M$ of $\mathcal{F}$ inherits a leafwise Riemannian metric, for which the path length metric d_L is complete. Let $\mathcal{N}_L = L \cap \mathcal{T}$ be the net defined by the coordinates transversal, and give $\mathcal{N}_L$ the restricted metric from d_L.

Then the inclusion $(\mathcal{N}_K, d_L) \subset (L, d_L)$ is a coarse isometry.

Definition: Fix $\lambda, R > 0$. An (λ, R)-quasi-tiling of a complete Riemannian manifold L is a collection $\{K_1, \ldots, K_\mu\}$ of a compact metric spaces with diameters at most $2R$, the coarse quasi-tiles, and a countable set of continuous maps into, $\{f_i \colon K_{\alpha_i} \to L \mid i \in \mathcal{I}\}$, with:

- Each f_i is a λ-coarse isometry onto its image.
- For any $x \in L$, there exist $i \in \mathcal{I}$ so that

$$B_L(x, R) \subset \mathrm{Pen}(f_i(K_{\alpha_i}), \lambda) \equiv \{y \in L \mid D_L(y, z) \text{ for some } z \in f_i(K_{\alpha_i})\}$$

$H_c(L, d_L, \lambda, R)$ denotes the least number of sets in an (λ, R)-quasi-tiling for L.

If no (λ, R)-quasi-tiling exists, then set $H_c(L, d_L, \lambda, R) = \infty$.

Define the λ-*coarse entropy*

$$h_\lambda(L, d_L) = \limsup_{R\to\infty} \frac{\ln\{H_c(L, d_L, \lambda, R)\}}{\mathrm{Vol}(B_L(x_0, R))}$$

and the *coarse entropy*

$$h_c(L, d_L) = \lim_{\lambda\to\infty} h_\lambda(L, d_L)$$

Lemma: $h_c(L, d_L)$ is a coarse invariant of the metic space (L, d_L).

Theorem: If (L, d_L) is quasi-isometric to a leaf of a C^1-foliation of a compact manifold, then $h_c(L, d_L) = 0$.

The inclusion $\mathcal{G}(L) \subset L$ of the leafwise Cayley graph, with geodesic edges, is a coarse isometry, hence

Corollary: $h_c(L, d_L) > 0 \quad \Longrightarrow \quad h(\mathcal{G}(L), d_{\mathcal{G}(L)}) > 0.$

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Combining these various results, we obtain:

Theorem: If (L, d_L) is a complete Riemannian manifold with metric d_L, and $h_c(L, d_L) > 0$, then (L, d_L) is not quasi-isometric to a leaf of a C^1-foliation of a compact manifold.

Moreover, the closure space $\mathfrak{X}(L)$ of the pattern space for the "leafwise Cayley graph" $\mathcal{G}(L)$ has infinite Hausdorff dimension.

Examples of complete Riemannian manifolds with $h(L, d_L) > 0$ can be built by explicit construction, using a geometric form of "fusion" as in the work of [Lukina, 2012].

For example, start with $L = \mathbb{H}^2$ the hyperbolic 2-disk.

Then as you go to infinity, attach lots of "water towers", lots and lots of water towers.

A "water tower" is a large sphere with a cylinder connector, where the diameter of the sphere grows increasingly large.

If we place these attachments systematically randomly, then this makes the coarse geometry "wild" at infinity, while preserving the homeomorphism class of the space.

See [Attie & Hurder, 1996] for the complicated explanation.

See [Zeghib, 1994] for the beautifully simple version.

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Thank you for your attention.