Minimal sets for foliations

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Given *one* derivative,

What can you do with one derivative?

Topological dynamics for flows (and foliations) is a very wild world.

But along with a derivative, sometimes arrives order.

Model example for this: attractors for *Axiom A* diffeomorphisms, and Williams' results on their topological structure.

Goal: Classify (exceptional) minimal sets for C^1-foliations.

Study the "simplest" cases.

Themes in classification of foliations:

$$2 \leftrightarrow 1 \leftrightarrow 0 \leftrightarrow 1 \leftrightarrow 2$$

- “2” — C^2-invariants, such as characteristic classes
- “1” — C^1-invariants, such as entropy & Lyapunov spectrum
- “0” — C^0-invariants, topological dynamics of foliations
- “1” — Lipshitz invariants, tempered cocycles, measure properties
- “2” — Classification of cycles in $B\Gamma_q^2$

Discuss some aspects about transitioning from “0” to “1”.

Classification of foliations in the 1970's and 80's

- Haefliger, Thurston $\Longrightarrow$

  Classifying spaces $B\Gamma_q^r$ *for* $r \geq 1$

- Bott–Haefliger, Gelfand–Fuks, Kamber–Tondeur $\Longrightarrow$

  Secondary classes and how to calculate them

- Mizutani, Morita, Tsuboi, Duminy, Sergiescu $\Longrightarrow$

  Decompose foliations, Godbillon-Vey class of constituents

- Ghys, Hector, Langevin, Moussu, Rosenberg, Roussarie, Cantwell, Conlon, Plante, Sullivan, Walczak, Williams, Inaba, Matsumoto, Mizutani, Nishimori, Tsuchiya $\Longrightarrow$

  Foliations as dynamical systems

A minimal set $\mathcal{Z} \subset M$ is *exceptional* if the intersection $\mathcal{Z} \cap \mathcal{T}$ is totally disconnected for every transversal $\mathcal{T}$ to $\mathcal{F}$.

Leaves of $\mathcal{F}$ are recurrent in a minimal set $\Longrightarrow$ $\mathcal{Z} \cap \mathcal{T}$ is Cantor set.

Holonomy of $\mathcal{F}$ along paths in $\mathcal{Z}$ defines pseudogroup on $\mathcal{Z} \cap \mathcal{T}$.

Problem: Study the properties of *minimal pseudogroup actions* on Cantor sets. Look for properties characterizing their "C^1-ness".

Or, a little more precisely:

Problem: Given a pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$, determine if it came from a minimal set in a C^r-foliation, $r \geq 1$.

The action must be *minimal* & *compactly generated*.

Definition: A pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$ is *compactly generated*, if there exists two collections of *clopen* subsets $\{U_1, \ldots, U_k\}$ and $\{V_1, \ldots, V_k\}$ of $\mathfrak{X}$ and homeomorphisms $\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ which generate all elements of $\mathcal{G}_{\mathfrak{X}}$.

$\mathcal{G}_{\mathfrak{X}}^*$ is defined to be the compositions of the generators $\{h_i \colon U_i \to V_i \mid 1 \leq i \leq k\}$ and their inverses, on the maximal domains for which the composition is defined.

There must be given a well-defined Lipshitz class of metrics on $\mathfrak{X}$.

$d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are *bi-Lipshitz equivalent*, if they satisfy a Lipshitz condition for some $C \geq 1$,

$$C^{-1} \cdot d_{\mathfrak{X}}(x,y) \ \leq \ d'_{\mathfrak{X}}(x,y) \ \leq \ C \cdot d_{\mathfrak{X}}(x,y) \quad \text{for all } x,y \in \mathfrak{X}$$

Lipshitz geometry of the pair $(\mathfrak{X}, d_{\mathfrak{X}})$ investigates the geometric properties common to all metrics in the Lipshitz class of the given metric $d_{\mathfrak{X}}$. Example: Hausdorff dimension.

Cantor set has many metrics, need not be "locally homogeneous," or satisfy a "doubling property" of Assouad, which is necessary if there is an embedding in some Euclidean space.

Definition: The action of a compactly-supported pseudogroup $\mathcal{G}_{\mathfrak{X}}$ is *Lipshitz* with respect to $d_{\mathfrak{X}}$ if there exists $C \geq 1$ such that for each $1 \leq i \leq k$ then for all $w, w' \in U_i = \mathrm{Dom}(h_i)$ we have

$$C^{-1} \cdot d_{\mathfrak{X}}(w, w') \leq d_{\mathfrak{X}}(h_i(w), h_i(w')) \leq C \cdot d_{\mathfrak{X}}(w, w') \ .$$

We then say that $\mathcal{G}^*_{\mathfrak{X}}$ is C-Lipshitz with respect to $d_{\mathfrak{X}}$.

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a compactly-generated pseudogroup acting on a Cantor set $\mathfrak{X}$. If $\mathcal{G}_{\mathfrak{X}}$ is the defined by the restriction of the holonomy of a C^1-foliation to a transversal $\mathcal{Z} \cap \mathcal{T}$, then $\mathfrak{X}$ admits a metric $d_{\mathfrak{X}}$ such that the action is Lipshitz.

Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup acting on a Cantor space $\mathfrak{X}$, and let $V \subset \mathfrak{X}$ be a clopen subset. $\mathcal{G}_{\mathfrak{X}}|V$ is defined as the restrictions of all maps in $\mathcal{G}_{\mathfrak{X}}$ with domain and range in V.

Definition: Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{X}$ via Lipshitz homeomorphisms with respect to the metric $d_{\mathfrak{X}}$. Likewise, let $\mathcal{G}_{\mathfrak{Y}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{Y}$ via Lipshitz homeomorphisms with respect to the metric $d_{\mathfrak{Y}}$. Then

1. $(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is *Morita equivalent* to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if there exist clopen subsets $V \subset \mathfrak{X}$ and $W \subset \mathfrak{Y}$, and a homeomorphism $h \colon V \to W$ which conjugates $\mathcal{G}_{\mathfrak{X}}|V$ to $\mathcal{G}_{\mathfrak{Y}}|W$.
2. $(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is *Lipshitz equivalent* to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if the conjugation h is Lipshitz.

Two very interesting problems:

Problem: Given a compactly-generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, and suppose there exists a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators are Lipshitz, when is there a Lipshitz equivalence to the pseudogroup defined by an exceptional minimal set of a C^1 foliation?

Problem: Classify the compactly-generated Lipshitz pseudogroups acting minimally on a Cantor set $\mathfrak{X}$, up to Lipshitz equivalence.

Classify means, for example, that we are looking for *effective invariants* that distinguish the actions.

There are several known "types" of standard examples, of compactly-generated pseudogroups $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, which we recall. But first, there are some "bad characters" that the Lipshitz condition rules out.

Theorem: There exist compactly-generated pseudogroups $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, such that there is no metric on $\mathfrak{X}$ for which the generators of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipshitz condition.

This is related to the construction of complete Riemannian manifolds which cannot be realized as the leaf of a C^1-foliation.

Sketch of proof – for details see *"Lipshitz matchbox manifolds"*, arXiv:1309.1512.

Begin by constructing the model for the Cantor set $\mathfrak{X}$, with $d_{\mathfrak{X}}$.

Let $G_\ell = \mathbb{Z}/(2^\ell\,\mathbb{Z})$ be the cyclic group of order 2^ℓ.

Let $p_{\ell+1} \colon G_{\ell+1} \to G_\ell$ be the natural quotient map. Set:

$$\mathfrak{X} \;=\; \varprojlim \;\{p_{\ell+1} \colon G_{\ell+1} \to G_\ell\} \;\subset \prod_{\ell \geq 1} \mathbb{Z}/(2^\ell\,\mathbb{Z}) \;.$$

Metric on $\mathfrak{X}$: $\overline{x} = (x_1, x_2, x_3, \ldots)$ and $\overline{y} = (y_1, y_2, y_3, \ldots)$, then

$$d_{\mathfrak{X}}\,(\overline{x}, \overline{y}) = \sum_{\ell=1}^{\infty} \; 3^{-\ell} \delta(x_\ell, y_\ell),$$

where $\delta(x_\ell, y_\ell) = 0$ if $x_\ell = y_\ell$, and is equal to 1 otherwise.

Define action $A\colon \mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$, where $\mathbb{Z}$ acts on each factor $\mathbb{Z}/(2^{\ell}\,\mathbb{Z})$ by translation.

Action of A on $\mathbb{Z}$ on $\mathfrak{X}$ is minimal.

Let $\sigma\colon \mathfrak{X} \to \mathfrak{X}$ be the shift map, $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.

σ is a $2 - 1$ map, and so is not invertible.

σ is a 3-times expanding map.

Partition $\mathfrak{X}$ into clopen subsets, for $i = 0, 1$,

$$U_1(i) = \{(i, x_2, x_3, \ldots) \mid 0 \leq x_j < 2^j \ , \ p_{j+1}(x_{j+1}) = x_j \ , \ j > 1\}.$$

$\mathrm{diam}_{\mathfrak{X}}(U_1(0)) = \mathrm{diam}_{\mathfrak{X}}(U_1(1)) = d_{\mathfrak{X}}(U_1(0), U_1(1)) = 1/3.$

Inverse map $\tau_i = \sigma_i^{-1}\colon \mathfrak{X} \to U_1(i)$ given by the usual formula for the section, $\tau_i(x_1, x_2, x_3, \ldots) = (i, x_1, x_2, x_3, \ldots)$.

For $\overline{x} \in \mathfrak{X}$, set $\overline{x}_\ell = (x_1, \ldots, x_\ell)$.

For $\ell \geq 1$, define the clopen neighborhood of $\overline{x}$,

$$\begin{aligned} U_\ell(\overline{x}) &= \{(x_1, \ldots, x_\ell, \xi_{\ell+1}, \xi_{\ell+2}, \ldots) \\ &\quad \mid 0 \leq \xi_j < 2^j \ , \ p_{j+1}(\xi_{j+1}) = \xi_j \ , \ j > \ell\}. \end{aligned}$$

The restriction $\sigma^\ell \colon U_\ell(\overline{x}) \to \mathfrak{X}$ is $1-1$ and onto, 3^ℓ-expansive.

$\mathrm{diam}_{\mathfrak{X}}(U_\ell(\overline{x})) = 3^{-\ell}/2$.

So far, this is just the standard shift model.

A standard example of a Cantor minimal action by affine group.

The key point: construct hypercontraction $\varphi\colon \mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$.

Choose two distinct points $\overline{y}, \overline{z} \in \mathfrak{X}$, and choose a sequence $\{\overline{x}_k \mid -\infty < k < \infty\} \subset \mathfrak{X} - \{\overline{y}, \overline{z}\}$ of distinct points with $\lim_{k\to\infty} \overline{x}_k = \overline{y}$ and $\lim_{k\to-\infty} \overline{x}_k = \overline{z}$.

Choose disjoint clopen neighborhoods $V_k \subset \mathfrak{X}$ of the points $\overline{x}_k$ recursively.

$\mathrm{diam}_{\mathfrak{X}}(V_k) = \mathrm{diam}_{\mathfrak{X}}(V_{-k}) < \rho_k/(3\,\ell_k!)$

ρ_k is distance between all previous choices.

Homeomorphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}$ such that for all $-\infty < k < \infty$,

the restriction $\varphi_k\colon V_k \to V_{k+1}$ is a homeomorphism onto, and

φ is defined to be the identity on the complement of the union $V = \cup\{V_k \mid -\infty < k < \infty\}$.

The map φ is a homeomorphism.

Let $\mathcal{G}_{\mathfrak{X}} = \langle A, \tau_1, \tau_2, \varphi \rangle$ be pseudogroup they generate.

Claim: There does not exists a metric $d'_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators $\{A, \tau_1, \tau_2, \varphi\}$ of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipshitz condition.

Proof: If such a metric $d'_{\mathfrak{X}}$ exists, then some power of the contractions τ_i are contractions for the new metric $d'_{\mathfrak{X}}$.

But then the Lipshitz condition on φ becomes impossible.

What is going on?

This is a "toy problem" for a broader geometric problem.

Return to minimal sets, and construct a graph which cannot be embedded quasi-isometrically in a C^1-foliation.

Definition: A *matchbox manifold* is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that the transverse model space $\mathfrak{X}$ is totally disconnected, and for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{X}_x \subset \mathfrak{X}$ is a clopen subset, hence is homeomorphic to a Cantor set.

All matchbox manifolds are assumed to be smooth with a given leafwise Riemannian metric.

Figure: Blue tips are points in Cantor set $\mathfrak{X}_x$

Examples

• Exceptional minimal set for foliation $\mathcal{F}$ on M, with metric induced from Riemannian metric on M.

• Given a repetitive, aperiodic tiling of the Euclidean space $\mathbb{R}^n$ with finite local complexity, the associated tiling space Ω is the closure of the set of translations by $\mathbb{R}^n$ of the given tiling, in an appropriate Gromov-Hausdorff topology on the space of tilings. Metric is induced from "tiling matching", so a type of ultra-metric.

• Ghys-Kenyon construction of laminations, for Cayley graphs of finitely-generated groups, and for foliated Cayley graphs.

[Ghys, 1999] *Laminations par surfaces de Riemann.*

[Blanc, 2001] *Propriétés génériques des laminations.*

[Lozano-Rojo, 2006] *Cayley foliated space of a graphed pseudogroup.*

[Lukina, 2012] *Hierarchy of graph matchbox manifolds.*

[Lozano-Rojo and Lukina, 2013] *Suspensions of Bernoulli shifts.*

Remark: $L \subset \mathcal{Z}$ a leaf of minimal set for foliation, Γ_L the graph of the pseudogroup restricted to L, then there is an associated graph matchbox manifold $\mathfrak{M}(\Gamma_L)$ which captures the dynamics of L in $\mathcal{Z}$, and is transversally Cantor set. Plus, can keep all leaves as sticks, no need to fatten them up, so really does look like matchboxes.

Matchbox dynamics captures dynamics of foliation minimal sets.

Example above, tilings, trees, etc:

$$\text{expansive} \longleftrightarrow \text{shifts on trees} \longleftrightarrow \text{parabolic groups}$$

Need to also also look for rotational part of geometry, as in the decomposition of a semi-simple Lie group $G = P \times K$:

$$\text{equicontinuous} \longleftrightarrow \text{Cantor rotations} \longleftrightarrow \text{compact groups}$$

The *weak solenoids* correspond to the Cantor rotations, or maximal compact factors in Bruhat decomposition.

Presentation is a collection $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$,

• each M_ℓ is a connected compact simplicial complex, dimension n,

• each "bonding map" $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers.

The *generalized solenoid*

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \; \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \; \subset \prod_{\ell \geq 0} \; M_\ell$$

$\mathcal{S}_\mathcal{P}$ is given the product topology.

Presentation is *stationary* if $M_\ell = M_0$ for all $\ell \geq 0$, and the bonding maps $p_\ell = p_1$ for all $\ell \geq 1$.

Definition: $\mathcal{S}_{\mathcal{P}}$ is a *weak solenoid* if for each $\ell \geq 0$, M_ℓ is a compact manifold without boundary, and the map $p_{\ell+1}$ is a proper covering map of degree $m_{\ell+1} > 1$.

Classic example: Vietoris solenoid, defined by tower of coverings:

$$\longrightarrow \mathbb{S}^1 \xrightarrow{n_{\ell+1}} \mathbb{S}^1 \xrightarrow{n_\ell} \cdots \xrightarrow{n_2} \mathbb{S}^1 \xrightarrow{n_1} \mathbb{S}^1$$

where all covering degrees $n_\ell > 1$.

Weak solenoids are the most general form of this construction.

Proposition: A weak solenoid is a matchbox manifold.

Remark: A generalized solenoid may be a matchbox manifold, such as for Williams solenoids, and Anderson-Putnam construction of finite approximations to tiling spaces. Or, it may not be.

Associated to a presentation: sequence of proper surjective maps

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0.$$

and a fibration map $\Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$ obtained by projection onto the ℓ-th factor. $\Pi_0 = \Pi_\ell \circ q_\ell \colon \mathcal{S}_\mathcal{P} \to M_0$ for all $\ell \geq 1$.

Choice of a basepoint $x \in \mathcal{S}_\mathcal{P}$ gives basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$.

$\mathcal{H}_\ell = image(q_\ell \colon \pi_1(M_\ell, x_\ell)) \subset \mathcal{H}_0$.

Definition: $\mathcal{S}_\mathcal{P}$ is a *McCord (or normal) solenoid* if for each $\ell \geq 1$, $\mathcal{H}_\ell$ is a normal subgroup of $\mathcal{H}_0$.

$\mathcal{P}$ normal presentation $\Longrightarrow$ fiber $\mathfrak{X}_x = (\Pi_0)^{-1}(x)$ of $\Pi_0 \colon \mathcal{S}_\mathcal{P} \to M_0$ is a Cantor group, and monodromy action of $\mathcal{H}_0$ on $\mathfrak{X}_x$ is minimal.

A continuum Ω is *homogeneous* if its group of homeomorphisms is point-transitive. Alex Clark and I proved the following in 2010.

Theorem: Let $\mathfrak{M}$ be a matchbox manifold.

• If $\mathfrak{M}$ has equicontinuous pseudogroup, then $\mathfrak{M}$ is homeomorphic to a weak solenoid as foliated spaces.

• If $\mathfrak{M}$ is homogeneous, then $\mathfrak{M}$ is homeomorphic to a McCord solenoid as foliated spaces.

This looks almost like the Molino Theorem for TP foliations!

Though one point was troubling...

Solenoids have many possible Lipshitz classes of metrics.

For a weak solenoid, choose a metric d_ℓ on each X_ℓ.

Choose a series $\{a_\ell \mid a_\ell > 0\}$ with total sum $< \infty$.

Define a metric on $\mathfrak{X}_x$ by setting, for $u, v \in \mathfrak{X}_x$ so

$u = (x_0, u_1, u_2, \ldots)$ and $v = (x_0, v_1, v_2, \ldots)$,

$$d_{\mathfrak{X}}(u, v) = a_1 d_1(u_1, v_1) + a_2 d_1(u_2, v_2) + \cdots$$

Problem: What is the classification of weak (or McCord) solenoids, up to Lipshitz equivalence?

This problem seems to be wide open.

We give a simple example, in the case of Vietoris solenoids.

Let m_ℓ be the covering degrees for a presentation $\mathcal{P}$ with base $M_0 = \mathbb{S}^1$, given by $m_\ell = 2$ for ℓ odd, and $m_\ell = 3$ for ℓ even.

Let n_ℓ be the covering degrees for a presentation $\mathcal{Q}$ with base $M_0 = \mathbb{S}^1$, given by $\{n_1, n_2, n_3, \ldots\} = \{2, 3, 2, 2, 3, 2, 2, 2, 2, 3, \ldots\}$. The ℓ-th cover of degree 3 is followed by 2^ℓ covers of degree 2.

Sequences are equivalent for *Baer classification* of solenoids,

$\Longrightarrow \mathcal{S}_{\mathcal{P}} \cong \mathcal{S}_{\mathcal{Q}}$.

But for the metrics they define, the solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are not Lipshitz equivalent as matchbox manifolds.

Recent result by Alex Clark, Olga Lukina and myself.

Theorem: Let $\mathfrak{M}$ be a minimal matchbox manifold without holonomy. Then there exists a presentation $\mathcal{P}$ by simplicial maps between compact branched manifolds such that $\mathfrak{M}$ is homeomorphic to $\mathcal{S}_{\mathcal{P}}$ as foliated spaces.

This implies a sort of generalization of the Denjoy/Sacksteder:

Corollary: Let $\mathfrak{M}$ be an exceptional minimal set for a C^1-foliation $\mathcal{F}$ of a compact manifold M. If all leaves of $\mathcal{F}|\mathfrak{M}$ are simply connected, then there is a Lipshitz homeomorphism of $\mathfrak{M}$ with the inverse limit space $\mathcal{S}_{\mathcal{P}}$ defined by a presentation $\mathcal{P}$, given by simplicial maps between compact branched manifolds.

Problem: How to "classify" C^1-minimal sets which are inverse limit spaces of branched manifolds.

Amidst the vast wasteland formed by the class of all minimal Cantor actions by compactly generated-pseudogroups:

• Study the *Lipshitz subclass*, and its classification by invariants such as *Bratteli diagrams* and o*rdered K-Theory*, *dimension properties*, and possibly other invariants such as secondary classes associated to their embeddings into C^1-foliations, which reflect their "inner C^1-ness".

• Study the *Zygmund subclass*, such as C^2-embedded normal solenoids, and their geometric and cohomological invariants, such as secondary classes associated to their embeddings into C^2-foliations, which reflect their "inner C^2-ness".

Thank you for your attention!