Cantor dynamics of renormalizable groups

Steve Hurder, UIC
Joint work with
Olga Lukina, University of Vienna
Wouter Van Limbeek, UIC

.

.

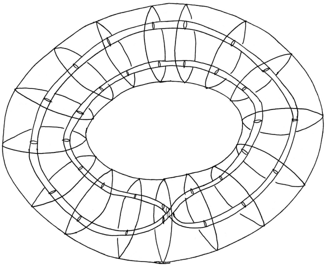
http://www.math.uic.edu/∼hurder/talks/Tokyo20200623.pdf

The Hirsch foliation is a codimension-one foliation $\mathcal{F}$ of a compact 3-manifold N.

With appropriate choices in its construction, the foliation $\mathcal{F}$ is real analytic with an exceptional minimal set.

- Morris Hirsch, *A stable analytic foliation with only exceptional minimal sets*, in **Dynamical Systems, Warwick, 1974**, Lect. Notes in Math. vol. 468, 1975, 9–10.

Construction of the Hirsch foliation:

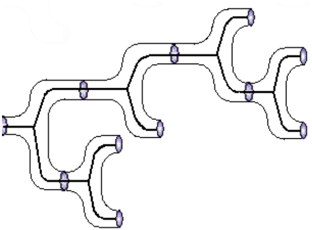


Get a foliated 3-manifold by gluing outer torus $\mathbb{T}^2$ to the inner torus $\mathbb{T}^2$ preserving foliation by circles, so is determined by a covering map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ of degree $d > 1$.

Choose the gluing map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ carefully, to obtain a foliation which is analytic and has an exceptional minimal set, whose pseudogroup dynamics is determined by the map ϕ.

Properties of the Hirsch foliation:

The generic leaves of $\mathcal{F}$ are "tree-like" surfaces.



There are also a finite number of leaves with "loops", corresponding to fixed points for the map $\phi \colon \mathbb{S}^1 \to \mathbb{S}^1$.

The geometry, dynamics and topology of “Hirsch-like” foliations have been well-studied, for example in:

- Alberto Pinto & Dennis Sullivan, *The circle and the solenoid*, **Discrete Contin. Dyn. Syst.**, vol. 16, 2006, 463–504.

- Bin Yu, *Affine Hirsch foliations on 3-manifolds*, **Algebr. Geom. Topol.**, vol. 17, 2017, 1743–1770.

- Sébastien Alvarez & Pablo Lessa, *The Teichmüller space of the Hirsch foliation*, **Ann. Inst. Fourier**, vol. 68, 2018, 1–51.

Definition: A closed connected manifold M is said to be non co-Hopfian if it admits a proper self-covering map $\phi\colon M \to M$.

- The circle $\mathbb{S}^1$ is non co-Hopfian.
- The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is non co-Hopfian.
- N closed connected manifold, then $\mathbb{S}^1 \times N$ is non co-Hopfian.
- The nilmanifold $N = \mathcal{H}/\Gamma$ where $\Gamma \subset \mathcal{H}$ is the integer lattice in the 3-dimensional Heisenberg Lie group $\mathcal{H}$ admits many inequivalent proper self-covering maps.

There are many other constructions of non co-Hopfian manifolds.

Non co-Hopfian manifolds have applications in dynamical systems, foliation theory, and spectral theory.

Constructions of generalized Hirsch foliations were given in

- Bis, Hurder & Shive, *Hirsch foliations in codimension greater than one*. **Foliations 2005**, World Sci. Publ., 2006, 71–108.

Theorem: Associated to a proper self-covering map $\phi\colon M \to M$, there is a generalized Hirsch foliation $\mathcal{F}$ on a closed manifold N, with codimension-q equal to the dimension of M.

Question 1: Which closed manifolds are non co-Hopfian?

Question 2: What are the dynamical properties of Hirsch foliations?

Question 3: What are the properties of minimal sets for Hirsch foliations?

The answer to Questions 2 and 3 depend on the choice of the proper covering map $\phi \colon M \to M$ of course.

We address Question 1, using ideas from algebra and dynamics.

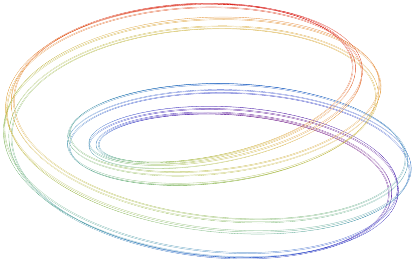
The Smale solenoid

For $m > 1$, let $\phi_m \colon \mathbb{S}^1 \to \mathbb{S}^1$, given by $\phi_m(e^{\iota\theta}) = e^{\iota m\theta}$.

ϕ_m is a proper self-covering map of the circle of degree m.

Iterate the map ϕ_m repeatedly to obtain the Smale solenoid:

$$\mathcal{S}_m \equiv \varprojlim \{\mathbb{S}^1 \xleftarrow{\phi_m} \mathbb{S}^1 \xleftarrow{\phi_m} \mathbb{S}^1 \xleftarrow{\phi_m} \cdots\} \subset \prod_{\ell \geq 0} \mathbb{S}^1 .$$



Associated to a proper self-map $\phi\colon M \to M$ we can form a generalized solenoid

$$\mathcal{S}_\phi \;\equiv\; \varprojlim \, \{M \xleftarrow{\phi} M \xleftarrow{\phi} M \xleftarrow{\phi} \cdots\} \subset \prod_{\ell \geq 0} M \ .$$

These are a special class of the <u>weak solenoids</u> introduced by Chris McCord in 1966.

Problem: Characterize the properties of the weak solenoids $\mathcal{S}_\phi$, and the dynamics of the induced shift map $\sigma_\phi\colon \mathcal{S}_\phi \to \mathcal{S}_\phi$.

Group chains

For the Smale solenoid, given the tower of maps

$$\mathcal{S}_m \equiv \varprojlim \{\mathbb{S}^1 \xleftarrow{\phi_m} \mathbb{S}^1 \xleftarrow{\phi_m} \mathbb{S}^1 \xleftarrow{\phi_m} \cdots\} \subset \prod_{\ell \geq 0} \mathbb{S}^1 ,$$

let $x_0 \in \mathbb{S}^1$ be the identity element, then $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$.

We get a chain of subgroups of finite index

$$\mathcal{G}_m = \{\mathbb{Z} \supset m \cdot \mathbb{Z} \supset m^2 \cdot \mathbb{Z} \supset \cdots\}$$

Next, do this for a non co-Hopfian manifold M of dimension $n > 1$.

Let $\phi\colon M \to M$ be a proper self-covering.

Choose a basepoint $x_1 \in M$ and set $x_0 = \phi(x_1)$. Then we have

$$\phi_*\colon \pi_1(M, x_1) \to \pi_1(M, x_0) \equiv \Gamma_0$$

Choose an isomorphism $\pi_1(M, x_1) \cong \pi_1(M, x_0)$.

- ⋆ ϕ_* induces a self-embedding $\varphi\colon \Gamma_0 \to \Gamma_0$.
- ⋆ Γ_0 is finitely generated.
- ⋆ $\varphi(\Gamma_0) \subset \Gamma_0$ is proper subgroup with finite index.
- ⋆ Group chain $\mathcal{G}_\varphi = \{\Gamma_0 \supset \Gamma_1 = \varphi(\Gamma_0) \supset \Gamma_2 = \varphi(\Gamma_1) \supset \cdots\}$.

A finite index inclusion $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is called a renormalization of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ in the percolation & physics literature.

Definition: Let Γ be a finitely generated group, then an inclusion $\varphi\colon \Gamma \to \Gamma$ with finite index image is called a renormalization of Γ.

Γ is said to be renormalizable if it admits a renormalization.

Γ is also called a finitely non-co-Hopfian group.

Fact: M is non co-Hopfian $\Leftrightarrow \pi_1(M, x)$ is renormalizable.

Questions:

1. What finitely-generated groups are renormalizable?

2. What are the invariants of renormalization maps?

Irreducibility:

Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization. Recursively define a descending chain of subgroups $\Gamma_{\ell+1} = \varphi(\Gamma_\ell)$ for $\ell \geq 0$, so

$$\Gamma \equiv \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

Let $\mathcal{G}_\varphi = \{\Gamma_\ell \equiv \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ be the descending chain of subgroups of finite index associated to φ, then

$$K(\varphi) = \bigcap_{\ell \geq 0} \Gamma_\ell$$

is called the kernel of the chain.

Definition: A renormalization $\varphi\colon \Gamma \to \Gamma$ is said to be irreducible if $K(\varphi)$ is the trivial group, and almost irreducible if $K(\varphi)$ is finite.

Definition: Γ is said to be <u>strongly scale-invariant</u> if there is an almost irreducible renormalization for Γ.

This terminology was introduced in

• *Nekrashevych and Pete*, **Scale-invariant groups**, Groups Geom. Dyn., 2011

Question: Is a strongly scale-invariant group Γ virtually nilpotent?

This question is inspired by a celebrated result of Gromov .

Example: Expanding manifolds

Let M be a closed Riemannian manifold. A smooth map $f: M \to M$ is expanding if there exists some $\lambda > 1$ such that

$$\|df(\vec{v})\| \geq \lambda \|\vec{v}\| \quad \text{for all } x \in M \text{ and } \vec{V} \in T_x M$$

The map f must be a proper covering.

Theorem: [Franks 1968] If M admits an expanding map, then $\Gamma_0 = \pi_1(M, x_0)$ has polynomial growth rate.

Theorem: [Gromov 1979] If Γ is a finitely generated group with polynomial growth rate, then Γ admits a nilpotent subgroup $\Lambda \subset \Gamma$ with finite index. i.e., Γ is virtually nilpotent.

Our work is motivated by a result in

- Van Limbeek, *Towers of regular self-covers and linear endomorphisms of tori*, **Geom. Topol.**, 2018.

Theorem: Let Γ be a strongly scale-invariant group, with a renormalization $\varphi\colon \Gamma \to \Gamma$ such that $\Gamma_\ell = \varphi^\ell(\Gamma)$ is normal in Γ. Then $\Gamma / K(\varphi)$ is abelian.

Question: Is there a weaker assumption than normality for the subgroups Γ_ℓ that yields a solution to the nilpotent question?

We approach this using ideas from Cantor dynamical systems,

- Hurder, Lukina & Van Limbeek, *Cantor dynamics of renormalizable groups*, **arxiv:2002.01565**

Construction of Cantor actions

Consider again the Smale solenoid. Fix the integer $m > 1$, so we have an embedding $\varphi\colon \mathbb{Z} \to \mathbb{Z}$, given by $\varphi(k) = m \cdot k$.

Then $\Gamma_\ell = m^\ell \cdot \mathbb{Z} \subset \mathbb{Z}$.

Pass to quotient groups and form the inverse limit space

$$\mathfrak{X} \equiv \varprojlim \{0 = \mathbb{Z}/\mathbb{Z} \xleftarrow{m_*} \mathbb{Z}/m\mathbb{Z} \xleftarrow{m_*} \mathbb{Z}/m^2\mathbb{Z} \xleftarrow{m_*} \cdots\}$$

The inverse limit $\mathfrak{X}$ is a Cantor group, the m-adic integers $\widehat{\mathbb{Z}}_m$.

The group $\Gamma = \mathbb{Z}$ acts by addition on each quotient group $\mathbb{Z}/m^\ell\mathbb{Z}$.

Get dynamical system $\mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$ which is m-adic odometer.

Let Γ be a finitely generated group.

Let $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$ be a group chain, where $\Gamma_0 = \Gamma$ and $\Gamma_{\ell+1} \subset \Gamma_\ell$ is a proper subgroup of finite index.

$X_\ell = \Gamma / \Gamma_\ell$ is a finite set with transitive left Γ-action.

Inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a surjection $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Define

$$\mathfrak{X} \equiv \varprojlim \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \ .$$

The product of finite sets is given the Tychonoff topology - cylinder sets generate the topology.

Then $\mathfrak{X}$ is a closed subset, so is a Cantor space with left Γ-action.

Obtain minimal Γ-action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$.

Called a <u>subodometer</u> by Cortez and Petite.

A Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \epsilon \quad \text{for all } g \in \Gamma.$$

For the ultrametric metric on $\mathfrak{X}$, the action Φ is isometric:

- $(\mathfrak{X}, \Gamma, \Phi)$ is an equicontinuous Cantor action.

Remark: A smooth equicontinuous action on a manifold is analogous to an isometric action.

Remark: A minimal equicontinuous Cantor action can also be viewed as a group action on a rooted tree.

Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous Cantor action.

This defines a homomorphism $\Phi\colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$

$\widehat{\Gamma} = \overline{\Phi(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ is the closure in uniform topology

Theorem: [Ellis, 1969] Φ equicontinuous implies that $\widehat{\Gamma}$ is a profinite group, compact and totally disconnected.

This result of Ellis is the analog in topological dynamics for the method used to study Riemannian pseudogroups in:

- André Haefliger and Éliane Salem, *Pseudogroupes d'holonomie des feuilletages riemanniens sur des variétés compactes* 1-*connexes*, in **Géométrie différentielle (Paris, 1986)**, 1988.

Lemma: Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization for Γ with associated Cantor action $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$. Then $\ker(\Phi_\varphi) \subset K(\varphi)$, where $\Phi_\varphi\colon \Gamma \to \widehat{\Gamma}_\varphi$ is the map to the completion.

Strategy: For $K(\varphi)$ finite, find conditions on renormalization $\varphi\colon \Gamma \to \Gamma$ which imply that $\widehat{\Gamma}_\varphi$ is a virtually nilpotent group, and hence Γ is virtually nilpotent.

Lemma: Φ_φ induces an equicontinuous action $\widehat{\Phi}_\varphi \colon \widehat{\Gamma} \times \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$.

For a sequence $\widehat{\gamma} = \{\Phi_\varphi(\gamma_i) \in \mathbf{Homeo}(\mathfrak{X}) \mid i > 0\} \in \widehat{\Gamma}$ which converges in the uniform topology of maps, given $x \in \mathfrak{X}_\varphi$ set $\widehat{\gamma} \cdot x = \lim \ \Phi_\varphi(\gamma_i)(x)$.

Lemma: Φ_φ minimal implies that $\widehat{\Gamma}_\varphi$ acts transitively on $\mathfrak{X}_\varphi$.

For $x \in \mathfrak{X}_\varphi$, define the isotropy subgroup

$$\mathcal{D}_x = \{\widehat{\gamma} \in \widehat{\Gamma}_\varphi \mid \widehat{\Phi}_\varphi(\widehat{\gamma})(x) = x\}$$

Isomorphism class of $\mathcal{D}_x$ is independent of choice of x and invariant of isomorphism of actions.

Proposition: $\mathfrak{X}_\varphi$ is a homogeneous space for $\widehat{\Gamma}_\varphi$. That is,

$$\mathfrak{X}_\varphi \cong \widehat{\Gamma}_\varphi / \mathcal{D}_x \quad \text{as left } \Gamma - \text{spaces}$$

Remark: If Γ is abelian group, then $\mathcal{D}_x$ is trivial, so $\mathfrak{X}_\varphi$ is a profinite group and Γ acts on $\mathfrak{X}_\varphi$ by group multiplication.

Say that $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$ is a generalized odometer.

Strategy: We obtain invariants of the self-embedding φ by studying the dynamics of the adjoint action of $\mathcal{D}_x$ on $\widehat{\Gamma}_\varphi$.

First, there is a canonical basepoint for $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$:

Proposition: There is a rescaling $\lambda_\varphi \colon X_\varphi \to X_\varphi$ whose image $U_1 = \lambda_\varphi(X_\varphi)$ is a clopen subset of X_φ. Moreover, the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is conjugate to the restricted action $(U_1, \Gamma_{U_1}, \Phi_{U_1})$.

Idea of proof: φ induces a map of quotients $\overline{\varphi} \colon \Gamma/\Gamma_\ell \to \Gamma_1/\Gamma_{\ell+1}$. This induces the shift map $\lambda_\varphi \colon X_\varphi \to U_1 \subset X_\varphi$.

Definition: $\mathcal{D}_\varphi \subset$ **Homeo**(X_φ) is the isotropy subgroup at the unique fixed-point x_φ of the contraction map λ_φ.

The study of invariants for the adjoint action of $\mathcal{D}_\varphi$ on $\widehat{\Gamma}_\varphi$ leads into analyzing the regularity properties of Cantor actions.

Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be a Cantor action of a countable group Γ.

The action is:

- ⋆ effective, or faithful, if $\Phi \colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$ has trivial kernel.
- ⋆ free if for all $x \in \mathfrak{X}$ and $g \in \Gamma$, $g \cdot x = x$ implies that $g = e$
- ⋆ isotropy group of $x \in \mathfrak{X}$ is $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$
- ⋆ $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and isotropy set

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\ g \in \Gamma\ ,\ g \neq id\ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)$$

Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is topologically free if $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X} \Longrightarrow \mathrm{Iso}(\Phi)$ has empty interior.

For Γ a countable group, this is a natural hypothesis to impose.

However, for a Cantor action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ where H is not countable, we introduce another definition of regularity.

First, recall the topology of Cantor space $\mathfrak{X}$ is generated by clopen subsets: U is closed and open. A non-empty clopen $U \subset \mathfrak{X}$ is adapted if the return times to U form a subgroup:

$$\Gamma_U = \{g \in \Gamma \mid \Phi(g)(U) = U\} \subset \Gamma$$

Lemma: For $x \in \mathfrak{X}$ and open $x \in V$, there is adapted U with $x \in U \subset V$.

Definition: An action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$, where

- H is a topological group and
- $\mathfrak{X}$ is a Cantor space

is quasi-analytic if for each clopen set $U \subset \mathfrak{X}$, $g \in H$

- if $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For H a countable group, this is equivalent to topologically free.

Profinite Actions:

- ⋆ $\varphi \colon \Gamma \to \Gamma$ is a renormalization for Γ
- ⋆ associated Cantor action $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$
- ⋆ induced profinite action $\widehat{\Phi}_\varphi \colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$

Here are our key results:

Theorem 1: The action $\widehat{\Phi}_\varphi$ is quasi-analytic.

Corollary 1: Let $\widehat{\gamma} \in \widehat{\Gamma}_\varphi$. The homeomorphism $\widehat{\Phi}_\varphi(\widehat{\gamma}) \colon \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$ is uniquely determined by its restriction to an adapted subset of $\mathfrak{X}$.

Theorem 2: A renormalization map φ induces a contraction map on the closure, $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

The proof of Theorem 2 looks "obvious", except that it isn't.

Here is the issue:

The renormalization φ induces a map $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \mathbf{Homeo}(U_1)$.

We need to show that the maps in the image of $\widehat{\varphi}$ have unique extensions to $\mathbf{Homeo}(X_\varphi)$.

This is exactly what Theorem 1 says is true.

Theorem 3: $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$

This connects the discriminant invariant for a Cantor action, with an invariant for a contraction profinite group.

The proof of Theorem 3 follows almost directly from the algebraic definition for $\mathcal{D}_\varphi$ developed in

- Jessica Dyer, *Dynamics of Equicontinuous Group Actions on Cantor Sets*, **Thesis UIC**, 2015.

Theorems 2 and 3 are applied to show that $\widehat{\Gamma}_\varphi$ is virtually nilpotent.

There is an extensive literature on the structure of profinite groups with an open contraction mapping, in particular by:

⋆ Baumgartner, Caprace, Reid, Wesolek, Willis, Wilson

The following result is based on results of

Udo Baumgartner & George Willis, and Colin Reid:

Theorem: Let $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ be a contraction map with open image. Then there is an isomorphism with a semi-direct product

$$\widehat{\Gamma}_\varphi \cong \mathcal{N}_\varphi \rtimes \mathcal{D}_\varphi$$

$$\begin{aligned}
\mathcal{N}_\varphi &= \{\widehat{g} \in \widehat{\Gamma}_\varphi \mid \lim_{\ell\to\infty} \widehat{\varphi}^\ell(\widehat{g}) = \widehat{e}\} \\
\mathcal{D}_\varphi &= \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi
\end{aligned}$$

Moreover, the contraction factor $\mathcal{N}_\varphi$ is pro-nilpotent.

We use this structure theorem for contraction maps to show:

Theorem [HLvL2020]: Let φ be a renormalization of the finitely generated group Γ. Suppose that

$$K(\varphi) = \bigcap_{\ell > 0} \varphi^{\ell}(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_{\varphi} = \bigcap_{n > 0} \widehat{\varphi}_0^{n}(\widehat{\Gamma}_{\varphi}) \subset \widehat{\Gamma}_{\varphi}$$

are both finite groups, then

- Γ is virtually nilpotent,
- If both groups are trivial, then Γ is nilpotent.

Remark: The normality assumption in Van Limbeek's Theorem is replaced by the assumption that $\mathcal{D}_{\varphi}$ is a finite group.

Next Steps:

$\star$ Let φ be an irreducible renormalization of a finitely generated group Γ. Show that $\mathcal{D}_\varphi$ is nilpotent, and thus Γ is virtually nilpotent.

This is true in all examples calculated. Need better understanding of closed subgroups of profinite groups to complete the proof.

$\star$ Develop general "formula" for calculating the discriminant invariant $\mathcal{D}_\varphi$ for renormalization map φ

Dynamics:

A proper self-covering proper $\phi\colon M \to M$ is called an endomorphism in the dynamical systems literature.

- *Michael Shub*, **Endomorphisms of compact differentiable manifolds**, Amer. J. Math., Vol. 91, 1969.

When ϕ is an expanding map on M, the induced dynamics on the weak solenoid $\mathcal{S}_\phi$ is of hyperbolic type, and well-studied.

Let $\varphi = \phi_* \colon \Gamma \to \Gamma$ be the renormalization associated to $\phi\colon M \to M$ which is not assumed to be expansive.

Question 1: What can be said about the dynamics on the minimal sets of the induced action on $\mathcal{S}_\phi$?

Question 2: Suppose the discriminant $\mathcal{D}_\varphi$ is a Cantor group. How does this influence the dynamics of the shift map on $\mathcal{S}_\phi$?

Problem 3: Let $\mathcal{F}_\phi$ be the Hirsch foliation associated to a proper self-covering $\phi\colon M \to M$. Show the discriminant group $\mathcal{D}_\varphi$ is a Morita equivalence invariant of the holonomy pseudogroup of $\mathcal{F}_\phi$.

The discriminant group $\mathcal{D}_\phi$ is an example of the phenomenon of shape dynamics discussed in Section 6 of:

- Hurder, *Lectures on Foliation Dynamics: Barcelona 2010*, in **Foliations: Dynamics, Geometry and Topology**, Advanced Courses in Mathematics CRM Barcelona, 2014.

U. Baumgartner and G. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, **Israel J. Math.**, 2004.

J. Dyer, S. Hurder and O. Lukina, *The discriminant invariant of Cantor group actions*, **Topology and Its Applications**, 2017.

J. Dyer, S. Hurder and O. Lukina, *Molino theory for matchbox manifolds*, **Pac. Jour. Math.**, 2016.

M. Gromov, *Groups of polynomial growth and expanding maps*, **Inst. Hautes Études Sci. Publ. Math.**, 1981.

S. Hurder and O. Lukina, *Wild solenoids*, **Trans. A.M.S.**, 2019.

S. Hurder and O. Lukina, *Orbit equivalence and classification of weak solenoids*, **Indiana Univ. Math. J.**, to appear; arXiv:1803.02098.

S. Hurder, O. Lukina and W. Van Limbeek, *Cantor dynamics of renormalizable groups*, **submitted**; arXiv:2002.01565.

V. Nekrashevych and G. Pete, *Scale-invariant groups*, **Groups Geom. Dyn.**, 2011.

C. Reid, *Endomorphisms of profinite groups*, **Groups Geom. Dyn.**, 2014.

M. Shub, *Expanding maps*, In **Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)**, Amer. Math. Soc., 1970.

W. Van Limbeek, *Towers of regular self-covers and linear endomorphisms of tori*, **Geom. Topol.**, 2018.