Homogeneous matchbox manifolds

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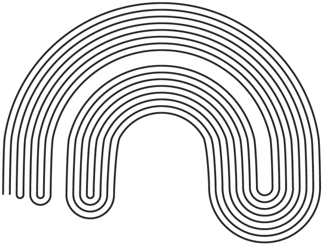
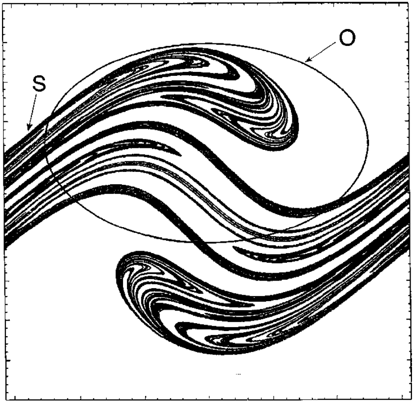
Examples: The circle $\mathbb{S}^1$ is decomposable. The Knaster Continuum (or *bucket handle*) is indecomposable.



This is one-half of a Smale Horseshoe. The 2-solenoid over $\mathbb{S}^1$ is a branched double-covering of it.

Continua. . .

Indecomposable continuum arise naturally as invariant closed sets of dynamical systems; e.g., attractors and minimal sets for diffeomorphisms.



[Picture courtesy Sanjuan, Kennedy, Grebogi & Yorke, "Indecomposable continua in dynamical systems with noise", Chaos 1997]

A Conjecture . . .

Definition: A space X is *homogeneous* if for every $x, y \in X$ there exists a *homeomorphism* $h \colon X \to X$ such that $h(x) = y$. Equivalently, X is homogeneous if the group $\mathrm{Homeo}(X)$ acts transitively on X.

Question: [Bing1960] If X is a homogeneous continuum and if every proper subcontinuum of X is an arc, must X then be a circle or a solenoid?

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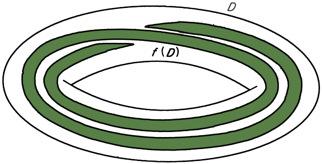
Question: [Bing1960] If X is a homogeneous continuum and if every proper subcontinuum of X is an arc, must X then be a circle or a solenoid?

Theorem: [Hagopian 1977] Let X be a homogeneous continuum such that every proper subcontinuum of X is an arc, then X is an inverse limit over the circle $\mathbb{S}^1$.



Matchbox manifolds

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We rephrase the context:

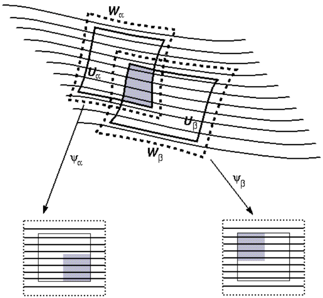
Definition: An n-dimensional *matchbox manifold* is a continuum $\mathfrak{M}$ which is a foliated space with leaf dimension n, and codimension zero.

$\mathfrak{M}$ is a foliated space if it admits a covering $\mathcal{U} = \{\varphi_i \mid 1 \leq i \leq \nu\}$ with foliated coordinate charts $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$. The compact metric spaces $\mathfrak{T}_i$ are totally disconnected $\Longleftrightarrow$ $\mathfrak{M}$ is a matchbox manifold.

The leaves of $\mathcal{F}$ are the path components of $\mathfrak{M}$.

Smooth matchbox manifolds

Definition: $\mathfrak{M}$ is a *smooth foliated space* if the leafwise transition functions for the foliation charts $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$ are C^∞, and vary continuously on the transverse parameter in the leafwise C^∞-topology.



Automorphisms of matchbox manifolds

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, with the transverse dynamics of the foliation $\mathcal{F}$ on the Cantor-like fibers $\mathfrak{T}_i$ representing fundamental groupoid data. They naturally appear in:

- dynamical systems, as minimal sets & attractors
- geometry, as laminations
- complex dynamics, as universal Riemann surfaces
- algebraic geometry, as models for "stacks".

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Haefliger Question: What are the topological invariants associated to a matchbox manifolds, and do they characterize them in some fashion?

A solution to the Bing Question

Theorem [Clark & Hurder 2009] Let $\mathfrak{M}$ be an orientable homogeneous smooth matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a McCord (or normal) solenoid. In particular, $\mathfrak{M}$ is minimal, so every leaf is dense.

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When the dimension of $\mathfrak{M}$ is $n = 1$ (that is, $\mathcal{F}$ is defined by a flow) then this recovers the result of Hagopian, but the proof is much closer in spirit to the later proof of this case by [Aarts, Hagopian and Oversteegen 1991].

The case where $\mathfrak{M}$ is given as a fibration over $\mathbb{T}^n$ with totally disconnected fibers was proven in [Clark, 2002].

Two applications

Here are two consequences of the Main Theorem:

Corollary: Let $\mathfrak{M}$ be an orientable homogeneous n-dimensional smooth matchbox manifold, which is embedded in a closed $(n+1)$-dimensional manifold. Then $\mathfrak{M}$ is itself a manifold.

For $\mathfrak{M}$ a homogeneous continuum with a non-singular flow, this was a question/conjecture of Bing, solved by [Thomas 1971]. Non-embedding for solenoids of dimension $n \geq 2$ was solved by [Clark & Fokkink, 2002]. Proofs use shape theory and Alexander-Spanier duality for cohomology.

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Corollary: Let $\mathfrak{M}$ be the tiling space associated to a tiling $\mathcal{P}$ of $\mathbb{R}^n$. If $\mathfrak{M}$ is homogeneous, then the tiling is periodic.

Generalized solenoids

Let M_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} M_\ell \xrightarrow{p_\ell} M_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0$$

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The p_ℓ are called the *bonding maps* for the solenoid

$$\mathcal{S} = \varprojlim \; \{p_\ell \colon M_\ell \to M_{\ell-1}\} \; \subset \; \prod_{\ell=0}^{\infty} M_\ell$$

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$$\mathcal{S} = \lim_{\longleftarrow} \; \{p_\ell \colon M_\ell \to M_{\ell-1}\} \; \subset \; \prod_{\ell=0}^{\infty} M_\ell$$

Choose basepoints $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$. Set $G_\ell = \pi_1(M_\ell, x_\ell)$.

Then we have a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \longrightarrow M_0$.

McCord solenoids

Definition: $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image H_ℓ of G_ℓ in $H_{\ell_0} \equiv G_{\ell_0}$ is a normal subgroup.

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Caution: There are constructions of inverse limits $\mathcal{S}$ as above where the bonding maps are not normal coverings, and the McCord condition does not hold, but $\mathcal{S}$ is homogeneous [Fokkink & Oversteegen 2002].

Our technique of proof of the main theorem for such examples presents the inverse limit space $\mathcal{S}$ as homeomorphic to a normal tower of coverings.

Effros Theorem

Let X be a separable and metrizable topological space. Let G be a topological group with identity e.

For $U \subseteq G$ and $x \in X$, let $Ux = \{gx \mid g \in U\}$.

Definition: An action of G on X is *transitive* if $Gx = X$ for all $x \in X$.

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Alternate proofs of have been given by [Ancel 1987] and [van Mill 2004]. Remarkably, Van Mill shows that Effros Theorem is equivalent to the *Open Mapping Principle* of Functional Analysis. This appeared in the American Mathematical Monthly, pages 801–806, 2004.

Interpretation for compact metric spaces

The metric on the group $\mathrm{Homeo}(X)$ for (X, d_X) a separable, locally compact, metric space is given by

$$\begin{aligned} d_H(f, g) &:= \sup\{d_X(f(x), g(x)) \mid x \in X\} \\ &\quad + \sup\left\{d_X\left(f^{-1}(x), g^{-1}(x)\right) \mid x \in X\right\} \end{aligned}$$

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Corollary: Let X be a homogeneous compact metric space. Then for any given $\epsilon > 0$ there is a corresponding $\delta > 0$ so that if $d_X(x,y) < \delta$, there is a homeomorphism $h \colon X \to X$ with $d_H(h, id_X) < \epsilon$ and $h(x) = y$.

In particular, for a homogeneous foliated space $\mathfrak{M}$ this conclusion holds.

This observation was used by [Aarts, Hagopian, & Oversteegen 1991] and [Clark 2002] in their study of matchbox manifolds.

Holonomy groupoids

Let $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$ for $1 \leq i \leq \nu$ be the covering of $\mathfrak{M}$ by foliation charts. For $U_i \cap U_j \neq \emptyset$ we obtain the holonomy transformation

$$h_{ji} \colon D(h_{ji}) \subset \mathfrak{T}_i \longrightarrow R(h_{ji}) \subset \mathfrak{T}_j$$

These transformations generate the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of $\mathfrak{M}$, modeled on the transverse metric space $\mathfrak{T} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_\nu$

Typical element of $\mathcal{G}_{\mathcal{F}}$ is a composition, for $\mathcal{I} = (i_0, i_1, \ldots, i_k)$ where $U_{i_\ell} \cap U_{i_{\ell-1}} \neq \emptyset$ for $1 \leq \ell \leq k$,

$$h_{\mathcal{I}} = h_{i_k i_{k-1}} \circ \cdots \circ h_{i_1 i_0} \colon D(h_{\mathcal{I}}) \subset \mathfrak{T}_{i_0} \longrightarrow R(h_{\mathcal{I}}) \subset \mathfrak{T}_{i_k}$$

$x \in \mathfrak{T}$ is a *point of holonomy* for $\mathcal{G}_{\mathcal{F}}$ if there exists some $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ with $x \in D(h_{\mathcal{I}})$ such that $h_{\mathcal{I}}(x) = x$ and the germ of $h_{\mathcal{I}}$ at x is non-trivial.

We say $\mathcal{F}$ is *without holonomy* if there are no points of holonomy.

Equicontinuous matchbox manifolds

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, y \in D(h_{\mathcal{I}}) \text{ with } d_{\mathfrak{X}}(x, y) < \delta \quad \Longrightarrow \quad d_{\mathfrak{X}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(y)) < \epsilon$$

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Theorem: A homogeneous matchbox manifold $\mathfrak{M}$ is equicontinuous without holonomy.

The proof relies on one basic observation and extensive technical analysis.

Lemma: Let $h \colon \mathfrak{M} \to \mathfrak{M}$ be a homeomorphism. Then h maps the leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. That is, every $h \in \mathrm{Homeo}(\mathfrak{M})$ is foliation-preserving.

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Theorem: An equicontinuous matchbox manifold $\mathfrak{M}$ is minimal.

Three Structure Theorems

We can now state the three main structure theorems.

Theorem 1: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy. Then $\mathfrak{M}$ is homeomorphic to a solenoid

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Theorem 2: Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then the bonding maps above can be chosen so that $q_\ell \colon M_\ell \longrightarrow M_0$ is a normal covering for all $\ell \geq 0$. That is, $\mathcal{S}$ is McCord.

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Theorem 3: Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then there exists a clopen subset $V \subset \mathfrak{T}$ such that the restricted groupoid $\mathcal{H}(\mathcal{F}, V) \equiv \mathcal{G}_{\mathcal{F}}|V$ is a group, and $\mathfrak{M}$ is homeomorphic to the suspension of the action of $\mathcal{H}(\mathcal{F}, V)$ on V. Thus, the fibers of the map $q_\infty \colon \mathfrak{M} \to M_0$ are homeomorphic to a profinite completion of $\mathcal{H}(\mathcal{F}, V)$.

Coding & Quasi-Tiling

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy.

Fix basepoint $w_0 \in int(\mathfrak{T}_1)$ with corresponding leaf $L_0 \subset \mathfrak{M}$.

The equivalence relation on $\mathfrak{T}$ induced by $\mathcal{F}$ is denoted Γ, and we have the following subsets:

$$\begin{aligned}
\Gamma_W &= \{(w, w') \mid w \in W \ , \ w' \in \mathcal{O}(w)\} \\
\Gamma_W^W &= \{(w, w') \mid w \in W \ , \ w' \in \mathcal{O}(w) \cap W\} \\
\Gamma_0 &= \left\{w' \in W \mid (w_0, w') \in \Gamma_W^W\right\} = \mathcal{O}(w_0) \cap W
\end{aligned}$$

Note that Γ_W^W is a groupoid, with object space W. The assumption that $\mathcal{F}$ is without holonomy implies Γ_W^W is equivalent to the groupoid of germs of local holonomy maps induced from the restriction of $\mathcal{G}_{\mathcal{F}}$ to W.

Equicontinuity & uniform domains

Proposition: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy. Given $\epsilon_* > 0$, then there exists $\delta_* > 0$ such that:

- for all $(w, w') \in \Gamma_W^W$ the corresponding holonomy map $h_{w,w'}$ satisfies $D_{\mathfrak{X}}(w, \delta_*) \subset D(h_{w,w'})$
- $d_{\mathfrak{X}}(h_{w,w'}(z), h_{w,w'}(z')) < \epsilon_*$ for all $z, z' \in D_{\mathfrak{X}}(w, \delta_*)$.

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- for all $(w, w') \in \Gamma_W^W$ the corresponding holonomy map $h_{w,w'}$ satisfies $D_{\mathfrak{T}}(w, \delta_*) \subset D(h_{w,w'})$
- $d_{\mathfrak{T}}(h_{w,w'}(z), h_{w,w'}(z')) < \epsilon_*$ for all $z, z' \in D_{\mathfrak{T}}(w, \delta_*)$.

Let $W \subset \mathfrak{T}_1$ be a clopen subset with $w_0 \in W$. Decompose W into clopen subsets of diameter $\epsilon_\ell > 0$,

$$W = W_1^\ell \cup \cdots \cup W_{\beta_\ell}^\ell$$

Set $\eta_\ell = \min \left\{ d_{\mathfrak{T}}(W_i^\ell, W_j^\ell) \mid 1 \leq i \neq j \leq \beta_\ell \right\} > 0$ and let $\delta_\ell > 0$ be the constant of equicontinuity as above.

The orbit coding function

- The code space $\mathcal{C}_\ell = \{1, \ldots, \beta_\ell\}$
- For $w \in W$, the $\mathcal{C}^\ell_w$-code of $u \in W$ is the function $C^\ell_{w,u} \colon \Gamma_w \to \mathcal{C}_\ell$ defined as: for $w' \in \Gamma_w$ set $C^\ell_{w,u}(w') = i$ if $h_{w,w'}(u) \in W^\ell_i$.
- Define $V^\ell = \left\{ u \in W^\ell_1 \mid C^\ell_{w_0,u}(w') = C^\ell_{w_0,w_0}(w') \text{ for all } w' \in \Gamma_0 \right\}$

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Lemma: If $u, v \in W$ with $d_{\mathfrak{X}}(u, v) < \delta_\ell$ then $C^\ell_{w,u}(w') = C^\ell_{w,v}(w')$ for all $w' \in \Gamma_w$. Hence, the function $C^\ell_w(u) = C^\ell_{w,u}$ is locally constant in u.

Thus, V^ℓ is open, and the translates of this set define a Γ_0-invariant clopen decomposition of W.

The coding decomposition

The Thomas tube $\widetilde{\mathfrak{N}}_\ell$ for $\mathfrak{M}$ is the "saturation" of V^ℓ by $\mathcal{F}$.

The saturation is necessarily all of $\mathfrak{M}$. But the tube structure comes with a vertical fibration, which allows for collapsing the tube in foliation charts. This is the basis of the main technical result:

The coding decomposition

The Thomas tube $\widetilde{\mathfrak{N}}_\ell$ for $\mathfrak{M}$ is the "saturation" of V^ℓ by $\mathcal{F}$.

The saturation is necessarily all of $\mathfrak{M}$. But the tube structure comes with a vertical fibration, which allows for collapsing the tube in foliation charts. This is the basis of the main technical result:

Theorem: For $\mathrm{diam}(V^\ell)$ sufficiently small, there is a quotient map $\Pi_\ell \colon \widetilde{\mathfrak{N}}_\ell \to M_\ell$ whose fibers are the transversal sections isotopic to V^ℓ, and whose base if a compact manifold. This yields compatible maps $\Pi_\ell \colon \mathfrak{M} \to M_\ell$ which induce the solenoid structure on $\mathfrak{M}$.

Furthermore, if $\mathfrak{M}$ is homogeneous, then $\mathrm{Homeo}(\mathfrak{M})$ acts transitively on the fibers of the tower induced by the maps $\Pi_\ell \colon \mathfrak{M} \to M_\ell$, hence the tower is normal.

Leeuwenbrug Conjecture

Conjecture: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and $V \subset \mathfrak{T}$ a clopen set. Then $\mathfrak{M}$ is characterized up to homeomorphism by the restricted groupoid $\mathcal{H}(\mathcal{F}, V) \equiv \mathcal{G}_{\mathcal{F}}|V$ and any partial quotient M_{ℓ}.

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