Topology and dynamics of Kuperberg minimal sets

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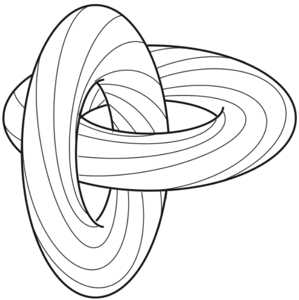
Seifert Theorem

Theorem: [Seifert, 1950] Let $\mathcal{X}$ be a C^1-vector field on $\mathbb{S}^3$, which is C^0-close to the Hopf fibration. Then the flow of $\mathcal{X}$ has a periodic orbit.



Seifert Conjecture

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Every closed oriented 3-manifold M admits a non-vanishing smooth vector field.

C^r-**Seifert Conjecture:** Every non-vanishing C^r-vector field $\mathcal{X}$ on a closed 3-manifold M has a periodic orbit.

Some history

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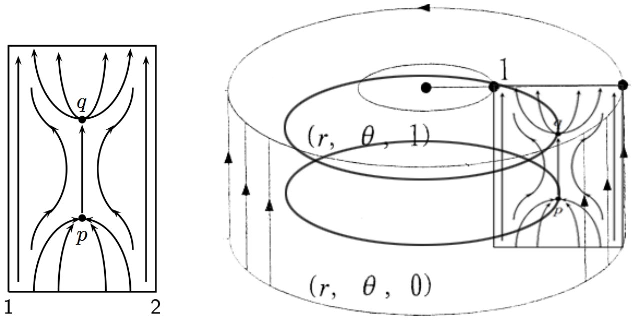
The key to the proof is the fundamental idea of a "plug", which has been the basis of all subsequent approaches to the Conjecture. A plug is a modification of the vertical flow in a coordinate box $(x, y, z) \in [-1, 1]^3 \cong U \subset M$. The modification can be anything, except it must agree with the vertical field on the boundary of U, and it must satisfy the:

Mirror Symmetry Property: An orbit entering a plug (from the bottom) either never leaves the plug (it is "trapped"), or exits the plug at the mirror image point at the top of the plug.

Wilson Plug

Wilson's fundamental idea was the construction of a plug which trapped content, and all trapped orbits have limit set a periodic orbit contained in the plug. The two periodic orbits are attractors:



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In early 1970's, Paul Schweitzer had two deep insights:

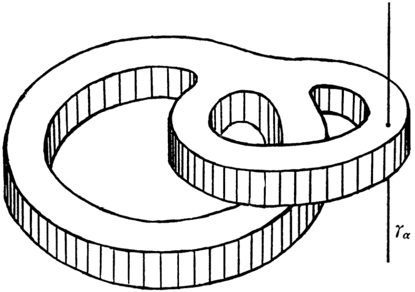
In the Wilson Plug:

• the periodic circles can be replaced by a minimal set for a flow without periodic orbits, such as the Denjoy minimal set;

• the new minimal set does not have to be in a planar flow, but may be contained in a surface flow which embeds in $\mathbb{R}^3$.

Schweitzer Plug

The circular orbits of the Wilson Plug are replaced by Denjoy minimal sets, embedded as follows:



Handel's Theorem

A minimal set K for a flow $\mathcal{X}$ on a 3-manifold M is said to be "surface-like" if there is a tamely embedded surface $\Sigma \hookrightarrow M$ whose image contains K.

Theorem: [Handel, 1980] Let $\mathcal{X}$ be a flow on a 3-manifold M such that its minimal sets are surface-like. Then $\mathcal{X}$ cannot be C^2.

Handel's analysis implies any construction of counter-examples to the Seifert Conjecture requires "3-dimensional dynamics".

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The real insight was to not try to find the "right" minimal set, as in the works of P. Schweitzer or J. Harrison [1982/1988], but rather to control the dynamics in the plug.

Kuperberg's "Big Idea"

Shigenori Matsumoto's summary:

そこで，どうしても W 内のふたつの周期軌道 T_1 と T_2 を予め破壊しておく必要がある．しかしそのために新しい部品を開発するのでは話は振り出しに戻ってしまう．Kuperberg の発想は，**W 内の周期軌道自身で自分達を破壊させる**というものである．敵同士が妨害工作をしあうようにわなを仕掛けた後は，何もせずに黙ってみていればよいということである．

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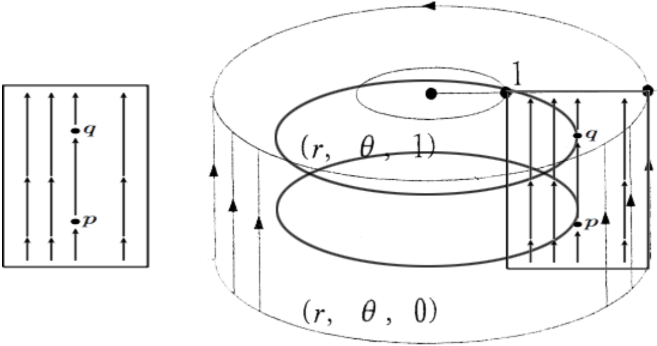
We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to let closed orbits demolish themselves. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

(transl. by Kiki Hudson Arai)

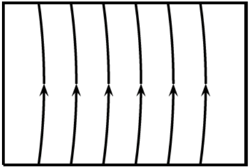
Modified Wilson Plug

The vector field $\mathcal{W} = f(r,z)\partial/\partial z + g(r,z)\partial/\partial\theta$ on the plug $(r,\theta,z) \in [1,3] \times \mathbb{S}^1 \times [-2,2] = W$ is radially symmetric, with $f(r,0) = 0$ and $g(r,z) = 0$ only near the boundary, and $g(r,z) = 1$ away from the boundary ∂W.

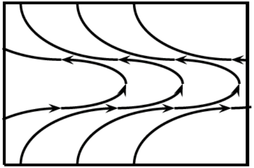


Orbits of modified Wilson

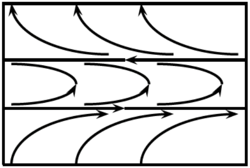
This modification of the standard Wilson flow is a fundamental technical point. The orbits of $\mathcal{W}$ appear like this:



$r \approx 1, 3$

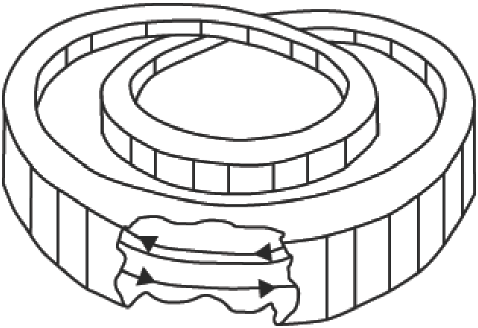


$r \approx 2$

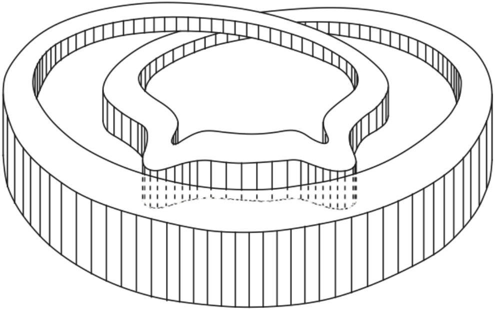


$r = 2$

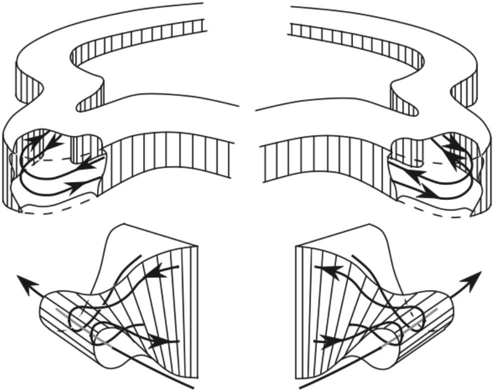
Embed Wilson as a double cover



Grow two horns

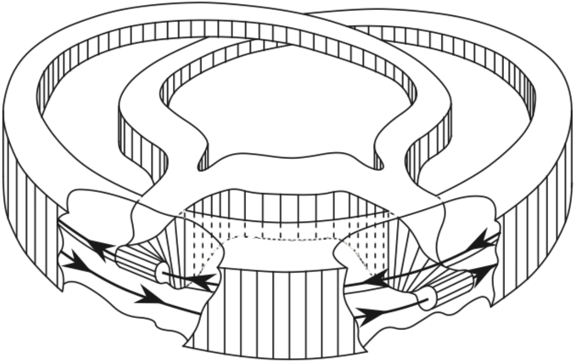


Twist the horns



Insert the horns

The vector field $\mathcal{W}$ induces a field $\mathcal{K}$ on the surgered manifold.
Then the Kuperberg Plug is pictured as:



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Theorem:[K. Kuberberg, 1994] The Kuperberg Plug is an aperiodic flow.

That is, there exists a non-empty subset of the entry and exit regions consisting of orbits which enter the plug, and do not exit.

The construction above can be modified so that it is analytic. Thus, the Seifert Conjecture is false for any degrees of smoothness.

Remarks

Kuperberg's manuscript was first put in circulation in 1993. It was discussed in November 1993 in a working seminar at Tokyo Tech Institute, attended by Conlon, Ghys, Hurder, Matsumoto, Schweitzer, Tsuboi, and a few others. During the subsequent months, the results and proofs were sharpened by Ghys, Matsumoto and Kuperberg. A complete "snapshot" of the theory in 1994 requires combining the following papers from Spring 1994:

- K. Kuperberg, "A smooth counterexample to the Seifert conjecture", Annals of Math, 1994.
- S. Matsumoto, "Kuperberg's C^∞ counterexample to the Seifert conjecture", Sūgaku, Mathematical Society of Japan, 1995.
- É. Ghys, "Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)", Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, 1995.

Minimal set for the Kuperberg Plug

Observe that if $x \in P$ has forward orbit trapped in the Kuperberg Plug P, then the ω-limit set ω_x is a closed invariant subset of the interior of P, hence contains a minimal set K for the flow.

Problem: Describe the dynamics of $\mathcal{X}$ restricted to K.

Problem: What is the topological shape of K?

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The proof that the Kuperberg flow is aperiodic follows from analysis of individual orbits of the flow. The questions above ask about more global aspects of the flow.

More dynamics for the Kuperberg Plug

Theorem:[Matsumoto, 1994] The Kuperberg flow has non-empty wandering set, hence is an aperiodic flow which "stops content".

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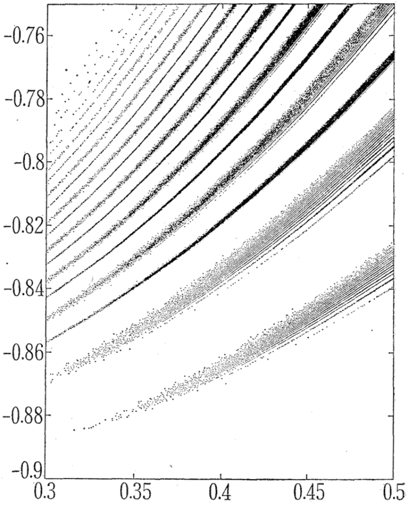
Theorem:[Ghys 1994] The Kuperberg flow has entropy 0.

Proof: If $\mathcal{X}$ had positive entropy, then by a Theorem of Katok, it would have a periodic orbit.

Computer model of K



Our project was to understand the global topology and dynamics of "the Kuperberg Plug". In actuality, there are many choices involved in the construction, so part of the project was to "redo" the published arguments, into a cohesive whole which identified the properties true for all such flows, and those properties which depend on choices.

Invariant zippered laminations

Theorem:[H & R] The Kuperberg flow preserves a "zippered lamination" $\overline{\mathcal{L}}$, where:

• $\overline{\mathcal{L}}$ is defined by the closure of the orbit of the image of the Reeb Cylinder in the Wilson Plug.

• The boundary $\mathcal{Z} = \partial\overline{\mathcal{L}}$ is the flow of a Cantor set (the *zipper*), defined by 1-dimensional ping-pong dynamics for an induced dynamical subsystem of the flow.

• The complement $\mathcal{L} = \overline{\mathcal{L}} - \mathcal{Z}$ is a 2-dimensional lamination with open leaves.

• The leaves of $\mathcal{L}$ are quasi-isometric to infinite trees (propellors).

• The orbits of all points not in $\overline{\mathcal{L}}$ are wandering.

Topological entropy

Theorem:[H & R] The Kuperberg flow has a topological entropy 0.

Proof: All orbits of the flow not in $\overline{\mathcal{L}}$ are wandering, while the orbits of points in $\overline{\mathcal{L}}$ have subexponential separation of points. Thus, the entropy of the flow is zero.

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Theorem:[H & R] The lamination $\mathcal{L}$ has positive *foliation* entropy.

Proof: The holonomy of $\mathcal{L}$ generates the ping-pong dynamics on its boundary $\partial\mathcal{L}$, hence has positive entropy.

Shape of a continua

For $\epsilon > 0$ let $N_\epsilon(K) = \{x \in P \mid d(x, K) < \epsilon\}$. Set $N_\ell = N_{1/\ell}(K)$.

Definition: The shape of K is $\mathcal{S}(K) = \varprojlim N_\ell$.

A continua $K \subset M$ is said to be stable if there exists $\ell_0 > 0$ such that for all $\ell > \ell_0$, the inclusion $N_\ell \subset N_{\ell_0}$ is a homotopy equivalence.

For example, the Denjoy minimal set has stable shape $\cong \mathbb{S}^1 \wedge \mathbb{S}^1$.

Shape of the minimal set

We then realize the original motivation for this work:

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Noting that the minimal set of a C^2-surface flow has stable shape, following Handel's Theorem, we propose:

Conjecture: If $\mathcal{X}$ is a C^2-vector field with no periodic orbits on a closed 3-manifold M, then the shape of each minimal set K for the flow is not stable.

Main Results: Homeomorphism type of the minimal set

Finally, we describe the unique minimal set K:

Theorem:[H & Rechtman] $K = \overline{\mathcal{L}}$ for a "generic" Kuperberg Plug.

By generic, we mean that the singularities for the vanishing of the vertical vector field $\mathcal{W}$ used to define $\mathcal{K}$ are quadratic type, and the insertion yields a quadratic radius function.

The two papers below contain particular constructions of Kuperberg Plugs which assert the existence of open disks in K.

• É. Ghys, "Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)", Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, 1995.

• G. Kuperberg and K. Kuperberg, "Generalized counterexamples to the Seifert conjecture" , Annals of Math, 1996.

Strategy of proofs

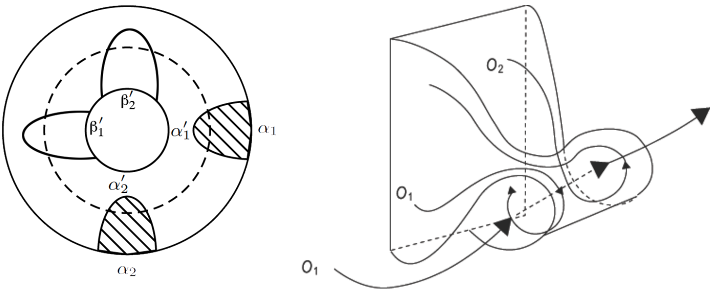
The proofs of these results require a careful analysis of the construction and dynamics of both the modified Wilson and Kuperberg Plugs, along with new observations.

The strategy in Kuperberg's original work, and in the subsequent published papers, is to analyze the behavior of individual orbits of the flow. This fundamental observation remains that by Kuperberg, that the insertion process breaks the orbits for the Kuperberg flow into segments, which are image of orbit segments from the modified Wilson Plug. One then analyzes the combinatorics of these orbit segments.

Our strategy is to consider the orbits of transversal segments to the Kuperberg flow, obtaining global dynamical information as well.

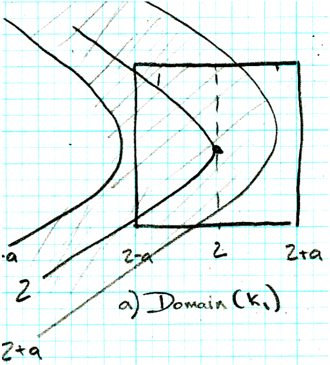
The insertion

Points entering the base of P are called *primary entry points*.
Points entering the insertion face are called *secondary entry points*.
Primary and secondary exit points are likewise defined.



Radius inequality

$\bar{r}(x) \geq r(x)$, where $\bar{r}(x)$ is the radius of the image of x in the inserted region, with equality only at the periodic orbit entry point. Thus, passing through the face of an insertion increases the radius.



Trapped orbits

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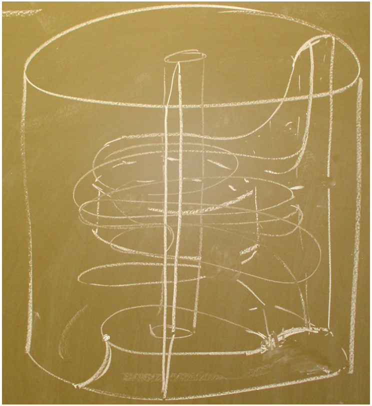
Proposition: If x is a primary entry point with $2 - \epsilon < r(x) \leq 2$, then the orbit of x is trapped in P.

Proposition: The orbit of an entry point x with $r(x) = 2$ returns to the vertical line $r = 2$, $\theta = \theta_0$, $-2 \leq z \leq -1$ infinitely often, with the sequence of points limiting on the image of the periodic orbit $p = (r = 2, \theta = \theta_0, z = -1)$. Its ω-limit set is the Cantor set $\mathcal{Z}$ (the zipper).

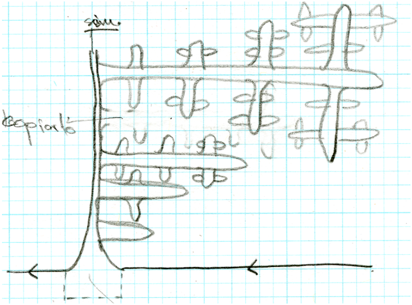
Tongues in Wilson Plug

We illustrate the complexity of studying the orbits of transverse segments to the Wilson flow:



Propellors in Kuperberg Plug

When a tongue as above is combined with the recursive Wilson dynamics of the Kuperberg flow, we obtain a “propellor”:

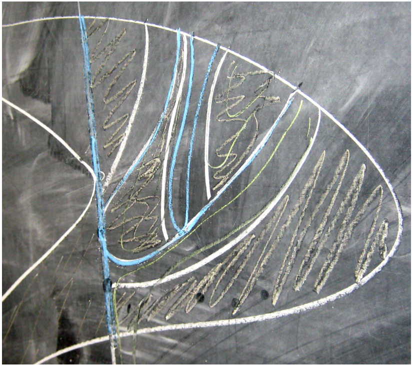


Propellors produce leaves of the lamination

This is a picture of level 0 of the lamination - a cone off the level 0 endpoints of the ping-pong game:

Re-insertion of the propellors

This is a rough picture of the first part of level 1 of the lamination. The zipper Cantor set is seen emerging along the vertical blue line:



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