Self-intersections of foliation cycles

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Realizing homology classes

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Thom [1953,1954] solved this problem (up to torsion) in terms of bordism classes.

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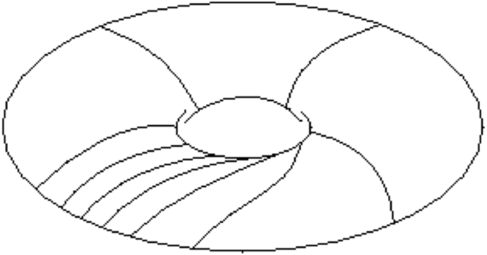
Example: Realize class $x = \alpha \cdot e_1 + \beta \cdot e_2 \in H_1(\mathbb{T}^2, \mathbb{R})$

For $\alpha, \beta \in \mathbb{R}$ construct foliation of $\mathbb{T}^2$ with slope $\lambda = \beta/\alpha$

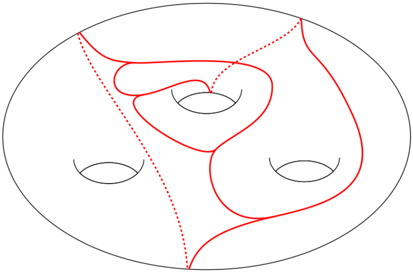


Another Example:

Lamination Λ embedded in surface Σ_g (carried by train track)

Leaf L defines asymptotic class $[L] \in H_1(\Sigma_g; \mathbb{R}) \cong H_1(\mathbb{T}^{2g}, \mathbb{R})$

Non-trivial if branched cover of algebraic Anosov map of $\mathbb{T}^2$.



Asymptotic cycles

L a complete Riemannian n-manifold is "closed at infinity" if there is $x \in L$ and sequence $R_\ell \to \infty$ so that

$$\rho(L) = \lim_{\ell \to \infty} \frac{|\partial B(x, R_\ell)|}{|B(x, R_\ell)|} = 0$$

$B(x, R) = \{y \in L \mid d_L(x, y) \leq R)$

$\partial B(x, R) = \{y \in L \mid R - 1 \leq d_L(x, y) \leq R)$

$|X|$ denotes Riemannian volume of $X \subset L$.

Asymptotic cycles

Theorem: Let L be oriented, then $\mathcal{C} = \{F_\ell = B(x, R_\ell) \mid \ell = 1, 2, \ldots\}$ defines an *asymptotic fundamental class* for the bounded n-forms on L.

$$\langle [\mathcal{C}], \psi \rangle \ = \ \lim_{i \to \infty} \frac{1}{|F_{\ell_i}|} \cdot \int_{F_{\ell_i}} \psi$$

Theorem: Let $\iota \colon L \to M$ be embedding with bounded geometry, and assume L is "closed at infinity" and oriented. Then it defines *asymptotic geometric cycle*, and homology class $[\iota\mathcal{C}] \in H_n(M, \mathbb{R})$.

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Problem: Which homology classes $x \in H_n(M; \mathbb{R})$ can be realized by an asymptotic geometric cycle $\iota \colon \mathcal{C} \to M$?

Foliated spaces

Definition: A *foliated space of dimension n* is a continuum Λ with a partition $\mathcal{F}$ into leaves, such that there exists a compact separable metric space $\mathfrak{X}$, and for each $x \in \Lambda$ there is a compact subset $\mathfrak{T}_x \subset \mathfrak{X}$, an open subset $U_x \subset \Lambda$ with $x \in U_x$, and a homeomorphism defined on the closure $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ such that for each $y \in U_x$ the connected component of $\mathcal{F}|U_x$ containing y is defined by $\varphi_x^{-1}((-1,1)^n \times w_y)$ for some $w_y \in \mathfrak{T}_x$.

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$\{\Lambda, \mathcal{F}\}$ is *smoothly embedded* in M if $\Lambda \subset M$, and for each $x \in \Lambda$, there exists a $C^{\infty,0}$-chart for M, $\psi_x \colon \overline{W}_x \to [-1,1]^m$ about x which restricts to a foliation chart for Λ.

Codimension $q = m - n$.

Foliated spaces

Remark: $L \subset \Lambda \subset M$ is embedded with bounded geometry. So L "closed at infinity" yields asymptotic geometric cycle in M.

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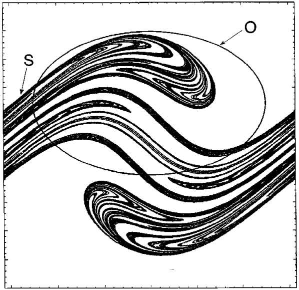
Remark: Embedded foliated spaces $\Lambda \subset M$ arise naturally as invariant (attractors) in differentiable dynamics. Examples include:

- Hyperbolic invariant set for Axiom A diffeomorphism $f: M \to M$.
- Orbit closures for diffeomorphism $f: M \to M$:



Transverse invariant measures

A *transverse invariant measure* μ for a foliated space $(\Lambda, \mathcal{F})$ is a family of finite Borel measures $\{\mu_\alpha \mid \alpha \in \mathcal{A}\}$ defined on a family of transversals $\mathfrak{T}_\alpha \subset \Lambda$ to $\mathcal{F}$, such that for each holonomy map $h_{\beta,\alpha}$ from an open subset of $\mathfrak{T}_\alpha$ to an open subset of $\mathfrak{T}_\beta$ and Borel subset $E \subset \mathrm{Domain}(h_{\beta,\alpha})$ then,

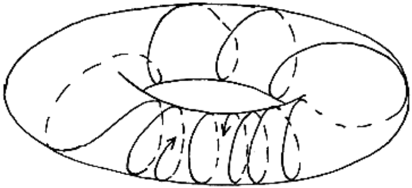
$$\mu_\beta(h_{\beta,\alpha}(E)) = \mu_\alpha(E)$$

Theorem: [Ruelle-Sullivan, Plante 1976] $L \subset \Lambda$ leaf which is "closed at infinity" yields transverse invariant measure μ for $(\Lambda, \mathcal{F})$.

Atomics

Definition: A measure μ has *atoms* if there exists $x \in \mathfrak{T}_\alpha$ such that $\mu_\alpha(\{x\}) > 0$.

$\mathcal{C} = \{F_\ell = B(x, R_\ell) \subset L \mid \ell = 1, 2, \ldots\}$ defines μ, then an atom for μ corresponds to a compact leaf $L_0 \subset \Lambda$, which is a type of Poincaré-Bendixson limit cycle for $\mathcal{F}$.



Intersections of cycles

Example: Let $M = \mathbb{S}^2 \times \mathbb{S}^2$, then obtain classes

$$x = [\mathbb{S}^2 \times \{y_0\}] \in H_2(M;\mathbb{Z}) \ , \ y = [\{x_0\} \times \mathbb{S}^2] \in H_2(M;\mathbb{Z})$$

Their intersection product $x \cap y = [1] \in H_0(M, \mathbb{Z})$

$x \in H_n(M, \mathbb{Z})$ & $y \in H_q(M, \mathbb{Z})$ represented by geometric cycles

$$\iota_x : C_x \to M \quad , \quad \iota_y : C_y \to M$$

Then $x \cap y \in H_0(M;\mathbb{Z})$ can be calculated via counting “signed” points of intersection.

Self-intersections of cycles

For $x \in H_n(M,\mathbb{Z})$ how to calculate $x \cap x \in H_{m-2q}(M;\mathbb{Z})$?

Step 1: Represent x by geometric cycle $\iota_x \colon C_x \to M$

Step 2: Choose perturbation $\iota'_x \colon C_x \to M$

Step 3: Count intersection homology classes $\iota_x(C_x) \cap \iota'_x(C_x)$

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Alternate approach: Construct closed form ω on M of degree $q = m - n$ which is Poincaré dual to $[\iota_x C_x]$

$$[\iota_x C_x] \in H_n(M; \mathbb{Z}) \to H_n(M; \mathbb{R}) \cong H^q(M; \mathbb{R}) \cong H^q_{deR}(M)$$

Then $([\iota_x C_x] \cap [\iota_x C_x])^* = [\omega \wedge \omega] \in H^{2q}_{deR}(M) \cong H_{m-2q}(M; \mathbb{R})$.

Main Theorem

Theorem: Let $\mathcal{F}$ be a $C^{\infty,0}$-foliation of a foliated space $\Lambda \subset M$ embedded in a closed oriented manifold M, such that the leaves of $\mathcal{F}$ are oriented, immersed C^1-submanifolds of M. Let μ be a transverse invariant measure for $\mathcal{F}$ without atoms. Let C_μ be the closed foliation n-current associated to μ. Then the self-intersection product $[C_\mu] \cap [C_\mu] \in H_{m-2q}(M; \mathbb{R})$ vanishes.

More precisely, for the Poincaré dual closed q-form ω_μ on M,

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Corollary: If $x \in H_n(M; \mathbb{R})$ can be realized by an asymptotic geometric cycle $L \subset \Lambda \subset M$ with no atoms, then $x \cap x = 0$.

Anosov diffeomorphisms

$f : M \to M$ is Anosov diffeomeorphism if there exists $\lambda > 1$, and

- $TM = E^- \oplus E^+$, E^- dimension n, E^+ dimension q, $n + q = m$
- $E^\pm$ are invariant under the differential Df,
- $Df|E^+$ is uniformly expanding: $\|DF(\vec{X})\| \geq \lambda \|\vec{X}\|$, $\vec{X} \in E^+$
- $Df|E^-$ is uniformly contracting: $\|DF(\vec{X})\| \leq \lambda^{-1} \|\vec{X}\|$, $\vec{X} \in E^-$

The distributions E^+ and E^- are uniquely integrable, giving foliations $\mathcal{F}^\pm$, whose leaves are smoothly immersed submanifolds with *polynomial growth rate*.

The foliations are $C^{\infty,0}$ - continuous, with smooth leaves - but rarely smooth unless the map f is algebraic.

Ruelle-Sullivan currents

Ruelle and Sullivan [1976] showed that the leaves $L^{\pm} \subset M$ for $\mathcal{F}^{\pm}$ define closed n-currents:

$$[C_{-}] \in H_n(M, \mathbb{R}) \quad \text{and} \quad [C_{+}] \in H_q(M, \mathbb{R})$$

with $[C_{-}] \cap [C_{+}] = 1$. Thus both classes are non-zero.

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The result holds more generally for Axiom A diffeomorphisms.

The leaves define invariant measures without atoms.

Main Theorem implies $[C_{\pm}] \cap [C_{\pm}] = 0$.

Example

Theorem: [Kleptsyn & Kudryashov (2009)] $M = \mathbb{S}^2 \times \mathbb{S}^2$ admits no Axiom A diffeomorphism $f : M \to M$.

Note that $H_2(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, so there does exists a smooth map $f : M \to M$ whose action on homology is hyperbolic.

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Proof: If exists, then $0 \neq [C_-] = \alpha \cdot e_1 + \beta \cdot e_2 \in H_2(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{R})$

$[C_-] \cap [C_-] = 0$ implies $\alpha = 0$, or $\beta = 0$.

If $[C_-] = \alpha \cdot e_1$ then $(f^\ell)_*[C_-] = \lambda^{-\ell} \cdot [C_-] \to 0$.

But e_1 is an integral class, so this is impossible.

Ditto for $[C_-] = \beta \cdot e_2$. □

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Shiraiwa [1973] proved using machinery of Axiom A dynamics.

Two key ideas

(1) A finite Borel measure μ on $\mathfrak{T}$ has no atoms iff the diagonal $\Delta \subset \mathfrak{T} \times \mathfrak{T}$ has measure 0 for $\mu \times \mu$.

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(2) For a closed $(m-2q)$-form ψ on M, the pairing

$$\langle [\psi], [C_\mu] \cap C_\mu] \rangle \; = \; \int_M \psi \wedge \omega_\mu \wedge \omega_\mu$$

reduces to calculating the mass of the diagonal in $\displaystyle\bigcup_{\alpha \in \mathcal{A}} \mathfrak{T}_\alpha \times \mathfrak{T}_\alpha$ for the measure $\mu \times \mu$.

Constructing the normal bundle

Each $L \subset \Lambda$ is smoothly embedded, with normal bundle $Q_L = TL^{\perp}$.
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$\Lambda \subset M$ is ANR, so there is open neighborhood $\Lambda \subset W \subset M$ and extension $\sigma_W \colon W \to G_m(TM)$.

Choose smooth approximation to σ_W, obtain smooth subbundle $Q \subset TM$, $\pi_W \colon Q \to W$, transverse to $T\mathcal{F}$.

Thom classes

Φ_W denotes a Thom form on $\pi_W \colon Q_W \to W$ with support contained in the unit disk subbundle Q^1_W.

- Φ_W is closed q-form with fiberwise compact support
- $\int_{Q_x} \Phi_W = 1$ for each fiber $Q_x = \pi_W^{-1}(x)$
- *Integration over the fiber* map, for $\Omega^p_{\pi,c}(Q_W)$ the space of p-forms on Q_W with fiberwise compact supports,

$$\int_\pi \; : \; \Omega^p_{\pi,c}(Q_W) \to \Omega^{p-q}(W)$$

Properties of fiberwise integration

- For $\phi \in \Omega^p(W)$ and $\widetilde{\psi} \in \Omega^p(Q_W)$ with bounded uniform norm,

$$\int_\pi \pi_W^* \phi \wedge \Phi_W = \phi \quad , \quad d \int_\pi \widetilde{\psi} \wedge \Phi_W = \int_\pi d\, \widetilde{\psi} \wedge \Phi_W$$

- Uniform norm estimate

$$\| \int_\pi \widetilde{\psi} \wedge \Phi_W \|_W \ \leq \ B_\Phi \cdot \|\widetilde{\psi}\|_{Q_W}$$

Renormalization

For $s > 0$, $\nu_s \colon Q_W \to Q_W$ is the fiberwise linear map defined by multiplication by s. Q_W^s denotes s-disk subbundle.

- ν_s maps Q_W^1 diffeomorphically to Q_W^s.
- Define $\Phi_W^s = \nu_{1/s}^*(\Phi_W)$, then $\nu_s^*(\Phi_W^s) = \Phi_W$.
- Φ_W^s is a smooth form on Q_W with support in Q_W^s,
- integral of Φ_W^s over each fiber of π_W equals 1.

Poincaré dual classes

$exp_M \colon TM \to M$ is geodesic exponential map.

$\exp^Q_W \colon Q \to M$ is restriction to $Q \subset TM$.

For compact subset $K \subset L$, there is push-forward map

$$\omega^s_K = \frac{1}{|K|} \cdot (\exp^Q_W)_*(\Phi^s_W) \in \Omega^q(M)$$

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Proposition: For each $0 < s < \epsilon_0$, and sequence of compact sets K_ℓ which are closed at infinity, then the following limit exists,

$$\lim_{\ell \to \infty} \omega^s_{K_\ell} = \omega^s$$

Poincaré dual to $[\mathcal{C}]$ where $\mathcal{C}$ is closed n-current defined by the K_ℓ.

Bounds

For a closed $(m-2q)$-form ψ on M, calculate

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Local embedding $\exp^Q_L : Q^s_L \to M$ is *recurrent*, so must estimate values of $\omega^s_{K_\ell} \wedge \omega^s_{K_\ell}$ where image overlaps. This estimate is most delicate, and is heart of extension of original result from C^1-foliations, to $C^{\infty,0}$-foliated spaces.

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Assumption of "no atoms" implies that tame estimates exist for integrals as $\ell \to \infty$, and the limit tends to 0.

Further applications

- Obstructions to existence of Axiom A diffeomorphisms

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- Higher-order intersection products for laminations dimensions at least 2, analogous to self-linking numbers for flows [Gambaudo and Ghys, Khesin, Kotschick and Vogel]