# Self-intersections of foliation cycles

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# Realizing homology classes

New, *improved* version of result with Yoshihito Mitsumatsu,

"The intersection product of transverse invariant measures", *Indiana Univ. Math. J.*, 1991.

Let $M$ be closed $m$-manifold.

$C$ a closed, oriented $n$ manifold for $0 < n < m$.

$\iota \colon C \to M$ defines class $[\iota C] \in H_n(M, \mathbb{Z})$.

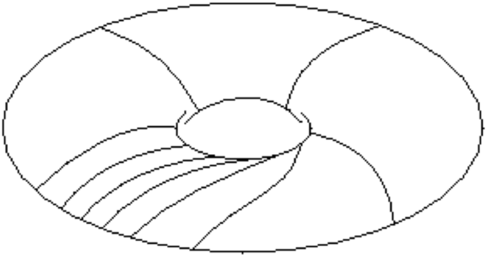
**Problem:** Which homology classes $x \in H_n(M; \mathbb{Z})$ can be realized by a geometric cycle $\iota \colon C \to M$?

Thom [1953,1954] solved this problem (up to torsion) in terms of bordism classes.

**Problem:** Which homology classes $[C] \in H_n(M;\mathbb{R})$ can be realized by a geometric cycle $\iota\colon C \to M$?

**Example:** Realize class $x = \alpha \cdot e_1 + \beta \cdot e_2 \in H_1(\mathbb{T}^2, \mathbb{R})$

For $\alpha, \beta \in \mathbb{R}$ construct foliation of $\mathbb{T}^2$ with slope $\lambda = \beta/\alpha$

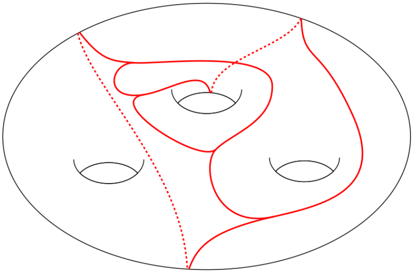


**Another Example:**

Lamination $\Lambda$ embedded in surface $\Sigma_g$ (carried by train track)

Leaf $L$ defines asymptotic class $[L] \in H_1(\Sigma_g; \mathbb{R}) \cong H_1(\mathbb{T}^{2g}, \mathbb{R})$

Non-trivial if branched cover of algebraic Anosov map of $\mathbb{T}^2$.



## Asymptotic cycles

$L$ a complete Riemannian $n$-manifold is "closed at infinity" if there is $x \in L$ and sequence $R_\ell \to \infty$ so that

$$\rho(L) = \lim_{\ell \to \infty} \frac{|\partial B(x, R_\ell)|}{|B(x, R_\ell)|} = 0$$

$B(x, R) = \{y \in L \mid d_L(x, y) \leq R)$

$\partial B(x, R) = \{y \in L \mid R - 1 \leq d_L(x, y) \leq R)$

$|X|$ denotes Riemannian volume of $X \subset L$.

## Asymptotic cycles

**Theorem:** Let $L$ be oriented, then $\mathcal{C} = \{F_\ell = B(x, R_\ell) \mid \ell = 1, 2, \ldots\}$ defines an *asymptotic fundamental class* for the bounded $n$-forms on $L$.

$$\langle [\mathcal{C}], \psi \rangle \;=\; \lim_{i \to \infty} \frac{1}{|F_{\ell_i}|} \cdot \int_{F_{\ell_i}} \psi$$

**Theorem:** Let $\iota \colon L \to M$ be embedding with bounded geometry, and assume $L$ is "closed at infinity" and oriented. Then it defines *asymptotic geometric cycle*, and homology class $[\iota\mathcal{C}] \in H_n(M, \mathbb{R})$.

**Problem:** Which homology classes $x \in H_n(M; \mathbb{R})$ can be realized by an asymptotic geometric cycle $\iota \colon \mathcal{C} \to M$?

# Foliated spaces

**Definition:** A *foliated space of dimension $n$* is a continuum $\Lambda$ with a partition $\mathcal{F}$ into leaves, such that there exists a compact separable metric space $\mathfrak{X}$, and for each $x \in \Lambda$ there is a compact subset $\mathfrak{T}_x \subset \mathfrak{X}$, an open subset $U_x \subset \Lambda$ with $x \in U_x$, and a homeomorphism defined on the closure $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ such that for each $y \in U_x$ the connected component of $\mathcal{F}|U_x$ containing $y$ is defined by $\varphi_x^{-1}((-1,1)^n \times w_y)$ for some $w_y \in \mathfrak{T}_x$.

$\{\Lambda, \mathcal{F}\}$ is *smoothly embedded* in $M$ if $\Lambda \subset M$, and for each $x \in \Lambda$, there exists a $C^{\infty,0}$-chart for $M$, $\psi_x \colon \overline{W}_x \to [-1,1]^m$ about $x$ which restricts to a foliation chart for $\Lambda$.

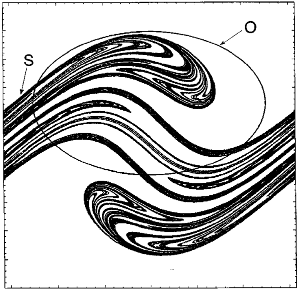
Codimension $q = m - n$.

# Foliated spaces

**Remark:** $L \subset \Lambda \subset M$ is embedded with bounded geometry. So $L$ "closed at infinity" yields asymptotic geometric cycle in $M$.

**Remark:** Embedded foliated spaces $\Lambda \subset M$ arise naturally as invariant (attractors) in differentiable dynamics. Examples include:

- Hyperbolic invariant set for Axiom A diffeomorphism $f : M \to M$.
- Orbit closures for diffeomorphism $f : M \to M$:



# Transverse invariant measures

A *transverse invariant measure* $\mu$ for a foliated space $(\Lambda, \mathcal{F})$ is a family of finite Borel measures $\{\mu_\alpha \mid \alpha \in \mathcal{A}\}$ defined on a family of transversals $\mathfrak{T}_\alpha \subset \Lambda$ to $\mathcal{F}$, such that for each holonomy map $h_{\beta,\alpha}$ from an open subset of $\mathfrak{T}_\alpha$ to an open subset of $\mathfrak{T}_\beta$ and Borel subset $E \subset \mathrm{Domain}(h_{\beta,\alpha})$ then,

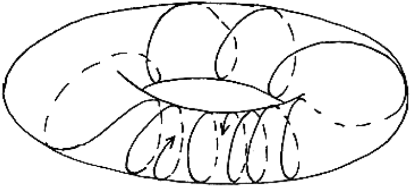
$$\mu_\beta(h_{\beta,\alpha}(E)) = \mu_\alpha(E)$$

**Theorem:** [Ruelle-Sullivan, Plante 1976] $L \subset \Lambda$ leaf which is "closed at infinity" yields transverse invariant measure $\mu$ for $(\Lambda, \mathcal{F})$.

# Atomics

**Definition:** A measure $\mu$ has *atoms* if there exists $x \in \mathfrak{T}_\alpha$ such that $\mu_\alpha(\{x\}) > 0$.

$\mathcal{C} = \{F_\ell = B(x, R_\ell) \subset L \mid \ell = 1, 2, \ldots\}$ defines $\mu$, then an atom for $\mu$ corresponds to a compact leaf $L_0 \subset \Lambda$, which is a type of Poincaré-Bendixson limit cycle for $\mathcal{F}$.



# Intersections of cycles

**Example:** Let $M = \mathbb{S}^2 \times \mathbb{S}^2$, then obtain classes

$$x = [\mathbb{S}^2 \times \{y_0\}] \in H_2(M;\mathbb{Z}) \ , \ y = [\{x_0\} \times \mathbb{S}^2] \in H_2(M;\mathbb{Z})$$

Their intersection product $x \cap y = [1] \in H_0(M, \mathbb{Z})$

$x \in H_n(M, \mathbb{Z})$ & $y \in H_q(M, \mathbb{Z})$ represented by geometric cycles

$$\iota_x : C_x \to M \quad , \quad \iota_y : C_y \to M$$

Then $x \cap y \in H_0(M;\mathbb{Z})$ can be calculated via counting "signed" points of intersection.

## Self-intersections of cycles

For $x \in H_n(M, \mathbb{Z})$ how to calculate $x \cap x \in H_{m-2q}(M; \mathbb{Z})$?

**Step 1:** Represent $x$ by geometric cycle $\iota_x \colon C_x \to M$

**Step 2:** Choose perturbation $\iota'_x \colon C_x \to M$

**Step 3:** Count intersection homology classes $\iota_x(C_x) \cap \iota'_x(C_x)$

Alternate approach: Construct closed form $\omega$ on $M$ of degree $q = m - n$ which is Poincaré dual to $[\iota_x C_x]$

$$[\iota_x C_x] \in H_n(M; \mathbb{Z}) \to H_n(M; \mathbb{R}) \cong H^q(M; \mathbb{R}) \cong H^q_{deR}(M)$$

Then $([\iota_x C_x] \cap [\iota_x C_x])^* = [\omega \wedge \omega] \in H^{2q}_{deR}(M) \cong H_{m-2q}(M; \mathbb{R})$.

# Main Theorem

**Theorem:** Let $\mathcal{F}$ be a $C^{\infty,0}$-foliation of a foliated space $\Lambda \subset M$ embedded in a closed oriented manifold $M$, such that the leaves of $\mathcal{F}$ are oriented, immersed $C^1$-submanifolds of $M$. Let $\mu$ be a transverse invariant measure for $\mathcal{F}$ without atoms. Let $C_\mu$ be the closed foliation $n$-current associated to $\mu$. Then the self-intersection product $[C_\mu] \cap [C_\mu] \in H_{m-2q}(M; \mathbb{R})$ vanishes.

More precisely, for the Poincaré dual closed $q$-form $\omega_\mu$ on $M$,

$$0 = [\omega_\mu \wedge \omega_\mu] \in H^{2q}_{deR}(M)$$

**Corollary:** If $x \in H_n(M; \mathbb{R})$ can be realized by an asymptotic geometric cycle $L \subset \Lambda \subset M$ with no atoms, then $x \cap x = 0$.

# Anosov diffeomorphisms

$f : M \to M$ is Anosov diffeomeorphism if there exists $\lambda > 1$, and

- $TM = E^- \oplus E^+$, $E^-$ dimension $n$, $E^+$ dimension $q$, $n + q = m$
- $E^\pm$ are invariant under the differential $Df$,
- $Df|E^+$ is uniformly expanding: $\|DF(\vec{X})\| \geq \lambda\|\vec{X}\|$, $\vec{X} \in E^+$
- $Df|E^-$ is uniformly contracting: $\|DF(\vec{X})\| \leq \lambda^{-1}\|\vec{X}\|$, $\vec{X} \in E^-$

The distributions $E^+$ and $E^-$ are uniquely integrable, giving foliations $\mathcal{F}^\pm$, whose leaves are smoothly immersed submanifolds with *polynomial growth rate*.

The foliations are $C^{\infty,0}$ - continuous, with smooth leaves - but rarely smooth unless the map $f$ is algebraic.

# Ruelle-Sullivan currents

Ruelle and Sullivan [1976] showed that the leaves $L^{\pm} \subset M$ for $\mathcal{F}^{\pm}$ define closed $n$-currents:

$$[C_{-}] \in H_n(M, \mathbb{R}) \quad \text{and} \quad [C_{+}] \in H_q(M, \mathbb{R})$$

with $[C_{-}] \cap [C_{+}] = 1.$ Thus both classes are non-zero.

The result holds more generally for Axiom A diffeomorphisms.

The leaves define invariant measures without atoms.

Main Theorem implies $[C_{\pm}] \cap [C_{\pm}] = 0$.

# Example

**Theorem:** [Kleptsyn & Kudryashov (2009)] $M = \mathbb{S}^2 \times \mathbb{S}^2$ admits no Axiom A diffeomorphism $f : M \to M$.

Note that $H_2(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, so there does exists a smooth map $f : M \to M$ whose action on homology is hyperbolic.

*Proof:* If exists, then $0 \neq [C_-] = \alpha \cdot e_1 + \beta \cdot e_2 \in H_2(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{R})$

$[C_-] \cap [C_-] = 0$ implies $\alpha = 0$, or $\beta = 0$.

If $[C_-] = \alpha \cdot e_1$ then $(f^\ell)_*[C_-] = \lambda^{-\ell} \cdot [C_-] \to 0$.

But $e_1$ is an integral class, so this is impossible.

Ditto for $[C_-] = \beta \cdot e_2$. $\square$

Shiraiwa [1973] proved using machinery of Axiom A dynamics.

# Two key ideas

**(1)** A finite Borel measure $\mu$ on $\mathfrak{T}$ has no atoms iff the diagonal $\Delta \subset \mathfrak{T} \times \mathfrak{T}$ has measure 0 for $\mu \times \mu$.

**(2)** For a closed $(m-2q)$-form $\psi$ on $M$, the pairing

$$\langle [\psi], [C_\mu] \cap C_\mu] \rangle \;=\; \int_M \psi \wedge \omega_\mu \wedge \omega_\mu$$

reduces to calculating the mass of the diagonal in $\displaystyle\bigcup_{\alpha \in \mathcal{A}} \mathfrak{T}_\alpha \times \mathfrak{T}_\alpha$

for the measure $\mu \times \mu$.

## Constructing the normal bundle

Each $L \subset \Lambda$ is smoothly embedded, with normal bundle $Q_L = TL^{\perp}$. Together these give $Q_\Lambda = T\mathcal{F}^{\perp} \subset TM$.

Let $G_m(TM)$ be the Grassmann bundle of $n$-dimensional subspaces of $TM$. Obtain $\sigma_\Lambda \colon \Lambda \to G_m(TM)$.

$\Lambda \subset M$ is $ANR$, so there is open neighborhood $\Lambda \subset W \subset M$ and extension $\sigma_W \colon W \to G_m(TM)$.

Choose smooth approximation to $\sigma_W$, obtain smooth subbundle $Q \subset TM$, $\pi_W \colon Q \to W$, transverse to $T\mathcal{F}$.

# Thom classes

$\Phi_W$ denotes a Thom form on $\pi_W \colon Q_W \to W$ with support contained in the unit disk subbundle $Q^1_W$.

- $\Phi_W$ is closed $q$-form with fiberwise compact support
- $\displaystyle\int_{Q_x} \Phi_W = 1$ for each fiber $Q_x = \pi_W^{-1}(x)$
- *Integration over the fiber* map, for $\Omega^p_{\pi,c}(Q_W)$ the space of $p$-forms on $Q_W$ with fiberwise compact supports,

$$\int_\pi \,:\, \Omega^p_{\pi,c}(Q_W) \to \Omega^{p-q}(W)$$

## Properties of fiberwise integration

- For $\phi \in \Omega^p(W)$ and $\widetilde{\psi} \in \Omega^p(Q_W)$ with bounded uniform norm,

$$\int_\pi \pi_W^* \phi \wedge \Phi_W = \phi \quad , \quad d \int_\pi \widetilde{\psi} \wedge \Phi_W = \int_\pi d\, \widetilde{\psi} \wedge \Phi_W$$

- Uniform norm estimate

$$\| \int_\pi \widetilde{\psi} \wedge \Phi_W \|_W \ \leq \ B_\Phi \cdot \|\widetilde{\psi}\|_{Q_W}$$

# Renormalization

For $s > 0$, $\nu_s \colon Q_W \to Q_W$ is the fiberwise linear map defined by multiplication by $s$. $Q_W^s$ denotes $s$-disk subbundle.

- $\nu_s$ maps $Q_W^1$ diffeomorphically to $Q_W^s$.
- Define $\Phi_W^s = \nu_{1/s}^*(\Phi_W)$, then $\nu_s^*(\Phi_W^s) = \Phi_W$.
- $\Phi_W^s$ is a smooth form on $Q_W$ with support in $Q_W^s$,
- integral of $\Phi_W^s$ over each fiber of $\pi_W$ equals 1.

# Poincaré dual classes

$exp_M \colon TM \to M$ is geodesic exponential map.

$\exp_W^Q \colon Q \to M$ is restriction to $Q \subset TM$.

For compact subset $K \subset L$, there is push-forward map

$$\omega_K^s = \frac{1}{|K|} \cdot (\exp_W^Q)_*(\Phi_W^s) \in \Omega^q(M)$$

**Proposition:** For each $0 < s < \epsilon_0$, and sequence of compact sets $K_\ell$ which are closed at infinity, then the following limit exists,

$$\lim_{\ell \to \infty} \omega_{K_\ell}^s = \omega^s$$

Poincaré dual to $[\mathcal{C}]$ where $\mathcal{C}$ is closed $n$-current defined by the $K_\ell$.

# Bounds

For a closed $(m-2q)$-form $\psi$ on $M$, calculate

$$\int_M \psi \wedge \omega^s_{K_\ell} \wedge \omega^s_{K_\ell}$$

Use exponential coordinates given by $\exp^Q_L \colon Q^s_L \to M$ to reduce to an integral over normal bundle $Q^s_L \to L$ if all $K_\ell \subset L$.

Local embedding $\exp^Q_L \colon Q^s_L \to M$ is *recurrent*, so must estimate values of $\omega^s_{K_\ell} \wedge \omega^s_{K_\ell}$ where image overlaps. This estimate is most delicate, and is heart of extension of original result from $C^1$-foliations, to $C^{\infty,0}$-foliated spaces.

Assumption of "no atoms" implies that tame estimates exist for integrals as $\ell \to \infty$, and the limit tends to 0.

# Further applications

- Obstructions to existence of Axiom A diffeomorphisms

- Higher-order intersection products for laminations dimensions at least 2, analogous to self-linking numbers for flows [Gambaudo and Ghys, Khesin, Kotschick and Vogel]