Coarse entropy and transverse dimension

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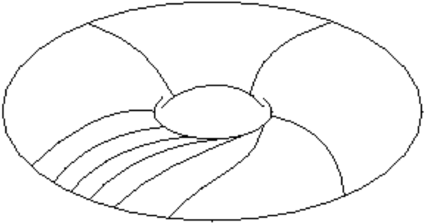
Foliations

William Thurston colloquially compared a foliation to the stripes on a zebra. In Chicago, we have a better comparison:

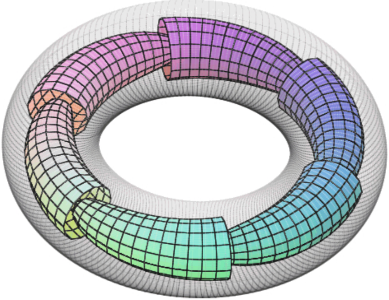
Foly Cow!

Wikipedia: A foliation is a kind of clothing worn on a manifold, cut from a stripy fabric. On each sufficiently small piece of the manifold, these stripes give the manifold a local product structure. This product structure does not have to be consistent outside local patches: a stripe followed around long enough might return to a different, nearby stripe.



A third definition is to give some of the basic examples, such as the

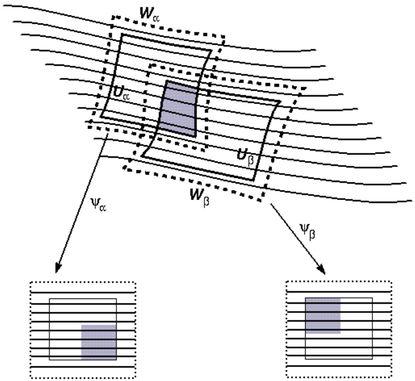
Reeb foliation



Historical paper by André Haefliger, available at *foliations.org*,

Naissance des feuilletages, d'Ehresmann-Reeb à Novikov

Definition: A foliation $\mathcal{F}$ of a manifold M is a "uniform partition" of M into submanifolds of constant dimension p and codimension q. More precisely, a smooth manifold of dimension n is *foliated* if there is a covering of M by coordinate charts whose change of coordinate functions preserve the horizontal level sets.



A leaf L of a foliation is formed by taking one plaque in a coordinate chart, then taking the increasing union of all plaques that successively intersect the previous collection.

A foliation $\mathcal{F}$ of a compact manifold M is also . . .

• a local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle for a "good covering". (Ehresmann, Haefliger)

• a dynamical system on M with multi-dimensional time.

• a groupoid $\mathcal{G}_{\mathcal{F}} \to M$ with fibers complete manifolds, the holonomy covers of leaves.

Each point of view has advantages and disadvantages.

Question: Given a connected open manifold L without boundary, is there a foliation $\mathcal{F}$ of a compact manifold M for which L is homeomorphic to a leaf of $\mathcal{F}$?

The geometry and topology of a leaf L is determined by the combinatorics of the intersections of the plaques in a finite covering of M by foliation charts.

The question above is analogous to asking about the topological and geometrical properties of finitely generated groups.

For foliations, the possibilities are much more extensive!

Example: Let Γ be the fundamental group of a compact connected manifold X. Suppose that Γ has an effective action on a compact manifold Y. This gives an injective representation $\rho\colon \Gamma \to \mathrm{Homeo}(Y)$. For the universal covering $L = \widetilde{X} \to X$, form

$$M_\Gamma = \{\widetilde{X} \times Y\}/(x, y) \sim (x \cdot \gamma^{-1}, \rho(\gamma) \cdot y)$$

Then M_Γ has a foliation with leaf L. The geometry and the topology of the leaf L thus is closely related to that of Γ.

The construction requires a finitely generated group Γ and a faithful representation into $\mathrm{Homeo}(Y)$. For example, if $Y = \mathbb{S}^1$, then this places strong restrictions on Γ.

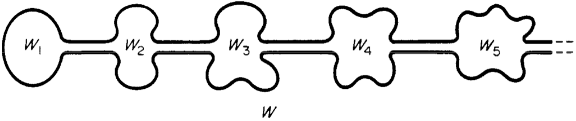
Theorem: [John Cantwell & Lawrence Conlon, Topology 1987]

Let L be an open, connected surface without boundary. Then L is diffeomorphic to a leaf of a smooth foliation $\mathcal{F}$ of a compact 3-manifold M.

Proof: The manifold L is homeomorphic to $\mathbb{R}^2 - K$ where K is a compact, totally disconnected set. They then construct a foliation with a leaf whose ends are homeomorphic to K.

Theorem: [Étienne Ghys, Topology 1985]

For all $d \geq 3$, there exists a non-compact manifold W of dimension d, which is not homeomorphic to a leaf of any foliation of *codimension-one* of a compact manifold M.



Proof: If such a foliation $\mathcal{F}$ exists, then M compact implies the end of W must be recurrent on itself. Choose the connected sum components W_i cleverly: choose an increasing list of primes $\{p_i \mid i = 1, 2, \ldots\}$, let $W_i = \mathbb{S}^3/(p_i \cdot \mathbb{Z})$ be the lens space with fundamental group $\mathbb{Z}/(p_i \cdot \mathbb{Z})$. Reeb Stability implies that in a codimension one manifold, each compact region W_i has a product neighborhood, so W cannot be end recurrent.

That's it, for what is known about non-leaves.

Consider the geometric version of this question.

Let (X, d_X) and (Y, d_Y) be metric spaces.

A homeomorphism $f : X \to Y$ is said to be a *quasi-isometry* if there exists constants $\lambda \geq 1$ and $C > 0$ so that for all $x, x' \in X$

$$\lambda^{-1} \cdot d_X(x, x') - C \leq d_Y(f(y), f(y')) \leq \lambda \cdot d_X(x, x') + C$$

Proposition: [Plante, Annals of Math 1976] Let L be a leaf of a foliation $\mathcal{F}$ of a compact manifold M. Then L has a complete Riemannian metric, unique up to quasi-isometry, with $C = 0$.

In his 1974 ICM address, Dennis Sullivan asked:

Question: Let L be a complete Riemannian smooth manifold without boundary. When is L *quasi-isometric* to a leaf of a C^r-foliation $\mathcal{F}_M$ of a compact smooth manifold M, for $r \geq 1$?

To answer this, you need some property of complete Riemannian manifolds which is an invariant of quasi-isometry, and which distinguishes when the manifold is a leaf.

The rate of volume growth for L – polynomial, subexponential, or exponential – is a quasi-invariant property of L. There exists foliations for which all of these types of growth occur.

There have been two types of obstructions, to date:

- For leaves with subexponential growth, the *average Euler class*

[Phillips & Sullivan, Topology 1981]

and the *average Pontrjagin classes*

[Januszkiewicz, Topology 1984]

- The *coarse entropy* of a complete Riemannian manifold

[Attie & Hurder, GTDS Seminar 1995, Topology 1996]

In this talk, we recall the definition of coarse entropy, and relate it to recent work of Olga Lukina.

We begin with a discussion of graph spaces.

Let $\mathcal{G}$ be a metric graph of finite type k. That is, there is a countable set of vertices $V(\mathcal{G})$ and edges $E(\mathcal{G})$ such that:

- each edge $e \in E(\mathcal{G})$ connects to two vertices, $\partial^+ e, \partial^- e \in V(\mathcal{G})$;
- each vertex $v \in V(\mathcal{G})$ is connected to at least one edge;
- each vertex $v \in V(\mathcal{G})$ is connected to no more than k edges;
- each edge has length 1.

The space $\mathcal{G}$ is given the path length metric, denoted $d_{\mathcal{G}}$.

Denote the closed ball by $B_{\mathcal{G}}(v, R) = \{x \in \mathcal{G} \mid d_{\mathcal{G}}(v, x) \leq R\}$.

Definition: For $R > 0$, an *R-quasi-tiling* of $\mathcal{G}$ consists of:

- a collection of vertices $\{v_1, \ldots, v_\mu\}$
- a countable set of isometries $\{f_i \colon B_{\mathcal{G}}(v_{\ell_i}, R) \to \mathcal{G} \mid i \in \mathcal{I}\}$
- so that the union of their images equals $\mathcal{G}$.

Let $H(\mathcal{G}, d_{\mathcal{G}}, R)$ denote the least number μ of vertices in an R-quasi-tiling of $\mathcal{G}$.

If no R-quasi-tiling exists, set $H(\mathcal{G}, d_{\mathcal{G}}, R) = \infty$.

Example: Let $\mathcal{G}$ be the Cayley graph of a finitely presented group Γ. Then $H(\mathcal{G}, d_{\mathcal{G}}, R) = 1$.

Consider $B_{\mathcal{G}}(v, R)$ as an open neighborhood of the graph with base vertex v_i in the box metric on pointed graphs.

If $B_{\mathcal{G}}(v', R)$ is the ball around another vertex v' which is isometric to $B_{\mathcal{G}}(v, R)$, then the pointed graph $(\mathcal{G}, v')$ is considered to be at most e^{-R} distance from $(\mathcal{G}, v)$.

The number $H(\mathcal{G}, d_{\mathcal{G}}, R)$ counts the number pointed trees which are e^{-R}-distinct up to isometry in this box metric.

Let $\#_v(S)$ be the number of vertices in a subgraph $S \subset \mathcal{G}$.

Example: Let $\mathcal{G}(\mathbb{F}^2)$ be the tree for the free group $\mathbb{F}^2 = \mathbb{Z} * \mathbb{Z}$. Then $\#_v(B_{\mathcal{G}}(v_0, R))$ equals the number of words in $\mathbb{F}^2$ with length at most R, hence is an exponential function of R.

Fix a base vertex $v_0 \in V(\mathcal{G})$.

Define the *pattern entropy* of the graph $(\mathcal{G}, d_{\mathcal{G}})$

$$h(\mathcal{G}, d_{\mathcal{G}}) = \limsup_{R \to \infty} \frac{\ln\{H(\mathcal{G}, d_{\mathcal{G}}, R)\}}{\#_v(B_{\mathcal{G}}(v_0, R))}$$

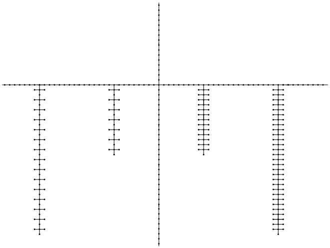
Proposition: The property $h(\mathcal{G}, d_{\mathcal{G}}) > 0$ is well-defined.

In Lukina's work, the sets $B_{\mathcal{G}}(v, R)$ are identified if they agree up to translation by the (isometric) group action.

In the above definition, the sets $B_{\mathcal{G}}(v, R)$ are identified is they are simply isometric. The number of sets needed to cover may be less.

The invariant $H(\mathcal{G}, d_{\mathcal{G}}, R)$ gives an uper bound estimate on the *box dimension* of the space of graphs, which gives an upper bound for its Hausdorff dimension.

With pseudogroup action:



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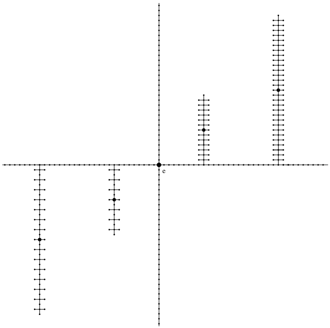
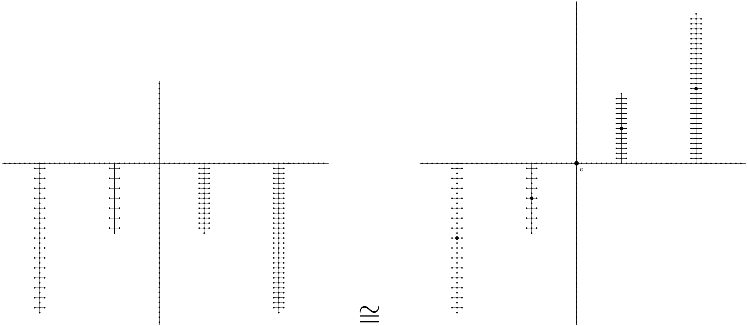


Without pseudogroup action



Let M be a compact Riemannian manifold, and $\mathcal{F}$ a foliation.

Fix a Riemannian metric on TM, with metric d_M

Given a leaf $L \subset M$, the inclusion induces a "leafwise" Riemannian metric whose associated path metric d_L is complete.

Example: Let $\mathcal{F}$ be the foliation defined by the flow of irrational slope on $\mathbb{T}^2$. Then each leaf is isometric to the Euclidean line $\mathbb{R}$, so has unbounded diameter, while $\mathbb{T}^2$ has bounded diameter.

By Plante, the quasi-isometry class of the metric d_L on L is unique.

We want to extend the notion of pattern entropy for graphs, to a type of pattern entropy for leaves.

For leaves of foliations, there is no reason why the corresponding notion of pattern entropy, defined analogously to the above, should be independent of the choices of Riemannian metric on M and covering of M by foliation charts, or even why the invariant should be finite at all.

The solution is to "coarsify" the pattern entropy.

That is, we allow a controlled amount of distortion in our tiling patterns, and then let this coarsening tend to infinity.

Let (X, d_X) and (Y, d_Y) be metric spaces.

For $\epsilon > 0$ we say that a subset $Z \subset Y$ is ϵ-dense, if for every $y \in Y$ there exists $z \in Z$ with $d_Y(y, z) < \epsilon$.

A set map $f : X \to Y$ is said to be a λ-*coarse isometry* if, for all $x, x' \in X$,

$$\lambda^{-1} \cdot d_X(x, x') - \lambda \leq d_Y(f(y), f(y')) \leq \lambda \cdot d_X(x, x') + \lambda$$

and the image $f(X) \subset Y$ is $\lambda > 0$ dense.

$f : X \to Y$ is *coarse isometry* if it is λ-coarse isometry for some λ.

Example: The inclusion $\mathbb{Z}^n \subset \mathbb{R}^n$ is a λ-coarse isometry for $\lambda \geq 1$.

Example: Ghys "non-leaf" W is coarse-isometric to $\mathbb{N}$.

A quasi-isometry is a coarse isometry, and a composition of coarse isometries is again a coarse isometry.

Question: Let $\mathcal{F}$ be foliation of a compact manifold M. What can be said about the set of coarse isometry classes of leaves if $\mathcal{F}$?

Question: If the set of coarse isometry classes of leaves of $\mathcal{F}$ is finite, what can be said about its topology/dynamics/geometry?

Definition: Fix $\lambda \geq 0, R > 0$. An (λ, R)-coarse-tiling of a complete Riemannian manifold L consists of:

- a collection of points $\{x_1, \ldots, x_\mu\} \subset L$
- a countable set of λ-coarse isometries
$\{f_i \colon B_L(x_{\ell_i}, R) \to L \mid i \in \mathcal{I}\}$
- such that the union of their images is λ-dense in L.

Let $H_c(L, d_L, \lambda, R)$ denote the least number μ of points in an (λ, R)-coarse-tiling for L.

If no (λ, R)-coarse-tiling for L exists, set $H_c(L, d_L, \lambda, R) = \infty$.

Define the λ-*coarse entropy*

$$h_\lambda(L, d_L) = \limsup_{R\to\infty} \frac{\ln\{H_c(L, d_L, \lambda, R)\}}{\mathrm{Vol}(B_L(x_0, R))}$$

and the *coarse entropy*

$$h_c(L, d_L) = \limsup_{\lambda\to\infty} \; h_\lambda(L, d_L)$$

Lemma: $h_c(L, d_L)$ is a coarse invariant of the metric space (L, d_L).

Theorem: Let (L, d_L) be a simply-connected complete Riemannian manifold which is coarse-isometric to a leaf of a C^1-foliation of a compact manifold. Then $h_c(L, d_L) = 0$.

Hence, if $h_c(L, d_L) > 0$, then (L, d_L) is not coarse-isometric to a leaf of a C^1-foliation of a compact manifold.

Idea of proof: Reeb stability theorem implies that for any compact subset K of a leaf of $\mathcal{F}$ there is a product neighborhood whose diameter $\epsilon > 0$ can be made arbitrarily small. Fix $\epsilon > 0$ small, so that all compact leaves in such a neighborhood are λ-coarse isometric, for some λ depending only on the geometry of $\mathcal{F}$.

Given $R > 0$, cover M by a finite collection product neighborhoods whose core compact sets have the form $B_{\mathcal{F}}(x_i, R)$, for some $\{x_\ell \mid 1 \leq \ell \leq \mu\}$. The C^1-hypothesis on $\mathcal{F}$ implies that its holonomy maps are Lipschitz, and so by the properties of holonomy, the number of these neighborhoods required is estimated above by e^{-kR} for some k.

Thus, each leaf has at most exponential growth for the number of coarse isometry types as a function of R.

Explicit constructions of complete manifolds with $h(L, d_L) > 0$ typically use an inductive combinatorial process.

The simplest construction was given by Abdelghani Zeghib, while listening to my talk in Tokyo (1993). The talk described a more "sophisticated" construction, using topological invariance of the Pontrjagin classes (and joint with Oliver Attie).

See [Attie & Hurder, 1996] for the complicated construction.

See [Zeghib, 1994] for the elegant construction, obtained by attaching to $\mathbb{H}^2$, random sequences of "bubbles" with increasing diameters, attached at points of $\mathbb{H}^2$ tending to infinity.

Key observation is that large bubbles are coarsely flat, so are coarse invariants of the geometry.

A complete Riemannian manifold with $h(L, d_L) > 0$ must be *very* wild at infinity. An alternate way to "measure" this complexity is to discretize the geometry.

Let $\mathcal{N}_L \subset L$ be a net. Then the inclusion $(\mathcal{N}_L, d_L) \subset (L, d_L)$ is a coarse isometry.

For example, let $\mathcal{F}$ be a foliation of a compact Riemannian manifold M. Let $\mathcal{T} \subset M$ be the union of the transversals defined by a covering of M by foliation charts. Then for a leaf $L \subset M$, $\mathcal{N}_L = L \cap \mathcal{T}$ is the net defined by the choice of transversal.

Definition: The Cayley graph of a leaf $\mathcal{G}(L)$ has vertices $V(L) = \mathcal{N}_L$, and an edge between two vertices if their corresponding plaques in the coordinate charts overlap. See [Lozano-Rojo, 2006] for example. We declare all edges to have length 1 as before, and give $\mathcal{G}(L)$ the path length metric $d_{\mathcal{G}(L)}$.

Proposition: The inclusion $(\mathcal{N}_K, d_{\mathcal{G}(L)}) \subset (L, d_L)$ is a coarse isometry. Hence, $h_c(\mathcal{G}(L), d_{\mathcal{G}(L)}) = h_c(L, d_L)$.

For the Cayley graph $\mathcal{G}_L$ and $\lambda \geq 1$, define the scaled metric $d_{L,\lambda} = \lambda^{-1} \cdot d_L$. The Cantor set $\mathbb{K}_L$ of subtrees of this graph is given the induced scaled metric.

$(\mathbb{K}_L, d_{L,\lambda})$ is a form of "universal transversal" for the discretized "dynamics" of the space L.

Theorem: If $h_c(L, d_L) > 0$, then for λ sufficiently large, the box dimension of the metric space $\{\mathbb{K}_L, d_{L,\lambda}\}$ is infinite.

Question: How are the λ-coarse pattern counting functions $h_\lambda(L, d_L)$ related to the dynamical properties of $\mathcal{F}$? It is dominated by the geometric entropy function for the pseudogroup dynamics.

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