Cantor dynamics of renormalizable groups

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- A compact closed connected manifold M is non-co-Hopfian if it admits a *proper* self-covering map $\Pi \colon M \to M$.

$\mathbb{S}^1$ is non-co-Hopfian. Pick an integer $m > 1$ then $\Pi_m \colon \mathbb{S}^1 \to \mathbb{S}^1$.

Let N be a compact closed connected manifold, then $\mathbb{S}^1 \times N$ is non-co-Hopfian.

Formulate a condition of "irreducibility".

Let M be non-co-Hopfian and $\Pi\colon M \to M$ a proper self-covering.

Choose a basepoint $x_1 \in M$ and set $x_0 = \Pi(x_1)$. Then we have

$$\Pi_*\colon \pi_1(M, x_1) \to \pi_1(M, x_0)$$

Since $\pi_1(M, x_1) \cong \pi_1(M, x_0)$, the map Π_* defines a self-embedding $\varphi\colon \Gamma_0 \to \Gamma_0$, where $\Gamma_0 \equiv \pi_1(M, x_0)$. Note that:

- Γ_0 is finitely generated.
- $\varphi(\Gamma_0) \subset \Gamma_0$ has finite index.

Definition: Let Γ be a finitely generated group, then an inclusion $\varphi\colon \Gamma \to \Gamma$ with finite index image is called a renormalization of Γ.

Γ is said to be renormalizable if it admits a renormalization.

Γ is also called a finitely non-co-Hopfian group in some literature.

For $\Gamma = \mathbb{Z}^n$ a finite index inclusion $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is called a renormalization of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ in the percolation & physics literature.

Question: What groups are renormalizable?

Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization. Then we recursively define a descending chain of subgroups $\Gamma_{\ell+1} = \varphi(\Gamma_\ell)$ for $\ell \geq 0$, so

$$\Gamma \equiv \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

Let $\mathcal{G}_\varphi = \{\Gamma_\ell \equiv \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ be the descending chain of subgroups of finite index associated to φ, then

$$K(\varphi) = \bigcap_{\ell \geq 0} \Gamma_\ell$$

is called the kernel of the chain.

Problem: Let Γ be a renormalizable group with group chain $\mathcal{G}_\varphi$ whose kernel $K(\varphi)$ is a finite group. What can be said about Γ?

Theorem: [Van Limbeek 2018] Let Γ be a renormalizable group , and suppose that each subgroup $\Gamma_\ell = \varphi^\ell(\Gamma)$ is normal in Γ.

Then $\Gamma/K(\varphi)$ is abelian.

⋆ *Van Limbeek*, **Towers of regular self-covers and linear endomorphisms of tori**, Geom. Topol., 2018.

The assumption that the images $\varphi^\ell(\Gamma)$ are all normal is very strong, and not satisfied for many renormalizable groups.

Example: Heisenberg group

$$\Gamma \;=\; \left\{ [x,y,z] = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

- $p, q > 1$. Define $\varphi[x,y,z] = [px, qy, pqz]$
- $\Gamma_\ell = \{[p^\ell x, q^\ell y, p^\ell q^\ell z] \mid x,y,z \in \mathbb{Z}\}$
- Γ_ℓ is not normal in Γ.
- $K(\varphi)$ is trivial group.

Example: Expanding manifolds

Let M be a closed Riemannian manifold. A smooth map $f\colon M \to M$ is expanding if there exists some $\lambda > 1$ such that

$$\|df(\vec{v})\| \geq \lambda \|\vec{v}\| \quad \text{for all } x \in M \text{ and } \vec{V} \in T_x M$$

The map f must be a proper covering.

Theorem: [Franks 1968] If M admits an expanding map then $\Gamma_0 = \pi_1(M, x_0)$ has polynomial growth rate.

Idea of proof: For some $\lambda_{min} > 1$, the ℓ-th power of the expanding map expands the inner radius of disks in the universal covering $\widetilde{M}$ at a rate bounded below by λ_{min}^{ℓ}. For some $\lambda_{max} \geq \lambda_{min}$, it expands their volumes at a rate bounded above by $\lambda_{max}^{n\ell}$. Thus the growth rate is bounded by a polynomial of degree $n = \dim M$.

Theorem: [Gromov 1979] If Γ is a finitely generated group with polynomial growth rate, then Γ admits a nilpotent subgroup $\Lambda \subset \Gamma$ with finite index. i.e., Γ is virtually nilpotent.

The idea of the proof of this celebrated result is not so simple. Gromov shows that there is a representation of Γ into a linear group, and then invokes the “Tits alternative” for linear groups.

Let $f \colon M \to M$ be an expanding map. Choose $x_0 \in M$ and set $\Gamma_0 = \pi_1(M, x_0)$. Then $\varphi = f_* \colon \Gamma_0 \to \Gamma_0$ is a renormalization of Γ_0 with $K(\varphi)$ the trivial group.

Gromov's Theorem motivates the following:

Conjecture: Γ a finitely generated group with a renormalization φ such that $K(\varphi)$ is trivial. Then Γ_0 is virtually nilpotent.

In the work of Nekrashevych & Pete [2011] they called a chain $\mathcal{G} = \{\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ with $K(\varphi)$ finite a strong scale, and asked if this implies that Γ_0 must be virtually nilpotent?

We give a positive solution to a modified version of this conjecture:

$\star$ *Hurder, Lukina & Van Limbeek*, **Cantor dynamics of renormalizable groups**, arxiv:2002.01565

Idea is to replace the metric space rescaling methods of Gromov, with induced actions on Cantor spaces, and then use the structure theory for profinite groups to again reduce the problem to linear algebra.

Some generalities:

Let Γ be a finitely generated group.

Let $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$ be a group chain, where $\Gamma_0 = \Gamma$ and $\Gamma_{\ell+1} \subset \Gamma_\ell$ is a proper subgroup of finite index.

For each $\ell > 0$, $X_\ell = \Gamma/\Gamma_\ell$ is a finite set with left Γ-action.

Inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a surjection $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$ so

$$\mathfrak{X} \equiv \varprojlim \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell$$

is a Cantor space with left Γ-action.

Obtain minimal Γ-action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$.

A Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x,y) < \delta \implies d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \epsilon \quad \text{for all } g \in \Gamma.$$

For Hamming (ultrametric) metric on $\mathfrak{X}$, the action Φ is isometric:

- $(\mathfrak{X}, \Gamma, \Phi)$ is an equicontinuous Cantor action.

Our study of these actions began with the work

⋆ *Jessica Dyer*, **Dynamics of Equicontinuous Group Actions on Cantor Sets**, 2015 UIC PhD.

Three approaches to the subject:

- Equicontinuous Cantor actions

  Dynamical systems approach

- Group actions on rooted trees

  Geometric group theory approach

- Clopen subset chains for profinite groups

  Representations into finite groups

Topology of Cantor space $\mathfrak{X}$ is generated by clopen subsets: U is closed and open. Non-empty clopen $U \subset \mathfrak{X}$ is adapted if the return times to U form a subgroup:

$$\Gamma_U = \{g \in \Gamma \mid \Phi(g)(U) = U\} \subset \Gamma$$

Proposition: A Cantor action $\Phi\colon \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if and only if all orbits are finite for its action on the set

$$\mathcal{CO} = \{U \subset \mathfrak{X} \mid U \text{ is clopen}\}$$

Corollary: Given $x \in \mathfrak{X}$ and open $x \in V$, there is adapted U with $x \in U \subset V$.

- Action of Γ on coset space $X_U = \Gamma/\Gamma_U$ is a finite approximation to action Φ, so gives a linear model.

- Equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$, same as homomorphism $\Phi\colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$
- $\widehat{\Gamma} = \overline{\Phi(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ - closure in uniform topology
- $\widehat{\Gamma}$ is separable profinite group, compact and totally disconnected
- $\widehat{\Phi}\colon \widehat{\Gamma} \times \mathfrak{X} \to \mathfrak{X}$ is transitive equicontinuous action
- For $x \in \mathfrak{X}$, define $\mathcal{D}_x = \{\widehat{g} \in \widehat{\Gamma} \mid \widehat{\Phi}(\widehat{g})(x) = x\}$
- $\mathcal{D}_x$ is called the <u>discriminant</u> of the action Φ
- Isomorphism class of $\mathcal{D}_x$ is independent of choice of x and invariant of isomorphism of actions.

Dyer formula for $\mathcal{D}$

$$C_\ell \;=\; \bigcap_{g \in \Gamma} \; g \; \Gamma_\ell \; g^{-1} \;\subset \Gamma_\ell .$$

is the largest normal subgroup (the *core*) of Γ_ℓ.

For all $\ell \geq 1$, $C_{\ell+1} \subset C_\ell$ and C_ℓ has finite index in Γ.

Quotient group $Q_\ell = \Gamma / C_\ell$ with identity element $e_\ell \in Q_\ell$.

There are natural quotient maps $q_{\ell+1} \colon Q_{\ell+1} \to Q_\ell$.

Form the inverse limit Cantor group

$$\widehat{\Gamma}_\infty = \varprojlim \; \{q_{\ell+1} \colon Q_{\ell+1} \to Q_\ell \mid \ell \geq 0\} \; .$$

Theorem: [Dyer 2015] There is a natural isomorphism $\widehat{\tau}\colon \widehat{\Gamma} \to \widehat{\Gamma}_\infty$ which identifies the discriminant group $\mathcal{D}_x$ with the inverse limit

$$\mathcal{D}_x \cong \mathcal{D}_\infty = \varprojlim \, \{q_{\ell+1}\colon \Gamma_{\ell+1}/C_{\ell+1} \to \Gamma_\ell/C_\ell \mid \ell \geq 0\} \subset \widehat{\Gamma}_\infty \ .$$

For each $\ell > 0$, have subgroup $\quad D_\ell \equiv \Gamma_\ell/C_\ell \subset \Gamma/C_\ell \equiv Q_\ell$

Q_ℓ is a finite approximation to $\widehat{\Gamma}$

D_ℓ is the fiber of the map $Q_\ell \to X_\ell = \Gamma/\Gamma_\ell$

D_ℓ is totally not normal.

Quotient by D_ℓ breaks the symmetry of Q_ℓ

For an equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$

1. $\mathcal{D}$ can be a Cantor group for a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ when Γ is 3-dimensional Heisenberg group.
2. Every finite group and every separable profinite group can be realized as $\mathcal{D}$ for a Cantor action by a torsion-free, finite index subgroup of $\mathbf{SL}(n, \mathbb{Z})$, $n \geq 3$.

   ⋆ *Dyer, Hurder & Lukina*, **Molino theory for matchbox manifolds**, Pacific J. Math. 2017
3. $\mathcal{D}$ can be wide-ranging for arboreal representations of absolute Galois groups of number fields and function fields.

   ⋆ *Lukina*, **Arboreal Cantor actions**, Jour. L. M. S., 2019
   ⋆ *Lukina*, **Galois groups and Cantor actions**, arXiv:1809.08475

We apply the above ideas to our situation.

- Let $\varphi \colon \Gamma \to \Gamma$ be a renormalization.
- Let $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ be the associated group chain.
- Let $\Phi_\varphi \colon \Gamma \times X_\varphi \to X_\varphi$ be the associated Cantor action.
- Let $\mathcal{D}_\varphi \subset \mathbf{Homeo}(X_\varphi)$ be the isotropy at unique fixed-point x_φ

Proposition: There is a rescaling $\lambda_\varphi \colon X_\varphi \to X_\varphi$ whose image $U_1 = \lambda_\varphi(X_\varphi)$ is clopen subset of X_φ. Moreover, the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is conjugate to the restricted action $(U_1, \Gamma_{U_1}, \Phi_{U_1})$.

Idea of proof: φ induces a map of quotients $\overline{\varphi} \colon \Gamma/\Gamma_\ell \to \Gamma_1/\Gamma_{\ell+1}$. This induces the shift map $\lambda_\varphi \colon X_\varphi \to U_1 \subset X_\varphi$.

Heisenberg group actions continued:

$$\Gamma = \left\{ [x,y,z] = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

Example 1:

- Let $m > 1$. Define $\varphi_1[x,y,z] = [mx, my, m^2 z]$
- $\Gamma_\ell = \left\{ [m^\ell x, m^\ell y, m^{2\ell} z] \mid x,y,z \in \mathbb{Z} \right\}$
- $\mathcal{D}_{\varphi_1}$ is the trivial group.
- X_{φ_1} is a profinite Heisenberg group,
- Φ_{φ_1} is left multiplication by Γ.

Example 2:

- $p, q > 1$ distinct primes. Define $\varphi_2[x, y, z] = [px, qy, pqz]$
- $\Gamma_\ell = \{[p^\ell x, q^\ell y, p^\ell q^\ell z] \mid x, y, z \in \mathbb{Z}\}$
- $\mathcal{D}_{\varphi_2} \cong \widehat{\mathbb{Z}}_q \times \widehat{\mathbb{Z}}_p$.
- $X_{\varphi_2} \cong \{[x, y, z] \mid x \in \widehat{\mathbb{Z}}_p \ , \ y \in \widehat{\mathbb{Z}}_q \ , \ z \in \widehat{\mathbb{Z}}_{pq}\}$.
- Φ_{φ_2} is left multiplication by Γ.

So for p, q and p', q' distinct pairs of primes, the Cantor actions associated to the corresponding renormalizations are not conjugate.

They are not even continuously orbit equivalent.

Conclusion: Renormalizable groups give rise to a large class of inequivalent minimal equicontinuous Cantor actions.

We want to apply the Cantor action approach to the question whether a renormalizable group must be virtually nilpotent. This requires new techniques from the profinite literature.

First, there is a map induced on closures:

- $\widehat{\Phi}_\varphi \colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ action of closure $\widehat{\Gamma}_\varphi \subset \mathbf{Homeo}(X_\varphi)$.

Here is our main result:

Theorem: A renormalization map φ induces a contraction map on the closure, $\widehat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

This has many implications; there is an extensive literature of the structure of profinite groups with a contraction mapping, by:

⋆ Baumgartner, Caprace, Reid, Wesolek, Willis, Wilson

Here is the issue: the renormalization φ naturally induces a map $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \mathbf{Homeo}(U_1)$. We need to show that the maps in the image of $\widehat{\varphi}$ have unique extensions to $\mathbf{Homeo}(X_\varphi)$.

Take detour into regularity properties of action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$

- effective, or faithful, if $\Phi_0\colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$ has trivial kernel.
- free if for all $x \in \mathfrak{X}$ and $g \in \Gamma$, $g \cdot x = x$ implies that $g = e$
- isotropy group of $x \in \mathfrak{X}$ is $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$
- $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and isotropy set

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\, g \in \Gamma \ ,\ g \neq id \ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)$$

Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is topologically free if $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X} \implies \mathrm{Iso}(\Phi)$ has empty interior.

For Γ a countable group, this is a natural hypothesis to impose.

Definition: An action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$, where

- H is a topological group and
- $\mathfrak{X}$ is a Cantor space

is quasi-analytic if for each clopen set $U \subset \mathfrak{X}$, $g \in H$

- if $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For H a countable group, this is equivalent to topologically free.

Let G be the profinite completion of Γ.

G is the inverse limit over the partially ordered set of finite-index normal subgroups of Γ.

G acts on X_φ via the projection $G \to \widehat{\Gamma}_\varphi$.

Here is the key technical result:

Theorem: The action $\widehat{\Phi}_\varphi \colon G \times X_\varphi \to X_\varphi$ is quasi-analytic.

Idea of proof. Suppose $\widehat{h} \in G$ and $\widehat{\Phi}_\varphi(\widehat{h})$ fixes $U \subset X_\varphi$.

By conjugating $\widehat{h}$ we can assume $U_\ell \subset U$ for some $\ell > 0$.

By universality, φ extends to a map $\widehat{\varphi}\colon G \to G$ with open image.

Get a chain of clopen subgroups $\widehat{V}_\ell = \widehat{\varphi}(G)$ for $\ell \geq 0$.

Then $\widehat{V}_\ell$ projects to clopen set $\widehat{U}_\ell \subset \widehat{\Gamma}_\varphi$

$\widehat{U}_\ell$ is stabilizer of $U_\ell = \lambda_\varphi^\ell(X_\varphi)$. So for all $x \in X_\varphi$,

$$\widehat{\Phi}_\varphi(\widehat{h}) \circ \lambda_\varphi^\ell(x) = \lambda_\varphi^\ell(x)$$

By some technical wizardry, we use this relation to show that $\widehat{\Phi}_\varphi(\widehat{h})$ acts as the identity on all of X_φ.

This shows that the action $\widehat{\Phi}_\varphi\colon G \times X_\varphi \to X_\varphi$ is quasi-analytic.

Corollary: The action $\widehat{\Phi}_\varphi\colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ is quasi-analytic.

Main result then follows.

Summary: Given a renormalization $\varphi\colon \Gamma \to \Gamma$, there is

- a group chain $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$
- a Cantor action $\Phi_\varphi\colon X_\varphi \to X_\varphi$
- a contraction map $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

We then have a result from previous works with Lukina:

Theorem: $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$

This connects the invariants for Cantor actions, with invariants for contraction profinite groups.

The following result is based on results of

Udo Baumgartner & George Willis, and Colin Reid:

Theorem: Let φ be a renormalization of the finitely generated group Γ. Then there is an isomorphism with a semi-direct product

$$\widehat{\Gamma}_\varphi \cong \mathcal{N}_\varphi \rtimes \mathcal{D}_\varphi$$

$$\begin{aligned} \mathcal{N}_\varphi &= \{\widehat{g} \in \widehat{\Gamma}_\varphi \mid \lim_{\ell \to \infty} \varphi^\ell(\widehat{g}) = \widehat{e}\} \\ \mathcal{D}_\varphi &= \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi \end{aligned}$$

Moreover, the contraction factor $\mathcal{N}_\varphi$ is pro-nilpotent.

Theorem [HLvL2020]: Let φ be a renormalization of the finitely generated group Γ. Suppose that

$$K(\varphi) = \bigcap_{\ell > 0} \varphi^{\ell}(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_{\varphi} = \bigcap_{n > 0} \widehat{\varphi}_0^n(\widehat{\Gamma}_{\varphi}) \subset \widehat{\Gamma}_{\varphi}$$

are both finite groups, then

- Γ is virtually nilpotent,
- If both groups are trivial, then Γ is nilpotent.

One expects further properties of renormalizable groups can be obtained from applying results on contraction groups and scales of automorphisms of totally disconnected locally compact groups to the contraction map $\widehat{\varphi} \colon \widehat{\Gamma}_{\varphi} \to \widehat{\Gamma}_{\varphi}$ induced by a renormalization.

Basic Problem: Let φ be a renormalization of the finitely generated group Γ. If $K(\varphi)$ is finite, show that $\mathcal{D}_\varphi$ must be virtually nilpotent.

This is true in all examples calculated. Need better understanding of closed subgroups of profinite groups to prove it in general.

Future Work: Let Γ be a renormalizable group. Characterize the space of renormalizations of Γ.

This is related to questions about research into properties of open subgroups of profinite groups, and scale invariant groups of Benjamini, and Nekrashevych & Pete.

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