Foliations
Thurston Zebras to Cantor Geometries

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Franz Kamber, *Philippe Tondeur*, and also *Kuo-Tsai Chen* in 1977-79;

plus Riemannian geometry courses by

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A Foly Cow!

Wikipedia: A foliation is a kind of clothing worn on a manifold, cut from a stripy fabric. On each sufficiently small piece of the manifold, these stripes give the manifold a local product structure. This product structure does not have to be consistent outside local patches: a stripe followed around long enough might return to a different, nearby stripe.

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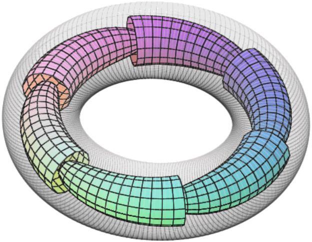
Forty years since the subject was energized by Thurston's works, 1971-75.

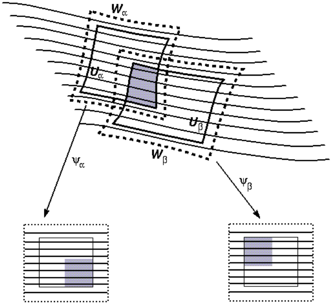
This was one theme at the meeting on *Geometry and Foliations 2013* last September in Tokyo: the contributions of Thurston and the subsequent developments of his ideas in the field. We ignored his famous comment, *that after his work [on foliations], all that was left to do was trivial*.

Historical paper by André Haefliger, available at *foliations.org*, *Naissance des feuilletages, d'Ehresmann-Reeb à Novikov*



Formal Definition

Definition: A foliation $\mathcal{F}$ of a manifold M is a "uniform partition" of M into submanifolds of constant dimension p and codimension q. More precisely, a smooth manifold of dimension n is *foliated* if there is a covering of M by coordinate charts whose change of coordinate functions preserve the leaves:



A foliation $\mathcal{F}$ of a compact manifold M is also ...

• a local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle for a "good covering". (Ehresmann, Haefliger)

• a dynamical system on M with multi-dimensional time.

• a groupoid $\mathcal{G}_{\mathcal{F}} \to M$ with fibers complete manifolds, the holonomy covers of leaves.

Each point of view has advantages and disadvantages.

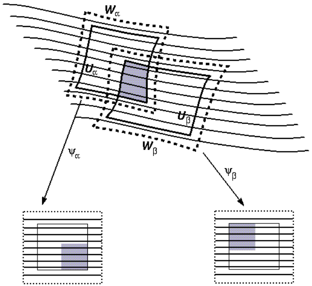
Problem: How to distinguish and classify foliations?

Convert to a discrete problem:

Pseudogroups

A section $\mathcal{T} \subset M$ for $\mathcal{F}$ is an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally.

Covering by coordinate charts provides local sections, whose union is a global transversal $\mathcal{T}$. The transverse holonomy along *leafwise paths* for $\mathcal{F}$ between points in $\mathcal{T}$ generates the pseudogroup $\mathcal{G}_{\mathcal{F}}$.



Groupoids

Definition: The groupoid of $\mathcal{G}_{\mathcal{F}}$ is the space of germs

$$\Gamma_{\mathcal{F}} = \{[g]_x \mid g \in \mathcal{G}_{\mathcal{F}} \ \& \ x \in D(g)\} \ , \ \mathcal{G}_{\mathcal{F}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

Definition: The *Haefliger groupoid* Γ_q^r is the collection of all germs of local C^r-diffeomorphisms between open subsets of $\mathbb{R}^q$, endowed with the sheaf topology. The objects of Γ_q^r are points of $\mathbb{R}^q$, and morphisms are germs of local C^r-diffeomorphisms of $\mathbb{R}^q$.

The derivative defines a groupoid transformation $\nu \colon \Gamma_q^r \to GL(\mathbb{R}^q)$.

This space is universal, it is non-Hausdorff, and it is simply huge!

Haefliger-Thurston classification

$B\Gamma_q^r =$ "classifying space" of (smooth) codimension q-foliations with transverse differentiability C^r, introduced by André Haefliger in 1970.

$B\Gamma_q^r \cong \|\Gamma_q^r\| \equiv$ the "semi-simplicial fat realization" of the groupoid Γ_q^r.

For $r \geq 1$, there is a natural map $B\nu \colon B\Gamma_q^r \to BGL(\mathbb{R}^q) \cong BO(\mathbb{R}^q)$ classifying the normal bundle to the universal foliation on $B\Gamma_q^r$.

Observation: [Haefliger] The foliation on $B\Gamma_q^r$ has a single leaf.

$\Rightarrow$ this is a really strange (non-Hausdorff) space.

- **Theorem:** (Haefliger) Each C^r-foliation $\mathcal{F}$ on M of codimension q determines a well-defined map $h_{\mathcal{F}}\colon M \to B\Gamma_q^r$ whose homotopy class in uniquely defined by $\mathcal{F}$.

Proof: Phillips Transversality, Gromov "h-Principle"

- **Theorem:** (Thurston) Each "natural" map $h_{\mathcal{F}}\colon M \to B\Gamma_q^r \times BO_p$ corresponds to a C^r-foliation $\mathcal{F}$ of codimension q on M, whose concordance class is determined by $h_{\mathcal{F}}$.

Proof: Draw lots of zebras; that is, very geometric at key steps...

Classification of $\mathcal{F}$ on M $\leftrightarrow$ calculate homotopy sets $[M, B\Gamma_q^r]$

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The Quest

Problem: Understand the homotopy type of the mysterious space $B\Gamma_q^r$!

Introduce $F\Gamma_q^r$, the homotopy fiber of the map $B\nu \colon B\Gamma_q^r \to BO(\mathbb{R}^q)$.

$F\Gamma_q^r$ is a topological measure of the difference between having a tangent subbundle $F \subset TM$ of rank p, and a C^r-foliation $\mathcal{F}$ on M with $F = T\mathcal{F}$.

Theorem: [Bott Vanishing Theorem, 1968]

$$B\nu^* \colon H^\ell(BO(\mathbb{R}^q); \mathbb{R}) \longrightarrow H^\ell(B\Gamma_q^r, \mathbb{R})$$

is the zero map for $r \geq 2$ and $\ell > 2q$.

$F\Gamma_q^r$ for $r \geq 2$ is definitely not trivial. In contrast, there is the amazing

Theorem: [Tsuboi, 1989] $F\Gamma_q^1$ has the weak homotopy type of a point.

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Godbillon-Vey class

Theorem: (Godbillon-Vey [1971]) For a C^2-foliation of codimension q, there is a cohomology class $GV(\mathcal{F}) = \Delta(h_1 \cdot c_1^q) \in H^{2q+1}(M;\mathbb{R})$, whose value depends only on the homotopy class of $h_{\mathcal{F}}\colon M \to B\Gamma_q^2$.

Proof:

- Assume $\mathcal{F}$ has oriented normal bundle $\Rightarrow$ there is a q-form ω with $ker(\omega) = T\mathcal{F}$.
- Froebenius Theorem implies the differential $d\omega = \eta \wedge \omega$ for 1-form η.
- Calculation shows $\eta \wedge (d\eta)^q$ is a closed form.
- $GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H^{2q+1}_{deR}(M) \cong H^{2q+1}(M;\mathbb{R})$.

- η is the *Reeb modular form* for $\mathcal{F}$.
- $h_1 = [\eta] \in H^1_{deR}(\mathcal{F})$ in leafwise cohomology is well-defined.
- $c_1 = [d\eta]$ represents the first *Chern class* of the normal bundle to $\mathcal{F}$.

Theorem: [Roussarie, 1971] The Godbillon-Vey class of the weak-stable foliation $\mathcal{F}$ on the unit sphere bundle M^3 for a Riemann surface Σ with curvature $\kappa = -1$ is non-zero.

Proof: $\mathcal{F}$ is also described as the foliation by cosets of the parabolic subgroup of upper triangular matrices in $SL(\mathbb{R}^2)$, acting on the unit tangent bundle viewed as the coset space $M = T^1\Sigma_g = \Gamma\backslash SL(\mathbb{R}^2)$.

- η is the contact form associated to the geodesic flow on M.
- Calculate $\eta \wedge d\eta$ is a multiple of the volume form.

Theorem: [Thurston, 1971] For each $\alpha \in \mathbb{R}$, there exists a foliation $\mathcal{F}_\alpha$ of codimension-one on $\mathbb{S}^3$ with $GV(\mathcal{F}_\alpha) = \alpha \in \mathbb{R} \cong H^3(M;\mathbb{R})$.

Proof: Glue foliations as above together, using clever a observation about flows on unit tangent bundles to Riemann surfaces defined by fundamental domains in $\mathbb{H}^2$ (the slopes on boundary tori can be controlled...)

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Secondary Classes

Theorem: (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972])

For each codimension q, and $r \geq 2$, there is a non-trivial space of secondary invariants $H^*(WO_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{array}{ccc} & & H^*(B\Gamma_q^r;\mathbb{R}) \\ & \tilde{\Delta} \nearrow & \downarrow h_{\mathcal{F}}^* \\ H^*(WO_q) & \stackrel{\Delta}{\longrightarrow} & H^*(M;\mathbb{R}) \end{array}$$

The image of Δ depends only on the homotopy class of $h_{\mathcal{F}} \colon M \to B\Gamma_q^2$.

Cartan approach of Kamber & Tondeur

- $\mathcal{F}$ determines a "partially flat connection" ∇ on the $GL(\mathbb{R}^q)$-bundle of frames orthogonal to the foliation tangent bundle $T\mathcal{F} \subset TM$.

- Construct a graded DGA, the"truncated Weil algebra" $W(gl(\mathbb{R}^q), O(q))_{2q}$ with $H^*(W(gl(\mathbb{R}^q), O(q))_{2q}) \cong H^*(WO_q)$.

- Calculate characteristic map $H^*(W(gl(\mathbb{R}^q), O(q))_{2q}) \to H^*(B\Gamma_q^2; \mathbb{R})$ using foliations defined by locally homogeneous spaces $M = \Gamma \backslash G$, where G is a semi-simple Lie group of higher rank, with foliation defined by right cosets of a Borel subgroup $P \subset G$.

- Key observation is that the non-triviality of the characteristic map follows from non-vanishing of the normal Euler class associated to the action of Γ on the compact quotient space $X = G/P$.

This became the model for almost all examples constructed in the 1970's of foliations which had non-trivial maps to $B\Gamma_q^2$.

Foliation dynamics

- A *continuous dynamical system* on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x,t) \mid t \in \mathbb{R}\}$ is thought of as the "time evolution" of the point $x \in M$. The trajectories of the points of M are necessarily *points*, *circles* or *lines* immersed in M, and the study of their aggregate and statistical behavior is the subject of foliation dynamics and ergodic theory for flows.

- *Foliation dynamics, ergodic theory* for foliations of leaf dimension ≥ 2. Replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the *dynamics* of $\mathcal{F}$ asks for properties of the limiting and statistical behavior of the collection of its leaves, and for ergodic theory look for measurable invariants.

Growth of leaves

An ergodic property of foliations: Riemannian metric on M yields complete Riemannian metrics on leaves.

For $x \in M$ and leaf L_x, consider leafwise balls of radius R,

$$B_{\mathcal{F}}(x, R) = \{y \in L_x \mid d_{\mathcal{F}}(x, y) \leq R\}$$

$$Gr(\mathcal{F}, x) = \limsup_{R \to \infty} \frac{\ln\{\mathrm{Vol}_{L_x}(B_{\mathcal{F}}(x, R))\}}{R} < \infty$$

$Gr(\mathcal{F}, x)$ is measurable orbit invariant.

L_x has *exponential growth* if $Gr(\mathcal{F}, x) > 0$.

Sullivan Conjecture:[1975] If $GV(\mathcal{F}) \neq 0$ then the set of leaves with exponential growth is non-empty.

Theorem: [Hurder, Jour. Diff. Geom. 1984]

If $GV(\mathcal{F}) \neq 0$ for $\mathcal{F}$ a codimension $q \geq 1$ foliation, then the set of leaves with exponential growth has positive Lebesgue measure.

Idea of Proof: Uses joint work with James Heitsch, inspired by work of Gerard Duminy, and *cocycle tempering* à la *Pesin Theory*.

The *Reeb modular class* $[\eta] \in H^1_{deR}(\mathcal{F})$ is an ergodic property of $\mathcal{F}$, and the operator norm of $[\eta]$ has estimate,

$$\|[\eta]\| \sim \int_M Gr(\mathcal{F}, x) \; dvol_M(x)$$

More refined ergodic invariants, use cocycles in spirit of Zimmer program.

Derivative map gives measurable cocycle $D\nu \colon \Gamma_{\mathcal{F}} \to GL(\mathbb{R}^n)$

For any parabolic subgroup $P \subset GL(\mathbb{R}^n)$ get an action on homogeneous space $X_P = GL(\mathbb{R}^n)/P$

Theorem:[Hurder & Katok, Annals of Math 1987]

- $[D\nu]_e \in H^1_{erg}(\Gamma_{\mathcal{F}}; X_P)$ is ergodic invariant
- Equivalence relation defined by $\mathcal{F}$ measurably amenable $\Rightarrow [D\nu]_e = 0$.
- If $[D\nu]_e = 0$ then many of the secondary classes vanish.

Conclusion:

$B\Gamma^2_q$ Foliation Classification $\Leftrightarrow$ Foliation Dynamics & Ergodic Properties.

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Foliation Dynamics & Ergodic Theory

Major theme of work in the field since 1980:

- Random walks on leaves, leafwise harmonic measures [Garnett, 1980] , [Ghys, 1995], [Candel, 2003], [Deroin & Klepsyn, 2007], [Deroin, Klepsyn & Navas, 2007]

- Foliation geometric entropy [Ghys, Langevin & Walczak, 1988]

- Lyapunov spectrum and foliation dynamics [Hurder, Walczak, 1988]

- Rigidity of Anosov systems [Hurder & Katok, 1990], [Hurder, 1992], [Katok, Spatzier, et al, 1993 onwards]

- Regularity of weak stable foliations [Hasselblatt, 1992 onwards], [Pugh, Shub &Wilkinson, 1997 onwards], etc

Mather-Thurston Theorem

$\mathrm{Diff}_c(\mathbb{R}^q)$ = compactly supported C^∞-diffeomorphisms of $\mathbb{R}^q$.

$\mathrm{Diff}_c^d(\mathbb{R}^q) = \mathrm{Diff}_c(\mathbb{R}^q)$ with discrete topology. HUGE!

Most amazing "unknown" theorem:

Theorem: [Mather & Thurston, 1974] There is a natural map $\sigma\colon \mathrm{Diff}_c^d(\mathbb{R}^q) \to \Omega^q F\Gamma_q^\infty$ which induces isomorphisms on homology

$$H^*(\Sigma^q \mathrm{Diff}_c^d(\mathbb{R}^q); \mathbb{Z}) \cong H^*(F\Gamma_q^\infty)$$

$F\Gamma_q^\infty$ is the homotopy fiber of the universal map $B\nu\colon B\Gamma_q^r \to BO(\mathbb{R}^q)$.

Question: What are the cycles in $H^*(\Sigma^q \mathrm{Diff}_c^d(\mathbb{R}^q); \mathbb{Z})$?

Propose to construct cycles using dynamical systems!

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Lipschitz Cantor Geometries

Only one "Cantor set", call it $\mathfrak{X}$, but has many metrics:

$d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are *Lipschitz equivalent*, if they satisfy a Lipschitz condition:

$$C^{-1} \cdot d_{\mathfrak{X}}(x,y) \ \leq \ d'_{\mathfrak{X}}(x,y) \ \leq \ C \cdot d_{\mathfrak{X}}(x,y) \quad \text{for all } x,y \in \mathfrak{X}$$

Definition: A compactly generated pseudogroup $\mathcal{G}_{\mathfrak{X}} = \langle g_1, \ldots, g_k \rangle$ acting on a Cantor set $\mathfrak{X}$ is *Lipschitz* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if there exists $C \geq 1$ such that for each $1 \leq i \leq k$ then

$$C^{-1} \cdot d_{\mathfrak{X}}(w,w') \leq d_{\mathfrak{X}}(h_i(w), h_i(w')) \leq C \cdot d_{\mathfrak{X}}(w,w') \ , \ \forall \, w, w' \in \mathrm{Dom}(g_i).$$

Definition: *Lipschitz Cantor Geometry* investigates properties of *minimal Lipschitz pseudogroups actions* on metric Cantor sets $(\mathfrak{X}, d_{\mathfrak{X}})$.

Work with coauthors Alex Clark and Olga Lukina during past 5 years has studied properties of *matchbox manifolds*, which are *foliated spaces* with transversal Cantor model $\mathfrak{X}$, and holonomy is pseudogroup action on $\mathfrak{X}$.

Definition: C^r *Cantor Geometry*, $r \geq 1$, investigates the properties of Cantor actions satisfying:

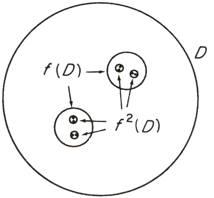
- $\mathfrak{X} \subset \mathbb{R}^q$ is an embedding,
- $\mathcal{G}_{\mathfrak{X}} = \langle g_1, \ldots, g_k \rangle$ is generated by restrictions of C^r-diffeomorphisms on some open neighborhood of $\mathfrak{X} \subset U \subset \mathbb{R}^q$.

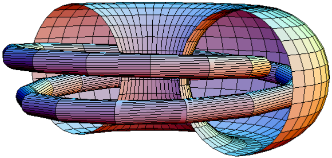
Why study these objects?

- Interesting in their own right – invariant sets for C^2-dynamical systems.

- For $r \geq 2$, examples which yield non-trivial classes in $H^*(F\Gamma_q^r; \mathbb{R})$, not in the image of the usual secondary characteristic maps.

Smoothly embedded solenoids





Cantor Set $\mathfrak{X}$ $\rightarrow$ *Solenoid in* $\mathbb{R}^3$

The holonomy along leaves of $\mathcal{F}$ "twist" to the points of $\mathfrak{X}$.

Pseudogroup maps on $\mathfrak{X}$ extend to open neighborhoods of $\mathfrak{X}$.

Solenoids naturally arise as hyperbolic attractors of smooth flows.
[Smale, 1967], [Gambaudo, Sullivan, Tresser, 1990 – onwards]

Theorem: [Clark & Hurder, 2011] For $p \geq 1$ and $q \geq 2n$, there exists commuting diffeomorphisms $\varphi_i \colon \mathbb{S}^q \to \mathbb{S}^q$, $1 \leq i \leq p$, so that the suspension of the induced action $\mathbb{Z}^p$ on $\mathbb{S}^q$ yields a smooth foliation $\mathcal{F}$ with solenoidal minimal set $\mathcal{S}$, such that:

- The leaves of $\mathcal{F}$ restricted to $\mathcal{S}$ are all isometric to $\mathbb{R}^p$
- The isotropy groups of periodic orbits form a profinite series

$$\cdots \Gamma_i \subset \cdots \Gamma_1 \subset \Gamma_0 = \mathbb{Z}^n$$

- $\mathfrak{X} \cong \varprojlim \, (\Gamma_0/\Gamma_i)$ is an "adic-completion" of $\mathbb{Z}^p$.
- Every open neighborhood of $\mathfrak{X}$ contains periodic open domains for the action of $\mathbb{Z}^p$ on $\mathbb{S}^q$

Question: What is relation to the quest?

Theorem: [Heitsch's Thesis, 1970]
The Bott Vanishing Theorem is false for $\mathbb{Z}$ coefficients.

Example: Let $(\mathbb{Z}/p\mathbb{Z})^m$ act on $\mathbb{D}^{2m}$ via rotations $\{\varphi_1, \ldots, \varphi_m\}$ with period p on each of the m-factors of $\mathbb{D}^2$.

Form the suspension flat bundle $\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi$. Then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \to H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \to H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Let $p \to \infty$, then observe associated $\lim_1$-terms not zero, so Pontrjagin classes do not vanish in $H^*(B\Gamma_q^2; \mathbb{Z})$.

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For any open neighborhood $Z \subset U$ of the solenoidal minimal set, there are infinite sequences of "Heitsch Examples" embedded in $\mathcal{F}|U$.

The restricted foliation $\mathcal{F}|U$ yields a map $h_Z \colon B\Gamma_{U|\mathcal{F}} \to B\Gamma_q^{\infty}$.

Theorem: Let $\mathcal{S}$ be the solenoidal minimal set above. Then the homotopy class of $h_Z \colon B\Gamma_{U|\mathcal{F}} \to B\Gamma_q^2$ is non-trivial.

Moreover, for $\ell > q/2$ there is a natural surjection

$$h_Z^* \colon H^{4\ell-1}(B\Gamma_q^2; \mathbb{R}) \to \mathcal{H}^{4\ell-1}(\mathcal{S}, \mathcal{F}; \mathbb{R})$$

Proof: The Cheeger-Simons classes derived from $H^*(BSO(q); \mathbb{R})$ are in the image of h_Z^* . . .

These solenoidal cycles are just the simplest type of iterated braids.

There are many more C^2 Cantor geometries...

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