Homeomorphisms of solenoids

Steve Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder

April 21, 2016

Dynamics of flows

Consider a non-singular C^1-vector field $\vec{V}$ on a compact n-manifold M, and its flow $\varphi_t \colon M \to M$ defined for all values of t.

Definition: A point $x \in M$ is *non-wandering* if its future and past orbits return infinitely often to a neighborhood of the point.

$\Omega \subset M$ denotes the set of all non-wandering points for the system.

A closed subset $\mathfrak{M} \subset M$ is *minimal* if it is flow invariant, and the orbit of every point in $\mathfrak{M}$ is dense in $\mathfrak{M}$.

Main Problems:

- Describe the topology of the non-wandering set Ω.
- Describe the dynamics of the flow on Ω.
- Describe the minimal sets of the flow, and their dynamics.

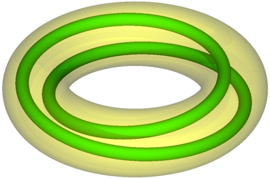
This program has been carried out for C^1-flows that satisfy Smale's *Axiom A* property, for example, and for *Morse-Smale* flows. Otherwise, simple questions such as what is the topology of the minimal sets for a flow are intractable.

Alternate approach: ask whether every flow is C^1-close to one of these standard types, as in the Palis Conjectures.

Example: a hyperbolic attractor

A *Smale solenoid* gives an example of a hyperbolic attractor:

Let $M = \mathbb{S}^1 \times \mathbb{D}^2$. The map $g : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{D}^2$ is defined by



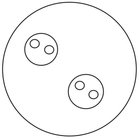
$$g(\theta, (y_1, y_2)) = \left(2\theta, \tfrac{1}{10} y_1 + \tfrac{1}{2}\cos\theta, \tfrac{1}{10} y_2 + \tfrac{1}{2}\sin\theta\right).$$

Let $C = \{0\} \times \mathbb{D}^2$ be a section.

Then $g^n(M) \cap C$ is the union of 2^n disjoint disks.

$\Lambda = \bigcap_{n\geq 1} g^n(M)$ is a hyperbolic attractor (called the *Smale* attractor): There are expanding ($\mathbb{S}^1$) and contracting ($\mathbb{D}^2$) directions.

$C \cap \Lambda$ is a Cantor set, and $(C \cap \Lambda, g)$ is a dynamical system on a totally disconnected set $C \cap \Lambda$.



Dynamics of foliations

Consider an integrable distribution $F \subset TM$ on a compact n-manifold M, and the foliation $\mathcal{F}$ whose leaves consist of the connected submanifolds of M which are tangent to F.

Definition: A point $x \in M$ is *non-wandering* if the leaf L_x through x returns infinitely often to a neighborhood of the point.

$\Omega \subset M$ denotes the set of all non-wandering points for the system.

The complement $M - \Omega$ consists of the *proper* leaves of $\mathcal{F}$.

A closed subset $\mathfrak{M} \subset M$ is *minimal* if it is a union of leaves, and each leaf $L \subset \mathfrak{M}$ is dense in $\mathfrak{M}$.

For example, a compact leaf is minimal.

Main Problems:

- Describe the topology of the non-wandering set Ω.
- Describe the dynamics of $\mathcal{F}$.
- Describe the minimal sets $\mathfrak{M}$ of $\mathcal{F}$ - what is their topology?
- Describe the dynamics of $\mathcal{F}|\mathfrak{M}$.

This program has been investigated in the case of C^2-foliations of codimension-one, in particular by Richard Sacksteder, John Cantwell & Larry Conlon, and Gilbert Hector.

Question: What hope is there to obtain meaningful results for C^r-foliations with codimension $q > 1$, and $r \geq 1$?

- S. Hurder, *Lectures on Foliation Dynamics*,

Foliations: Dynamics, Geometry and Topology,
Advanced Courses in Mathematics CRM Barcelona, Springer, 2014.

Look at special cases:

- Minimal sets with positive entropy
- Minimal sets with equicontinuous or distal dynamics
- Exceptional minimal sets

A minimal set $\mathfrak{M}$ is *exceptional* if its intersections with open transversals $\mathcal{T} \subset M$ to $\mathcal{F}$ have closures which are Cantor sets.

The dynamics of $\mathcal{F}|\mathfrak{M}$ is *equicontinuous* if for each transversal $\mathcal{T}$ to $\mathcal{F}$ has a metric such that the induced return map of the foliation to the set $\mathcal{T} \cap \mathfrak{M}$ is equicontinuous.

This is analogous to saying that $\mathcal{F}|\mathfrak{M}$ is a topological Riemannian foliation, except that when $\mathfrak{M}$ is exceptional, there is no normal bundle to $\mathcal{F}|\mathfrak{M}$.

Question: Can we classify the equicontinuous foliations on exceptional minimal sets?

Matchbox manifolds

A compact connected metrizable space $\mathfrak{M}$ is a n-dimensional *matchbox manifold* if it admits an atlas $\{U_i\}_{1 \leq i \leq \nu}$, where U_i is an open set equipped with a homeomorphism

$$\varphi_i : \overline{U}_i \to [-1, 1]^n \times X_i,$$

and $U_i = \varphi_i^{-1}((-1, 1)^n \times X_i$, where X_i is totally disconnected.

The term a *matchbox manifold* is due to **Aarts and Martens 1988**, who studied 1-dimensional matchbox manifolds (for $n = 1$).

Theorem: [Clark & Hurder, 2012] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then:

- There exists an open transversal $\mathcal{T}$ such that the induced return map on $\mathcal{T} \cap \mathfrak{M}$ is given by an equicontinuous action of a finitely generated group Γ on a Cantor set X.
- $\mathfrak{M}$ is foliated homeomorphic to a weak solenoid $\mathcal{S}$ which admits a Cantor fibration $\pi \colon \mathcal{S} \to M_0$ over a compact manifold M_0 with fiber X.

Theorem: [Clark, Hurder & Lukina, 2013] An equicontinuous $\mathfrak{M}$ is classified (up to foliated homeomorphism) by the induced holonomy action $\varphi \colon \Gamma \times X \to X$ on a transversal Cantor set X.

Problem: Understand the group of self-homeomorphisms of an n-dimensional matchbox manifold, and the maps between matchbox manifolds.

Solenoids

Let M_0 be a connected closed manifold, and, recursively, let $f^i_{i-1} : M_i \to M_{i-1}$ be a finite-to-one covering map. Then

$$M_\infty = \varprojlim \{f^i_{i-1} : M_i \to M_{i-1}\} = \{(y_0, y_1, y_2, \ldots) \mid f^i_{i-1}(y_i) = y_{i-1}\}$$

is a compact connected metrizable space called a *solenoid*.

Let $x_i \in M_i$, then the cylindrical clopen set (the fiber)

$$F_i = \{(f^i_0(x_i), \ldots, x_i, y_{i+1}, \ldots) \in M_\infty\} \text{ is a Cantor set } .$$

Theorem (McCord 1966): A solenoid is a matchbox manifold.

Example: Let $M_i = \mathbb{T}^n$, and let $f^i_{i-1} : \mathbb{T}^n \to \mathbb{T}^n$ be a proper covering. The inverse limit $\mathbb{T}_\infty$ is an abelian topological group.

Theorem: [Bowen & Franks 1976, Gambaudo & Tresser 1990] If $\mathcal{S}$ is a solenoid with base $\mathbb{S}^1$, and the degrees of the covering maps tend to ∞ sufficiently fast, then $\mathcal{S}$ is homeomorphic to the minimal set of a smooth flow on a 3-manifold.

Theorem: [Clark & Hurder 2010] If $\mathcal{S}$ is a solenoid with base $\mathbb{T}^n$, and the degrees of the covering maps tend to ∞ sufficiently fast, and are not too wild as matrices, then $\mathcal{S}$ is homeomorphic to the minimal set of a smooth foliation.

For other classes of solenoids, it remains an open problem, whether they can be realized as minimal sets of C^r-foliations for $r \geq 1$.

Problem: Study the properties of solenoids as equicontinuous dynamical systems.

Let $\pi\colon \mathcal{S} \to M_0$ be a weak solenoid defined by the system of maps $\{f_0^i\colon M_i \to M_0 \mid i > 0\}$. Choose a basepoint $x_0 \in M_0$, and basepoints $x \in M_i$ such that $f_0^i(x_i) = x_0$ then let

$$x = \lim x_i \in X = \pi^{-1}(x_0) \subset \mathcal{S} \ .$$

Define $G = G_0 = \pi_1(M_0, x_0)$, and let $G_i \subset G$ be the subgroup defined by $G_i = \mathrm{Image}\{(f_0^i)_\#\colon \pi_1(M_i, x_i) \to \pi_1(M_0, x_0)\}$.

The collection $\{G_i\}_{i \geq 0}$ forms a group chain in G.

Each $X_i = G_0/G_i$ is a finite set with a left action of G. It is a group if G_i is normal in G. The Cantor fiber of $\mathcal{S}$ is identified with

$$X \cong \varprojlim \ \{X_i \to X_{i-1}\} = \varprojlim \ \{G/G_i \to G/G_{i-1}\} = G_\infty$$

The left G-action $\Phi\colon G \to Aut(X)$ is minimal and equicontinuous. We say that (X, G) is an *equicontinuous minimal Cantor action*.

Homogeneous solenoids

A topological space $\mathfrak{M}$ is *homogeneous* if for every $x, y \in \mathfrak{M}$ there is a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$.

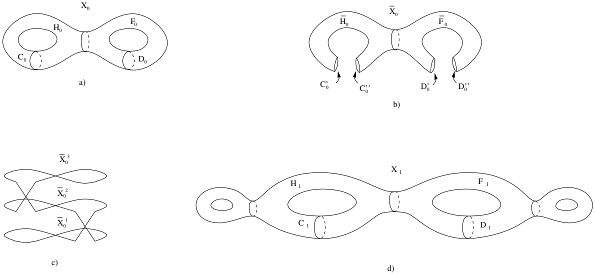
McCord 1965: If G_i is a normal subgroup of G for all $i \geq 0$, then G/G_i is a group, and G_∞ is a profinite group. Then M_∞ is homogeneous.

Example: Let $M_i = \mathbb{T}^n$, and let $f^i_{i-1} : \mathbb{T}^n \to \mathbb{T}^n$ be a proper covering. Then the inverse limit $\mathbb{T}_\infty$ is a homogeneous matchbox manifold.

Are there solenoids which are non-homogeneous?

Non-homogeneous solenoids

Schori 1966: Not all solenoids are homogeneous.



M_∞ is the inverse limit of 3-to-1 coverings of the genus 2 surface Σ_2. Every leaf in M_∞ is a non-compact surface of infinite genus. Schori found a closed loop in Σ_2, which, depending on the point in the fiber of $M_\infty \to \Sigma_2$, lifts either to a closed loop, or to a non-closed curve. So M_∞ is not homogeneous.

Rogers & Tollefson 1971: There exists a closed manifold M_0 and a group chain $\{H_i\}_{i\geq 0}$ with $H_0 = \pi_1(M_0, x_0)$ such that H_i is not a normal subgroup of H_0, but the associated solenoid M_∞ is a homogeneous space.

That is, it is possible to obtain a homogeneous solenoid as an inverse limit of non-regular coverings.

Question: How one can determine if a solenoid is homogeneous?

Equivalent group chains

Rogers and Tollefson suggested to use the following notion to study the problem of homogeneity.

Definition: (Equivalent group chains) Group chains $\{G_i\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ are *equivalent* if, possibly for a subsequence, one has

$$G_0 = H_0 \supset G_1 \supset H_1 \supset G_2 \supset H_2 \supset \cdots$$

Dynamically, $\{G_i\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ are equivalent if and only if there is a conjugacy $h \colon G_\infty \to H_\infty$ with $h(eG_i) = (eH_i)$.

If $\{G_i\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ are not equivalent, there may still exist a conjugacy $h \colon G_\infty \to H_\infty$, but it cannot preserve the basepoint.

A criterion for homogeneity

A group chain $\{H_i\}_{i\geq 0}$ is *weakly normal* if it is equivalent to a group chain $\{G_i\}_{i\geq 0}$ such that there is $i_0 \geq 0$ such that $i \geq i_0$

$$G_i \subset G_{i_0} \subset N(G_i), \ N(G_i) \text{ is the normalizer of } G_i \text{ in } G_0.$$

Theorem: (Fokkink and Oversteegen 2002) A solenoid $\mathcal{S}$ is homogeneous if and only if its associated group chain $\{H_i\}_{i\geq 0}$ is *weakly normal*.

Topologically, this means that, possibly restricting to clopen subset of the fiber, one can represent the solenoid as the inverse limit of a sequence of regular coverings of a closed manifold.

Ellis semigroup

Let $\Phi\colon G \to Homeo(X) : g \mapsto \phi_g$ be a group action.

Let $\phi(G) = \{\phi_g \mid g \in G\} \subset Maps(X, X)$.

Theorem: (Ellis 1969) The closure $\mathfrak{G} = \overline{\phi(G)} \subset Maps(X, X)$ in the topology of *pointwise convergence* on maps has a structure as a semigroup, called the *enveloping (Ellis) semigroup*.

Ellis 1969, see also **Auslander 1988**: If the action (X, G) is *equicontinuous*, then the Ellis semigroup $\mathfrak{G}$ is a group.

Note: The action (G_∞, G) is equicontinuous, so its Ellis semigroup is a group. Denote by

$$\mathfrak{G}_x = \{h \in \mathfrak{G} \mid h(x) = x\}$$

the isotropy subgroup of the $\mathfrak{G}$-action at $x \in X$.

A representation of the Ellis (semi)group

Remark: In general, it is quite difficult to compute the Ellis semigroup of an action (X, G). Let $\{G_i\}_{i\geq 0}$ be a group chain in G such that (X, G) is conjugate to (G_∞, G).

Let $C_i = \bigcap_{g\in G} gG_ig^{-1}$, then C_i is a normal subgroup of G, and

$$C_\infty = \varprojlim\{G/C_i \to G/C_{i-1}\} \text{ is a profinite group.}$$

Theorem: [Dyer, Hurder, Lukina 2015] The profinite group C_∞ is isomorphic to the Ellis group $\mathfrak{G}$ of the action (X, G). Also, if $\mathfrak{G}_x$ is the isotropy group of the $\mathfrak{G}$-action on X, then

$$\mathfrak{G}_x \cong \varprojlim\{G_i/C_i \to G_{i-1}/C_{i-1}\} \equiv \mathcal{D}_x.$$

$\mathcal{D}_x$ is called the *discriminant group* of the action.

A criterion for homogeneity

Since $\mathfrak{G}_x$ is a profinite group, it is either finite, or a Cantor group.

Proposition: [Dyer, Hurder, Lukina 2015] The isotropy group $\mathfrak{G}_x$ of the Ellis group action on X is a normal subgroup of $\mathfrak{G}_x$ if and only if $\mathfrak{G}_x$ is trivial.

Therefore, if the isotropy group $\mathfrak{G}_x$ is non-trivial, then $X \cong \mathfrak{G}/\mathfrak{G}_x$ does not admit group structure.

Corollary: Let $\mathfrak{M}$ be a solenoid with associated group chain $\{H_i\}_{i\geq 0}$. Then $\mathfrak{M}$ is homogeneous if and only if the isotropy group $\mathfrak{G}_x$ of the Ellis group action on X is trivial.

Advantages of the Ellis group approach

Computing the isotropy group $\mathfrak{G}_x$ has the following advantages as opposed to working with equivalence classes of group chains:

1. Unlike the normality properties of group chains, finiteness of the discriminant group $\mathcal{D}_x$ does not depend on the choice of a group chain.
2. In many examples, one can explicitly compute the quotients G_i/C_i and the discriminant group $\mathcal{D}_x$, while it may be quite difficult to show that there is no chain of normal subgroups equivalent to a given group chain $\{G_i\}_{i\geq 0}$.

Virtually homogeneous solenoids

We now have a criterion for a solenoid to be homogeneous, and we know that there are solenoids which are non-homogeneous.

Question: Are there solenoids which are 'less' non-homogeneous than other ones? That is, is there any way to quantify the degree of non-homogeneity?

We introduce the notion of a *virtually homogeneous solenoid*:

Definition: A solenoid $\mathfrak{M} \to M_0$ is *virtually homogeneous*, if there exists a finite-to-one covering $p\colon N_0 \to M_0$ such that the pullback solenoid $\mathfrak{N} = p^*\mathfrak{M}$ is homogeneous.

A criterion for virtual homogeneity

Recall that $\mathfrak{G} = \lim_{\longleftarrow}\{G/C_i \to G/C_{i-1}\}$ is the Ellis group of the action, and $\mathcal{D}_x$ is the discriminant group at x.

If $\mathcal{D}_x$ is non-trivial, it is either a finite group or a Cantor group.

Theorem: (Dyer, Hurder, Lukina 2015) If $\mathcal{D}_x$ is a finite group, then a solenoid $\mathfrak{M}$ with associated group chain $\{H_i\}_{i\geq 0}$ is virtually homogeneous.

Proof. One can show that $\mathcal{D}_x$ is finite if and only if $\{H_i\}_{i\geq 0}$ is equivalent to a chain $\{G_i\}_{i\geq 0}$ such that $H_i \cap C_1 = C_i$. Then use Galois theory to construct a covering of the base manifold.

The discrete Heisenberg group

Let $\mathcal{H} = (\mathbb{R}^3, *)$ be the Heisenberg group, and $\mathcal{H} = (\mathbb{Z}^3, *)$ be the discrete Heisenberg group, with

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy').$$

Then $M_0 = \mathbb{H}/\mathcal{H}$ is a closed 3-manifold, and $\pi_1(M_0, 0) = \mathcal{H}$.

Group chains in the Heisenberg group, giving homogeneous actions, were classified by **Lightwood, Sahin and Ugarcovici 2014**.

We are interested in group chains which give non-homogeneous actions.

A virtually homogeneous solenoid

Let p, q be distinct primes, and let $M_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix}$. Then $\{G_n\} = \{M_n\mathbb{Z}^2 \times p\mathbb{Z}\}$ defines a nested group chain. Let G_∞ be the corresponding inverse limit space. Let $M_n = \mathbb{H}/G_n$.

Proposition: The solenoid defined by the group action $(G_\infty, \mathbb{H})$ is virtually homogeneous, but not weakly normal.

Proof. Let $L_n = \begin{pmatrix} qp^n & p^2q^n \\ p^{n+1} & pq^{n+1} \end{pmatrix}$. Then $C_n = L_n\mathbb{Z}^2 \times p\mathbb{Z}$ is a normal subgroup of index p in G_n. Since p is a prime, C_n is a maximal normal subgroup of G_n. Then for $n \geq 1$ we have $|G_n/C_n| = p$, and it follows that $\mathcal{D}_x$ is nontrivial and finite.

Based on the works

- *J. Dyer, S. Hurder and O. Lukina*, The discriminant invariant of Cantor group actions, arXiv: 1509.06227.
- *J. Dyer, S. Hurder and O. Lukina*, Growth and homogeneity of matchbox manifolds, arXiv: 1602.00784.

Other works on equicontinuous and non-equicontinuous matchbox manifolds

- *A. Clark, R. Fokkink and O.Lukina*, The Schreier continuum and ends, Houston J. Math., 40:569-599, 2014.
- *A. Clark and S. Hurder*, Embedding solenoids in foliations, Topology Appl., 158:1249–1270, 2011..
- *A. Clark and S. Hurder*, Homogeneous matchbox manifolds, Trans. AMS, 365:3151–3191, 2013.
- *A. Clark, S. Hurder and O. Lukina*, Classifying matchbox manifolds, arXiv: 1311.0226.
- *A. Clark, S. Hurder and O. Lukina*, Shape of matchbox manifolds, Indagationes Mathematicae, 25:669–712, 2014.
- *A. Clark, S. Hurder and O. Lukina*, Voronoi tessellations of matchbox manifolds, Top. Proc. 41:167–259, 2013.
- *O. Lukina*, Hierarchy of graph matchbox manifolds, Topology Appl., 159:3461–3485, 2012.
- *O. Lukina*, Hausdorff dimension of matchbox manifolds, arXiv: 1407.0693.

¡Gracias por su atención!