Homeomorphisms of solenoidal manifolds

Steve Hurder

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University of Illinois at Chicago
www.math.uic.edu/∼hurder

A report on joint works with

Olga Lukina, Mathematical Institute, Leiden University

⋆ Clark & Hurder, *Homogeneous matchbox manifolds*, **Trans. A.M.S.**, 365, 2013.

⋆ Clark, Hurder & Lukina, *Classifying matchbox manifolds*, **Geom. & Top.**, 23, 2019

⋆ Dyer, Hurder & Lukina, *Molino theory for matchbox manifolds*, **Pac. J. Math.**, 289, 2017

⋆ Hurder & Lukina, *Wild solenoids*, **Trans. A.M.S.**, 371, 2019

⋆ Hurder, Lukina & van Limbeek, *Cantor dynamics of renormalizable groups*, **Groups, Geom., and Dynamics**, 15, 2021

⋆ Hurder & Lukina, *Type invariants for solenoidal manifolds*, preprint, 2023

A *continuum* is a compact, connected, non-empty metric space.

Examples include compact manifolds, finite CW complexes, laminations of compact manifolds, Hawaiian Earrings, etc

A topological space X is *homogeneous* if for every $x, y \in X$, there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$.

Theorem. [Bing, 1960] Let X be a homogeneous, circle-like continuum that contains an arc. Then either X is homeomorphic to a circle, or to a Vietoris solenoid.

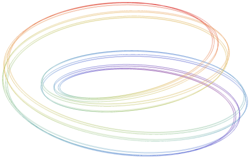
Vietoris solenoids

Choose a sequence of integers $\vec{m} = (m_1, m_2, \ldots)$ with $m_\ell > 1$.

Form the chain of coverings of the circle

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \cdots$$

$$\mathcal{S}(\vec{m}) = \varprojlim \, \{m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$



The *supernatural number*, or *Steinitz number*, defined by $\vec{m}$

$$\xi(\vec{m}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} \ ,$$

lcm denotes the least common multiple of the collection of integers. A Steinitz number ξ can be written uniquely as the formal product over the set of primes Π,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)}$$

The *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ.

Two Steinitz numbers ξ and ξ' are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$

The type associated to a Steinitz number ξ is the asymptotic equivalence class of ξ, denoted by $\tau[\xi]$.

Lemma. ξ and ξ' satisfy $\xi \overset{a}{\sim} \xi'$ if and only if their characteristic functions χ, χ' satisfy

- $\chi(p) = \chi'(p)$ for all but finitely many primes $p \in \Pi$
- $\chi(p) = \infty$ if and only iff $\chi'(p) = \infty$ for all primes $p \in \Pi$.

Theorem. [Bing, 1960] 1-dimensional solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$ are homeomorphic if $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$.

Theorem. [McCord, 1965] If 1-dimensional solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$ are homeomorphic then $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$.

Conclusion: A Vietoris solenoid is completely determined up to homeomorphism by the type $\tau[\xi(\vec{m})]$.

Exercise. Write down two strings $\vec{m}$ and $\vec{m'}$ with $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m'})$. Explicitly construct a homeomorphism between $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$.

Theorem. [Clark & H, 2013] Let X be a homogeneous continuum, and suppose that every for every $x \in X$, the connected component of a neighborhood of x is an n-disk. Then M is homeomorphic to a McCord solenoid of dimension n.

There is a *presentation* $\mathcal{P}$ consisting of:

- ⋆ M_ℓ is compact, closed, connected, n-dimensional manifold,
- ⋆ $p_{\ell+1} \colon M_{\ell+1} \to M_\ell$ is a *proper* covering map.

$$M \cong \mathcal{S}(\mathcal{P}) = \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell$$

M is called

- a *weak solenoid* (McCord) or
- a *solenoidal manifold* (Sullivan)

Questions:

(1) What is $\mathcal{S}(\mathcal{P})$, for a general covering sequence $\mathcal{P}$?

$\star$ D. Sullivan, *Solenoidal manifolds*, **J. Singul.**, 9:203–205, 2014.

$\star$ A. Verjovsky, *Low-dimensional solenoidal manifolds*, arXiv:2203.10032v2

(2) When are $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ homeomorphic?

(3) What is the structure of $\mathrm{Homeo}(\mathcal{S}(\mathcal{P}))$?

$q_\ell \colon M_\ell \to M_0$ is covering map, $m_\ell = \mathrm{degree}(q_\ell)$

$\xi(\mathcal{P}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\}$

Theorem. [H & Lukina, 2023] If solenoidal manifolds $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are homeomorphic then $\xi(\mathcal{P}) \overset{a}{\sim} \xi(\mathcal{P}')$.

Remark. For $n \geq 2$, the converse is false in so many ways, and opens the door to many questions

Need to take a "deep dive" into what homeomorphism between solenoidal manifolds means.

Projections $\widehat{p}_\ell \colon \mathcal{S}(\mathcal{P}) \to M_\ell$

$q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \to M_0$

Choose a basepoint $x \in \mathcal{S}(\mathcal{P})$, set $x_\ell = \widehat{p}_\ell(x) \in M_\ell$

Fundamental groups $\pi_1(M_\ell, x_\ell)$

$(q_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to \Gamma_\ell \subset \Gamma = \pi_1(M_0, x_0)$

Group chain $\mathcal{G}(\mathcal{P}) = \{\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$

- Each inclusion $\Gamma_\ell \subset \Gamma$ has finite index
- Γ_ℓ is not assumed to be normal in Γ
- Γ acts transitively on finite set $X_\ell = \Gamma/\Gamma_\ell$
- $C_\ell \subset \Gamma_\ell$ is kernel of action map $\Phi_\ell \colon \Gamma \to \mathrm{Aut}(X_\ell)$
- C_ℓ is normal in Γ
- $Q_\ell = \Gamma/C_\ell$ is finite group, acts on X_ℓ.

$$X_\infty \equiv \varprojlim \; \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \; .$$

$$\widehat{\Gamma}_\infty \equiv \varprojlim \; \{p_{\ell+1} \colon Q_{\ell+1} \to Q_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} Q_\ell \; .$$

The fiber $q_\ell^{-1}(x_0) \subset M_\ell$ is identified with $X_\ell = \Gamma/\Gamma_\ell$ as a Γ-space.

Action $\Phi \colon \Gamma \times X_\infty \to X_\infty$ is identified with monodromy of fibration $\widehat{p} \colon \mathcal{S}(\mathcal{P}) \to M_0$.

- X_∞ and $\widehat{\Gamma}_\infty$ are Cantor sets
- Action Φ is equicontinuous and minimal - a Cantor action.

Theorem. Suppose that $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are homeomorphic, then $\Phi\colon \Gamma \times X_\infty \to X_\infty$ and $\Phi'\colon \Gamma' \times X'_\infty \to X'_\infty$ are *return equivalent*.

Theorem. [Clark, H & Lukina, 2019] Suppose that $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are solenoidal manifolds of the same dimension, and their monodromy actions are return equivalent. If the base manifolds M_0 and M'_0 are *strongly Borel* (ie all finite coverings satisfy the Borel Conjecture), and each space contains a simply connected leaf, then $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are homeomorphic.

Remark. There are examples which show this is about as sharp of converse as can be expected.

$\star$ The study of the homeomorphism problem reduces to the study return equivalence between minimal equicontinuous Cantor actions.

Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ a minimal equicontinuous Cantor action

Properties:

$U \subset \mathfrak{X}$ clopen means closed and open.

U *adapted* if $U \neq \emptyset$, and for all $\gamma \in \Gamma$, $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma \cdot U = U$

$\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$

Γ_U is subgroup of finite index in Γ, as Γ translates U to give a partition of $\mathfrak{X}$

Γ_U defines finite covering of M_0 when $\Gamma = \pi_1(M_0, x_0)$

Fact. Action Φ is equicontinuous iff action of Γ on collection of clopen subsets has finite orbits.

$\Rightarrow$ adapted clopen sets form a subbasis for the topology of $\mathfrak{X}$.

Definition. Actions $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ and $\Phi'\colon \Gamma' \times \mathfrak{X}' \to \mathfrak{X}'$ are *return equivalent* if there exists adapted sets $U \subset \mathfrak{X}$ and $U' \subset \mathfrak{X}'$ and homeomorphism $h\colon U \to U'$ that conjugates the subgroups

$$\begin{aligned} \mathcal{H}(U) &= \mathrm{Image}\{\Phi_U\colon \Gamma_U \to \mathrm{Homeo(U)}\} \\ \mathcal{H}'(U') &= \mathrm{Image}\{\Phi'_{U'}\colon \Gamma'_{U'} \to \mathrm{Homeo(U')}\} \end{aligned}$$

The subtlety in this definition is that the map $\Phi_U\colon \Gamma_U \to \mathcal{H}(U)$ may have kernel, and likewise for $\Phi'_{U'}$.

That is, return equivalence is *lossy*.

Definition. Define the group of germs of return equivalences

$$\mathcal{E}_x(\mathfrak{X}, \Gamma, \Phi) = \{h\colon U \to V \mid x \in U, V \subset \mathfrak{X}\}/\sim$$

Example. Let $\mathfrak{G}(\Gamma)$ be the full profinite completion of Γ with respect to all normal subgroups of finite index, basepoint $\widehat{e} \in \mathfrak{G}(\Gamma)$.

Theorem. $\mathcal{E}_{\widehat{e}}(\mathfrak{G}(\Gamma), \Gamma, \Phi) \cong \mathrm{Comm}(\Gamma)$, where $\mathrm{Comm}(\Gamma)$ is the group of abstract commensurators of Γ.

$\star$ E. Bering, IV and D. Studenmund, *Topological Models of Abstract Commensurators*, arXiv:2108.10586v1.

Problem. Calculate $\mathcal{E}_x(\mathfrak{X}, \Gamma, \Phi)$ in general.

The action Φ is equicontinuous, so isometric on $\mathfrak{X}$

$\Rightarrow$ closure $\mathfrak{G}(\Phi) = \overline{\Phi(\Gamma)} \subset \mathrm{Homeo}(\mathfrak{X})$ is profinite group

Action Φ is minimal $\Rightarrow$ induced action $\widehat{\Phi}\colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ is transitive

Isotropy group $\mathfrak{D}_x = \{\widehat{\phi} \in \mathfrak{G}(\Phi) \mid \widehat{\phi}(x) = x\}$ is closed subgroup

$\mathfrak{D}_x$ is called *discriminant* of action by H & Lukina, or *parabolic subgroup* by Nekrashevych

Guiding Principle. Dynamics of action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ determined by action of $\mathfrak{D}_x$ on $\mathfrak{X}$

The action Φ is:

- *free* if $\mathfrak{D}_x$ is identity $\Rightarrow \mathfrak{X} = \mathfrak{G}(\Phi)/\mathfrak{D}_x$ is group
- *topologically free* if $id \neq \widehat{g} \in \mathfrak{X}_x$ fixes no open set in $\mathfrak{X}$
- *stable* if $\exists\ \epsilon > 0$ such that action $\widehat{\Phi}$ on ϵ-neighborhood of x is topologically free
- *wild* if $\forall\ \epsilon > 0$ there exists $id \neq \widehat{g} \in \mathfrak{D}_x$ and $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$ so that action $\widehat{\Phi}(\widehat{g})$ on U fixes open subset $V \subset U$

Theorem. The property that an action is wild or stable is an invariant of return equivalence.

Remark. For each of these cases, the study of $\mathcal{E}_x(\mathfrak{X}, \Gamma, \Phi)$ has special techniques & invariants.

- The *asymptotic discriminant* of an equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$, distinguishes the actions up to local conjugacy.
- In particular, using the asymptotic discriminant, one divides all actions into *stable* and *wild*.
- Direct limit group invariants, which distinguish different classes of wild actions.
- Prime spectrum of actions (*type* and *typeset*).

Abundance of actions on trees/Cantor sets

Theorem. [Dyer, H. & Lukina, 2017] Every finite group, and every separable profinite group, can be realized as the stable discriminant of an action of a torsion-free finite index subgroup of $\mathrm{SL}(n, \mathbb{Z})$, for $n \geq 5$, on a Cantor set.

Theorem. [H. & Lukina, 2019] There exists uncountably many wild actions on Cantor sets of the same torsion-free subgroup of $\mathrm{SL}(n, \mathbb{Z})$, for $n \geq 5$, with distinct asymptotic discriminants.

The study of $\mathcal{E}_x(\mathfrak{X}, \Gamma, \Phi)$ depends on the algebraic structure of Γ:

- ⋆ Γ is torsion-free abelian
- ⋆ Γ is torsion-free nilpotent
- ⋆ Γ is torsion-free arithmetic lattice in higher rank linear group
- ⋆ Γ is an automatic group acting on a tree boundary
- ⋆ Γ is a branch group acting on a tree boundary
- ⋆ Γ is an absolute Galois group with arboreal action on roots

Each of these cases deserves its own discussion.

The notions of type and typeset for $(\mathfrak{X}, \Gamma, \Phi)$ are inspired by the case for Γ abelian.

Let $\mathcal{G}(\mathcal{P}) = \{\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain for action.

Set $m_\ell = \#\{\Gamma_\ell / \Gamma_{\ell+1}\}$

Set $\xi(\mathcal{G}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\}$

Theorem. [H. & Lukina, 2023] The type $\tau[\xi(\mathcal{G})]$ depends only on the return equivalence class of the action (X_∞, Γ, ϕ) it determines.

Let $\Pi = \{2, 3, 5, \ldots\}$ denote the set of primes.

For $\xi = \prod_{p \in \Pi} p^{\chi(p)}$, define:

$$
\begin{aligned}
\pi(\xi) &= \{p \in \Pi \mid \chi(p) > 0\}, \text{prime spectrum of } \xi \ ; \\
\pi_f(\xi) &= \{p \in \Pi \mid 0 < \chi(p) < \infty\}, \text{finite prime spectrumof } \xi \ ; \\
\pi_\infty(\xi) &= \{p \in \Pi \mid \chi(p) = \infty\} \text{infinite prime spectrum of } \xi
\end{aligned}
$$

Note that if $\xi \overset{a}{\sim} \xi'$, then $\pi_\infty(\xi) = \pi_\infty(\xi')$.

The property that $\pi_f(\xi)$ is an *infinite* set is preserved by asymptotic equivalence of Steinitz numbers, so is an invariant of type.

The *typeset* for $(\mathfrak{X}, \Gamma, \Phi)$.

$\mathfrak{G}(\Phi)$ the profinite closure of the action.

For $\gamma \in \Gamma$ we obtain a subgroup $\langle\gamma\rangle \subset \mathfrak{G}(\Phi)$ whose closure $\mathcal{A}(\gamma) = \overline{\langle\gamma\rangle}$ is a compact abelian group,

Get an abelian action $(\mathcal{A}(\gamma), \mathbb{Z}, \Phi_\gamma)$

Definition. $\tau[\gamma] = \tau[\xi(\mathcal{A}(\gamma), \mathbb{Z}, \Phi_\gamma)]$

Definition. The *typeset* for $(\mathfrak{X}, \Gamma, \Phi)$ is the collection of types

$$\mathcal{T}[\mathfrak{X}, \Gamma, \Phi] = \{\tau[\gamma] \mid \gamma \in \Gamma\}$$

Theorem. [H. & Lukina, 2023] The *commensurable equivalence class* of the typeset $\mathcal{T}[\mathfrak{X}, \Gamma, \Phi]$ is an invariant of the return equivalence class of $(\mathfrak{X}, \Gamma, \Phi)$.

The type and typeset were introduced for the classification of dense subgroups of $\mathbb{Q}^n$.

⋆ [Baer, 1937], [Butler, 1965], [Arnold, 1982], [Fuchs, 2015]

This problem is equivalent, via Pontrjagin Duality, to the classification of profinite groups defined by a group chain $\mathcal{G} = \{\Gamma_\ell \mid \ell > 0\}$ in $\Gamma_0 = \mathbb{Z}^n$. That is, for the study of homeomorphisms between solenoidal manifolds with base $\mathbb{T}^n$.

It is known that this problem is not "solvable":

⋆ S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, **J. Amer. Math. Soc.**, 16, 2003.

The classification problem for dense subgroups of $\mathbb{Q}^n$ has been solved for special cases.

Definition. $A \subset \mathbb{Q}^n$ is a *Butler group* if it can be written as a sum $A = A_1 + \cdots + A_s$ of rank 1 subgroups.

In this case the classification problem is much more tractable:

⋆ D. Arnold and C. Vinsonhaler, *Isomorphism invariants for abelian groups*, **Trans. A.M.S.**, 330, 1992.

⋆ S. Thomas, *The classification problem for finite rank Butler groups*, in **Models, modules and abelian groups**, 2008.

Problem. Find classes of Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ for which the return equivalence group $\mathcal{E}(\mathfrak{X}, \Gamma, \Phi)$ can be calculated.