Cantor dynamics of renormalizable groups

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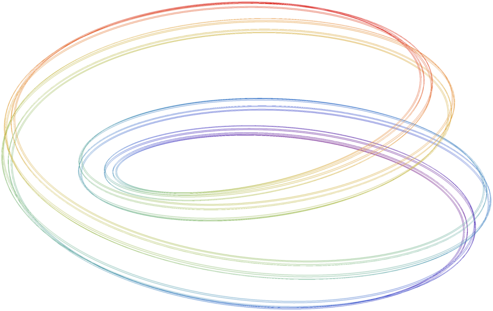
The Smale solenoid

For $m > 1$, let $\Pi_m \colon \mathbb{S}^1 \to \mathbb{S}^1$, given by $\Pi_m(e^{\iota\theta}) = e^{\iota m\theta}$.

Π_m is a proper self-covering map of the circle of degree m.

Iterate the map Π_m repeatedly to obtain the Smale solenoid:

$$\mathcal{S}_m \equiv \varprojlim \{\mathbb{S}^1 \xleftarrow{\Pi_m} \mathbb{S}^1 \xleftarrow{\Pi_m} \mathbb{S}^1 \xleftarrow{\Pi_m} \cdots\} \subset \prod_{\ell \geq 0} \mathbb{S}^1 .$$



Definition: A closed connected manifold M is said to be non co-Hopfian if it admits a proper self-covering map.

The circle $\mathbb{S}^1$ is non co-Hopfian.

The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is non co-Hopfian.

N closed connected manifold, then $\mathbb{S}^1 \times N$ is non co-Hopfian.

Associated to a proper self-map $\Pi \colon M \to M$ we can form a generalized solenoid $\mathcal{S}_\Pi$ as before. These are a special class of the weak solenoids introduced by Chris McCord in 1966.

More generally, non co-Hopfian manifolds have applications in dynamical systems, foliation theory, and spectral theory.

Problem: Characterize the non co-Hopfian manifolds.

Group chains

For the Smale solenoid, given the tower of maps

$$\mathcal{S}_m \equiv \varprojlim \{\mathbb{S}^1 \xleftarrow{\Pi_m} \mathbb{S}^1 \xleftarrow{\Pi_m} \mathbb{S}^1 \xleftarrow{\Pi_m} \cdots\} \subset \prod_{\ell \geq 0} \mathbb{S}^1 \ ,$$

let $x_0 \in \mathbb{S}^1$ be the identity element, then $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$.

We get a chain of subgroups of finite index

$$\mathcal{G}_m = \{\mathbb{Z} \supset m \cdot \mathbb{Z} \supset m^2 \cdot \mathbb{Z} \supset \cdots\}$$

Next, do this for a non co-Hopfian manifold of dimension $n > 1$.

Let M be non co-Hopfian and $\Pi \colon M \to M$ a proper self-covering.

Choose a basepoint $x_1 \in M$ and set $x_0 = \Pi(x_1)$. Then we have

$$\Pi_* \colon \pi_1(M, x_1) \to \pi_1(M, x_0) \equiv \Gamma_0$$

Choose an isomorphism $\pi_1(M, x_1) \cong \pi_1(M, x_0)$.

- ⋆ Π_* induces a self-embedding $\varphi \colon \Gamma_0 \to \Gamma_0$.
- ⋆ Γ_0 is finitely generated.
- ⋆ $\varphi(\Gamma_0) \subset \Gamma_0$ is proper subgroup with finite index.
- ⋆ Group chain $\mathcal{G}_\varphi = \{\Gamma_0 \supset \Gamma_1 = \varphi(\Gamma_0) \supset \Gamma_2 = \varphi(\Gamma_1) \supset \cdots\}$.

A finite index inclusion $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is called a renormalization of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ in the percolation & physics literature.

Definition: Let Γ be a finitely generated group, then an inclusion $\varphi\colon \Gamma \to \Gamma$ with finite index image is called a renormalization of Γ.

Γ is said to be renormalizable if it admits a renormalization.

Γ is also called a finitely non-co-Hopfian group.

Fact: M is non co-Hopfian $\Leftrightarrow \pi_1(M, x)$ is renormalizable.

Questions:

1. What groups are renormalizable?

2. What are the invariants of renormalization maps?

Irreducibility:

Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization. Recursively define a descending chain of subgroups $\Gamma_{\ell+1} = \varphi(\Gamma_\ell)$ for $\ell \geq 0$, so

$$\Gamma \equiv \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

Let $\mathcal{G}_\varphi = \{\Gamma_\ell \equiv \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ be the descending chain of subgroups of finite index associated to φ, then

$$K(\varphi) = \bigcap_{\ell \geq 0} \Gamma_\ell$$

is called the kernel of the chain.

Definition: A renormalization $\varphi\colon \Gamma \to \Gamma$ is said to be irreducible if $K(\varphi)$ is the trivial group, and almost irreducible if $K(\varphi)$ is finite.

In the terminology of Benjamini, and Nekrashevych and Pete,

⋆ *Scale-invariant groups*, **Groups Geom. Dyn.**, 2011

Definition: Γ is said to be strongly scale-invariant if there is an almost irreducible renormalization for Γ.

Question: Is a strongly scale-invariant group Γ virtually nilpotent?

This question is inspired by a celebrated result of Gromov .

Example: Expanding manifolds

Let M be a closed Riemannian manifold. A smooth map $f\colon M \to M$ is expanding if there exists some $\lambda > 1$ such that

$$\|df(\vec{v})\| \geq \lambda \|\vec{v}\| \quad \text{for all } x \in M \text{ and } \vec{V} \in T_x M$$

The map f must be a proper covering.

Theorem: [Franks 1968] If M admits an expanding map, then $\Gamma_0 = \pi_1(M, x_0)$ has polynomial growth rate.

Theorem: [Gromov 1979] If Γ is a finitely generated group with polynomial growth rate, then Γ admits a nilpotent subgroup $\Lambda \subset \Gamma$ with finite index. i.e., Γ is virtually nilpotent.

Our work was motivated by the following result from

• *Van Limbeek*, **Towers of regular self-covers and linear endomorphisms of tori**, Geom. Topol., 2018.

Theorem: Let Γ be a strongly scale-invariant group, with a renormalization $\varphi\colon \Gamma \to \Gamma$ such that $\Gamma_\ell = \varphi^\ell(\Gamma)$ is <u>normal</u> in Γ. Then $\Gamma/K(\varphi)$ is abelian.

Question: Is there a weaker assumption than normality for the subgroups Γ_ℓ that yields a solution to the nilpotent question?

Our approach uses ideas from Cantor dynamical systems,

• *Hurder, Lukina & Van Limbeek*, **Cantor dynamics of renormalizable groups**, arxiv:2002.01565

Construction of Cantor actions

Consider again the Smale solenoid. Fix the integer $m > 1$, so we have an embedding $\varphi \colon \mathbb{Z} \to \mathbb{Z}$, given by $\varphi(k) = m \cdot k$.

Then $\Gamma_\ell = m^\ell \cdot \mathbb{Z} \subset \mathbb{Z}$.

Pass to quotient groups and form the inverse limit space

$$\mathfrak{X} \equiv \varprojlim \{0 = \mathbb{Z}/\mathbb{Z} \xleftarrow{m_*} \mathbb{Z}/m\mathbb{Z} \xleftarrow{m_*} \mathbb{Z}/m^2\mathbb{Z} \xleftarrow{m_*} \cdots\}$$

The inverse limit $\mathfrak{X}$ is a Cantor group, the m-adic integers $\widehat{\mathbb{Z}}_m$.

The group $\Gamma = \mathbb{Z}$ acts by addition on each quotient group $\mathbb{Z}/m^\ell\mathbb{Z}$.

Get dynamical system $\mathbb{Z} \times \mathfrak{X} \to \mathfrak{X}$ which is m-adic odometer.

Let Γ be a finitely generated group.

Let $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$ be a group chain, where $\Gamma_0 = \Gamma$ and $\Gamma_{\ell+1} \subset \Gamma_\ell$ is a proper subgroup of finite index.

$X_\ell = \Gamma/\Gamma_\ell$ is a finite set with transitive left Γ-action.

Inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a surjection $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Define

$$\mathfrak{X} \; \equiv \; \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} \, X_\ell \; .$$

The product of finite sets is given the Tychonoff topology - cylinder sets generate the topology.

Then $\mathfrak{X}$ is a closed subset, so is a Cantor space with left Γ-action.

Obtain minimal Γ-action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$.

Called a <u>subodometer</u> by Cortez and Petite.

A Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \epsilon \quad \text{for all } g \in \Gamma.$$

For ultrametric metric on $\mathfrak{X}$, the action Φ is isometric:

- $(\mathfrak{X}, \Gamma, \Phi)$ is an equicontinuous Cantor action.

Remark: A minimal equicontinuous Cantor action can also be viewed as a group action on a rooted tree.

Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous Cantor action.

This defines a homomorphism $\Phi\colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$

$\widehat{\Gamma} = \overline{\Phi(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ is the closure in uniform topology

Proposition: [Ellis, 1969] Φ equicontinuous implies that $\widehat{\Gamma}$ is a profinite group, compact and totally disconnected.

Lemma: Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization for Γ with associated Cantor action $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$. Then $\ker(\Phi_\varphi) \subset K(\varphi)$, where $\Phi_\varphi\colon \Gamma \to \widehat{\Gamma}_\varphi$ is the map to the completion.

Strategy: For $K(\varphi)$ finite, find conditions on renormalization $\varphi\colon \Gamma \to \Gamma$ which imply that $\widehat{\Gamma}_\varphi$ is a virtually nilpotent group, and hence Γ is virtually nilpotent.

Lemma: Φ_φ induces an equicontinuous action $\widehat{\Phi}_\varphi \colon \widehat{\Gamma} \times \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$.

For a sequence $\widehat{\gamma} = \{\Phi_\varphi(\gamma_i) \in \mathbf{Homeo}(\mathfrak{X}) \mid i > 0\} \in \widehat{\Gamma}$ which converges in the uniform topology of maps, given $x \in \mathfrak{X}_\varphi$ set $\widehat{\gamma} \cdot x = \lim\ \Phi_\varphi(\gamma_i)(x)$.

Lemma: Φ_φ minimal implies that $\widehat{\Gamma}_\varphi$ acts transitively on $\mathfrak{X}_\varphi$.

For $x \in \mathfrak{X}_\varphi$, define the isotropy subgroup

$$\mathcal{D}_x = \{\widehat{\gamma} \in \widehat{\Gamma}_\varphi \mid \widehat{\Phi}_\varphi(\widehat{\gamma})(x) = x\}$$

Isomorphism class of $\mathcal{D}_x$ is independent of choice of x and invariant of isomorphism of actions.

Proposition: $\mathfrak{X}_\varphi$ is a homogeneous space for $\widehat{\Gamma}_\varphi$. That is,

$$\mathfrak{X}_\varphi \cong \widehat{\Gamma}_\varphi/\mathcal{D}_x \quad \text{as left } \Gamma - \text{spaces}$$

Remark: If Γ is abelian group, then $\mathcal{D}_x$ is trivial, so $\mathfrak{X}_\varphi$ is a profinite group and the action of Γ on it is by "addition".

Then $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$ is a generalized odometer.

Remark: We obtain invariants of the self-embedding φ by studying the dynamics of the adjoint action of $\mathcal{D}_x$ on $\widehat{\Gamma}_\varphi$.

We first observe there is a canonical basepoint for $(\mathfrak{X}_\varphi, \Gamma, \Phi_\varphi)$.

Proposition: There is a rescaling $\lambda_\varphi \colon X_\varphi \to X_\varphi$ whose image $U_1 = \lambda_\varphi(X_\varphi)$ is a clopen subset of X_φ. Moreover, the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is conjugate to the restricted action $(U_1, \Gamma_{U_1}, \Phi_{U_1})$.

Idea of proof: φ induces a map of quotients $\overline{\varphi} \colon \Gamma/\Gamma_\ell \to \Gamma_1/\Gamma_{\ell+1}$. This induces the shift map $\lambda_\varphi \colon X_\varphi \to U_1 \subset X_\varphi$.

- Let $\mathcal{D}_\varphi \subset \mathbf{Homeo}(X_\varphi)$ be the isotropy subgroup at the unique fixed-point x_φ of the contraction map λ_φ.

The study of invariants for the adjoint action of $\mathcal{D}_\varphi$ on $\widehat{\Gamma}_\varphi$ leads into analyzing the regularity properties of Cantor actions.

Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be a Cantor action of a countable group Γ.

The action is:

- ⋆ effective, or faithful, if $\Phi \colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$ has trivial kernel.
- ⋆ free if for all $x \in \mathfrak{X}$ and $g \in \Gamma$, $g \cdot x = x$ implies that $g = e$
- ⋆ isotropy group of $x \in \mathfrak{X}$ is $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$
- ⋆ $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and isotropy set

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\ g \in \Gamma\ ,\ g \neq id\ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)$$

Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is topologically free if $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X} \implies \mathrm{Iso}(\Phi)$ has empty interior.

For Γ a countable group, this is a natural hypothesis to impose.

However, for a Cantor action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ where H is not countable, we introduce another definition of regularity.

First, recall the topology of Cantor space $\mathfrak{X}$ is generated by clopen subsets: U is closed and open. A non-empty clopen $U \subset \mathfrak{X}$ is adapted if the return times to U form a subgroup:

$$\Gamma_U = \{g \in \Gamma \mid \Phi(g)(U) = U\} \subset \Gamma$$

Lemma: For $x \in \mathfrak{X}$ and open $x \in V$, there is adapted U with $x \in U \subset V$.

Definition: An action $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$, where

- H is a topological group and
- $\mathfrak{X}$ is a Cantor space

is quasi-analytic if for each clopen set $U \subset \mathfrak{X}$, $g \in H$

- if $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For H a countable group, this is equivalent to topologically free.

Here is our key technical result:

Theorem 1: The action $\widehat{\Phi}_\varphi \colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ is quasi-analytic.

Corollary: Let $\widehat{\gamma} \in \widehat{\Gamma}_\varphi$. The homeomorphism $\widehat{\Phi}_\varphi(\widehat{\gamma}) \colon \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$ is uniquely determined by its restriction to an adapted subset of $\mathfrak{X}$.

Return now to study of $\widehat{\Gamma}_\varphi$ associated to a renormalization $\varphi\colon \Gamma \to \Gamma$. Here is our two main (technical) results:

Theorem 2: A renormalization map φ induces a contraction map on the closure, $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

Theorem 3: $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$

This connects discriminant invariants for Cantor actions, with invariants for contraction profinite groups.

The proof of Theorem 3 follows almost directly from the algebraic definition for $\mathcal{D}_\varphi$ developed in

$\star$ *Jessica Dyer*, **Dynamics of Equicontinuous Group Actions on Cantor Sets**, 2015 UIC PhD.

The proof of Theorem 2 looks equally "obvious", except that it isn't. Here is the issue:

The renormalization φ naturally induces a map $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \mathbf{Homeo}(U_1)$. We need to show that the maps in the image of $\widehat{\varphi}$ have unique extensions to $\mathbf{Homeo}(X_\varphi)$.

This is exactly what Theorem 1 says is true.

Theorems 2 and 3 are applied to show that $\widehat{\Gamma}_\varphi$ is virtually nilpotent.

There is an extensive literature on the structure of profinite groups with an open contraction mapping, in particular by:

⋆ Baumgartner, Caprace, Reid, Wesolek, Willis, Wilson

The following result is based on results of

Udo Baumgartner & George Willis, and Colin Reid:

Theorem: Let $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ be a contraction map with open image. Then there is an isomorphism with a semi-direct product

$$\widehat{\Gamma}_\varphi \cong \mathcal{N}_\varphi \rtimes \mathcal{D}_\varphi$$

$$\begin{aligned} \mathcal{N}_\varphi &= \{\widehat{g} \in \widehat{\Gamma}_\varphi \mid \lim_{\ell\to\infty} \widehat{\varphi}^\ell(\widehat{g}) = \widehat{e}\} \\ \mathcal{D}_\varphi &= \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi \end{aligned}$$

Moreover, the contraction factor $\mathcal{N}_\varphi$ is pro-nilpotent.

We use this structure theorem for contraction maps to show:

Theorem [HLvL2020]: Let φ be a renormalization of the finitely generated group Γ. Suppose that

$$K(\varphi) = \bigcap_{\ell > 0} \varphi^{\ell}(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_{\varphi} = \bigcap_{n > 0} \widehat{\varphi}_0^{n}(\widehat{\Gamma}_{\varphi}) \subset \widehat{\Gamma}_{\varphi}$$

are both finite groups, then

- Γ is virtually nilpotent,
- If both groups are trivial, then Γ is nilpotent.

Remark: The normality assumption in Van Limbeek's Theorem is replaced by the assumption that $\mathcal{D}_{\varphi}$ is a finite group.

Basic Problem: Let φ be an irreducible renormalization of the finitely generated group Γ. Show that $\mathcal{D}_\varphi$ is nilpotent.

This is true in all examples calculated. Need better understanding of closed subgroups of profinite groups to prove it in general.

One expects further properties of renormalizable groups can be obtained from applying results on contraction groups and scales of automorphisms of totally disconnected locally compact groups to the contraction map $\widehat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ induced by a renormalization.

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