Type invariants for solenoidal manifolds

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Motivation

Problem 1: Study properties of foliated spaces which are transversally totally disconnected. These are called matchbox manifolds, or solenoidal manifolds in works of Sullivan.

Problem 2: Determine when a foliated space is homeomorphic to a minimal set of a C^r-foliation.

Problem 3: Classify the foliated spaces $(\mathfrak{M}, \mathcal{F})$ for which the transverse holonomy maps are equicontinuous, so preserve an ultra-metric on the transverse Cantor space.

⋆ We focus on some recent contributions to Problem 3.

Problem 3 asks to study the properties of Riemannian foliated spaces, analogous to Riemannian foliations on smooth manifolds.

Series of works by Jesús Álvarez López, Alberto Candel, Ramón Barral Lijó, and Manuel Moreira Galicia, including:

- *Equicontinuous foliated spaces*, by Álvarez López & Candel, **Mathematische Zeitschrift**, vol. 263, 2009.
- *Topological Molino's theory*, by Álvarez López & Moreira Galicia, **Pacific Journal of Math.**, vol. 280, 2016.
- *Molino's description and foliated homogeneity*, by Álvarez López & Barral Lijó, **Topology and its Applications**, vol. 260, 2019.

Here we focus on the structure of these spaces.

Weak solenoids

Theorem: [Clark & Hurder 2013] Let $(\mathfrak{M}, \mathcal{F})$ be an equicontinuous foliated space. Then $\mathfrak{M}$ is homeomorphic to a *weak solenoid*, in the sense of

- *Inverse limit sequences with covering maps*, by M.C. McCord, **Trans. Amer. Math. Soc.**, vol. 114, 1965

That is, there is given a collection $\mathcal{P} = \{ q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$, where each M_ℓ is a compact connected manifold without boundary of dimension $n \geq 1$, and q_ℓ is a proper covering map. The inverse limit

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{q_\ell \colon M_\ell \to M_{\ell-1}\} \; \subset \prod_{\ell \geq 0} \; M_\ell$$

is the *solenoidal manifold* associated to $\mathcal{P}$.

The theorem asserts that $\mathfrak{M}$ is foliated homeomorphic to $\mathcal{S}_{\mathcal{P}}$ for some choice of M_0 and tower of coverings $\mathcal{P}$.

Problem: Let $\mathcal{P}$ be a presentation,

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

What are the invariants of homeomorphism between two spaces which are the inverse limits of such towers of coverings?

Facts about the inverse limit $\mathcal{S}_{\mathcal{P}}$

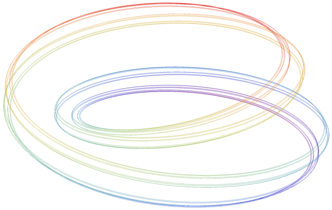
- $\star$ $\mathcal{S}_{\mathcal{P}}$ is compact and connected but not locally connected,
- $\star$ There is a fibration map $p_\ell \colon \mathcal{S}_{\mathcal{P}} \to M_\ell$ for each $\ell \geq 0$
- $\star$ The fiber $\mathfrak{X} = p_0^{-1}(x_0)$ is a Cantor space
- $\star$ $\mathcal{S}_{\mathcal{P}}$ is foliated by leaves which cover M_0
- $\star$ The holonomy pseudogroup of the foliation $\mathcal{F}_{\mathcal{P}}$ preserves an ultra-metric on the Cantor transversals.

Example: The Vietoris - van Dantzig Solenoid:

$$\mathcal{S}(\vec{m}) = \varprojlim \{ q_\ell : \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 1 \}$$

where q_ℓ is a covering map of the circle $\mathbb{S}^1$ of degree $m_\ell > 1$, and $\vec{m} = (m_1, m_2, \ldots)$ denotes the collection of covering degrees.

When $m_i = 2$ for all $i \geq 1$ we get the *Smale attractor*, which is a minimal set for a C^∞-foliation:



Supernatural numbers

A *Steinitz* (or *supernatural*) number is a formal infinite product

$$\Pi[\vec{m}] = \prod_{0 \leq i < \infty} m_i = \prod_{p \in \pi} p^{n(p)} \quad , \quad 0 \leq n(p) \leq \infty$$

where π denotes the collection of all primes.

Steinitz numbers Π and Π' are *asymptotically equivalent*, written $\Pi \overset{a}{\sim} \Pi'$, if there exists integers $m, m' > 0$ such that $m \cdot \Pi = m' \cdot \Pi'$.

The asymptotic equivalence class of Π is denote by Π_a

Definition: A <u>type</u> is a class Π_a of a Steinitz number Π.

Definition: Given Steinitz number $\Pi = \prod_{p \in \pi} \ p^{n(p)}$, define:

- $\star$ $\pi(\Pi) = \{p \in \pi \mid 0 < n(p)\}$, *prime spectrum of* Π
- $\star$ $\pi_f(\Pi) = \{p \in \pi \mid 0 < n(p) < \infty\}$, *finite prime spectrum of* Π
- $\star$ $\pi_\infty(\Pi) = \{p \in \pi \mid n(p) = \infty\}$, *infinite prime spectrum of* Π

Note that if $\Pi \overset{a}{\sim} \Pi'$, then $\pi_\infty(\Pi) = \pi_\infty(\Pi')$. The property that $\pi_f(\Pi)$ is an *infinite* set is also preserved by asymptotic equivalence.

Definition: Let $\mathcal{S}(\vec{m})$ be a 1-dimensional solenoid defined by a sequence of covering degrees $\vec{m}$. Then the type of $\mathcal{S}(\vec{m})$ is the asymptotic equivalence class $\Pi_a[\vec{m}]$.

[Bing, 1960] and [McCord, 1965] showed:

Theorem: Solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m'})$ are homeomorphic if and only if they have the same type.

[Aarts & Fokkink, 1991] gave a new proof of this result.

Their novel method of proof suggests the following:

Question: Does "type" make sense for all weak solenoids, and if so, what does it tell you about these foliated spaces?

Types of weak solenoids

Here is our first result.

Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid defined by a presentation
$\mathcal{P} = \{\, q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$.

Let m_ℓ denote the degree of the covering map q_ℓ.

Set $\vec{m} = \{m_1, m_2, \ldots\}$.

Definition: The *type* of $\mathcal{S}_{\mathcal{P}}$ is the type of $\Pi[\vec{m}]$.

Theorem 1 [Hurder & Lukina 2021]: The type of $\mathcal{S}_{\mathcal{P}}$ is an invariant of the homeomorphism class of the continuum $\mathcal{S}_{\mathcal{P}}$.

Corollary: The infinite prime spectrum $\pi_\infty(\Pi[\vec{m}])$ is an invariant of the homeomorphism class of $\mathcal{S}_{\mathcal{P}}$. Also, the property that $\pi_f(\Pi[\vec{m}])$ is an *infinite* set is preserved by homeomorphism.

For $\mathcal{P} = \{ q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$, set $\Gamma = \pi_1(M_0, x_0)$.

For non-trivial $\gamma \in \pi_1(M_0, x_0)$ there is a well defined type $\Pi_a[\gamma]$, defined as the type of the 1-dimensional solenoid in $\mathcal{S}_\mathcal{P}$ that covers a closed curve representing γ.

Definition: The collection

$$\mathbf{TS}[\mathcal{S}_\mathcal{P}] = \{\Pi_a[\gamma] \mid \gamma \in \pi_1(M_0, x_0)\}$$

is called the *type set* of $\mathcal{S}_\mathcal{P}$.

Theorem 2 [Hurder & Lukina 2022]: The type set $\mathbf{TS}[\mathcal{S}_\mathcal{P}]$ is an invariant of the homeomorphism class of $\mathcal{S}_\mathcal{P}$.

The theorems follow from the results in the works

- *Classifying matchbox manifolds*, by Clark, Hurder & Lukina, **Geometry & Topology**, vol. 23, 2019.

- *Pro-groups and generalizations of a theorem of Bing*, by Clark, Hurder & Lukina, **Topology and its Applications**, vol. 271, 2020.

We explain how this works.

Cantor actions

Given the weak solenoid

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{q_\ell \colon M_\ell \to M_{\ell-1}\} \; \subset \prod_{\ell \geq 0} \; M_\ell$$

for $x_0 \in M_0$, the fiber $\mathfrak{X} = p_0^{-1}(x_0)$ is a Cantor set.

The monodromy along leaves of the foliation of $\mathcal{S}_{\mathcal{P}}$ gives an action of $\Gamma = \pi_1(M_0, x_0)$ on $\mathfrak{X}$. The action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is minimal and equicontinuous.

Theorem: $\mathcal{S}_{\mathcal{P}}$ is homeomorphic to the suspension foliation for the action $(\mathfrak{X}, \Gamma, \Phi)$.

$\star$ Suffices to study the properties of minimal equicontinuous Cantor actions of finitely generated groups.

Assume that $(\mathfrak{X}, \Gamma, \Phi)$ is a minimal equicontinuous Cantor action.

Definition: $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, \Gamma, \Phi)$ if U is a *non-empty clopen* subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$.

- Given $x \in \mathfrak{X}$ and clopen set $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

- For U adapted, the set of "return times" to U,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\}$$

is a subgroup of Γ, called the *stabilizer* of U.

- $\mathcal{H}_U = \Phi(\Gamma_U) \subset \mathbf{Homeo}(U)$ is the monodromy of Γ_U

Definition: Minimal equicontinuous Cantor actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are return equivalent if there exists an adapted set $U_1 \subset \mathfrak{X}_1$ for the action Φ_1 and an adapted set $U_2 \subset \mathfrak{X}_2$ for the action Φ_2, and a homeomorphism $h \colon U_1 \to U_2$ which induces an isomorphism of the monodromy groups $\mathcal{H}_{U_1}$ with $\mathcal{H}_{U_2}$.

This is the version of Morita equivalence appropriate for equicontinuous actions on Cantor sets.

Theorem [Clark, Hurder & Lukina, 2019]: If solenoidal manifolds $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ are homeomorphic, then their monodromy Cantor actions are *return equivalent*.

If the actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are *topologically free*, then this result implies that the groups Γ_1 and Γ_2 are commensurable.

That is, there are subgroups $H_1 \subset \Gamma_1$ and $H_2 \subset \Gamma_2$ such that $H_1 \cong H_2$. For example, for the 1-dimensional solenoid, two subgroups of $\mathbb{Z}$ are commensurable.

Maps between inverse limits

The second ingredient we need are some properties of homeomorphisms between inverse limits, derived from:

- *Mappings of inverse limits*, Mioduszewski, **Coll. Math.**, vol. 10, 1963.

- *Approximation of maps of inverse limit spaces by induced maps*, Fort & McCord, **Fund. Math.**, vol. 59, 1966.

- *Maps between weak solenoidal spaces*, Rogers & Tollefson, **Coll. Math.**, vol. 23, 1971.

- *Homogeneous inverse limit spaces with non-regular covering maps as bonding maps*, Rogers & Tollefson, **Proc. A.M.S**, vol. 29, 1971.

Theorem: Suppose there are two presentations as weak solenoids

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \tag{1}$$
$$\mathcal{Q} = \{q_{\ell+1} \colon N_{\ell+1} \to N_\ell \mid \ell \geq 0\} \ , \tag{2}$$

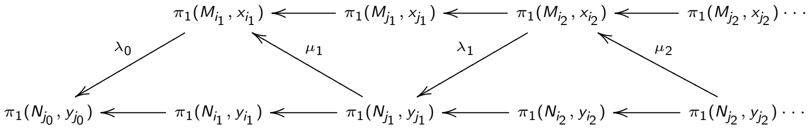
a homeomorphism $\Phi \colon \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{Q}$ and $\epsilon > 0$. Then there exists a decreasing sequence $\widehat{\epsilon} = \{\epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots\}$ with $\epsilon_0 \leq \epsilon$,

1. an intertwined increasing sequence $0 \leq j_0 < i_0 < j_1 < i_1 < j_2 < i_2 < \cdots$,
2. covering maps $\lambda_\ell \colon M_{i_{\ell+1}} \to N_{j_\ell}$ for $\ell \geq 0$,
3. covering maps $\mu_\ell \colon N_{j_\ell} \to M_{i_\ell}$ for $\ell \geq 0$,

such that the following diagram $\widehat{\epsilon}$-commutes:

$$\begin{array}{ccccccccc} M_{i_0} & \longleftarrow & M_{i_1} & \longleftarrow & M_{j_1} & \longleftarrow & M_{i_2} & \longleftarrow & M_{j_2} \cdots \\ & \swarrow_{\lambda_0} & & \nwarrow^{\mu_1} & & \swarrow_{\lambda_1} & & \nwarrow^{\mu_2} & \\ N_{j_0} & \longleftarrow & N_{i_1} & \longleftarrow & N_{j_1} & \longleftarrow & N_{i_2} & \longleftarrow & N_{j_2} \cdots \end{array} \tag{3}$$

Corollary: There is a commutative diagram of groups:



(4)

That is, for $\Gamma_\ell = \pi_1(M_{i_\ell}, x_{i_\ell})$ and $\Gamma'_\ell = \pi_1(N_{j_\ell}, y_{j_\ell})$, the descending group chains $\{\Gamma_\ell \mid \ell > 0\}$ and $\{\Gamma'_\ell \mid \ell > 0\}$ are intertwined.

The proof of Theorem 1 then follows almost immediately.

- *The prime spectrum of solenoidal manifolds*, Hurder & Lukina, arXiv:2103.06825.

Theorem 2 is motivated by a result from the theory of finite rank subgroups of $\mathbb{Q}^k$, which has a massive theory:

- **Abelian groups**, by László Fuchs, Springer Monographs in Mathematics, 2015.

A finite rank subgroup of $\mathbb{Q}^k$ corresponds to a descending chain in $\mathbb{Z}^k$ by Pontrjagin duality, as described in

- *$\mathbb{Z}^d$-odometers and cohomology*, Giordano, Putnam,& Skau, **Groups, Geometry, and Dynamics**, vol. 13, 2019.

So one asks, whether any of this classification theory for subgroups of $\mathbb{Q}^k$ is applicable to the study of weak solenoids?

Some, and maybe more than some.

Denote by $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ a properly descending chain of finite index subgroups

Choose $\gamma \in \Gamma$ such that $H_0 = \langle \gamma \rangle \subset \Gamma$ is free abelian. Then for $H_\ell = \Gamma_\ell \cap H_0$, we get a descending chain

$$\mathbb{Z} \cong H_0 \supset H_1 \supset H_2 \supset H_3 \supset \cdots$$

Definition: Set $\Pi[\gamma] = LCM \ \{\#(H_0/H_\ell) \mid \ell > 0\}$.

Definition: The collection of types

$$\mathbf{TS}[\mathcal{G}] = \{\Pi_a[\gamma] \mid \gamma \in \Gamma\}$$

is called the *type set* of $\mathcal{G}$.

For $\mathcal{G}$ the group chain associated to a tower of coverings $\mathcal{P}$, it follows in a straightforward manner that the type set $\mathbf{TS}[\mathcal{S}_{\mathcal{P}}] = \{\Pi_a[\gamma] \mid \gamma \in \pi_1(M_0, x_0)\}$ is an invariant for homeomorphisms between weak solenoids.

The story does not stop here, as the set of types have a partial order, yielding what is called the *type graph*:

• *Type graph*, Mutzbauer, **Abelian group theory (Honolulu, Hawaii, 1983)**, Lecture Notes in Math., Vol. 1006, 1983.

Question: Do the type invariants impose any restrictions on whether a weak solenoid can be the minimal set of a foliation?

Realization

Theorem: Let $M_0 = \mathbb{T}^k$ and $\mathcal{P} = \{ q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1 \}$ be a tower of coverings of M_0. Then there exists a C^0 foliation $\mathcal{F}_{\mathcal{P}}$ of $M_0 \times \mathbb{D}^q$, for $q \geq 2k$, with minimal set homeomorphic to $\mathcal{S}_{\mathcal{P}}$.

Moreover, for $r \geq 1$, if $\mathcal{P}$ satisfies certain technical growth conditions, then $\mathcal{F}_{\mathcal{P}}$ can be constructed as a C^r foliation.

- *Embedding solenoids in foliations*, by Clark & Hurder, **Topology and its Applications**, vol. 158, 2011.

$\star$ This suggests that any type set can be realized as a foliated minimal set, for abelian groups. For non-abelian groups, such as torsion-free nilpotent groups, the story seems more complicated.