Aperodicity at the boundary of chaos

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Genericity of Dynamics

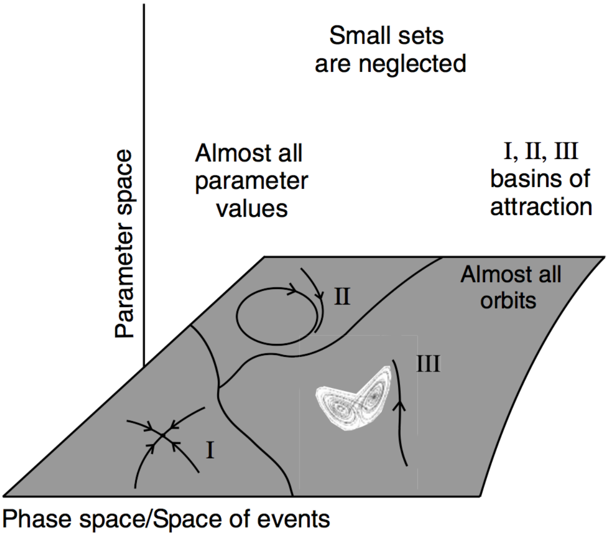
Weak Palis Conjecture: M compact manifold, then the space of $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms ($r \geq 1$) contains a dense open set which decomposes as the union $\mathcal{MS} \cup \mathcal{I}$ of two disjoint open sets:

- $\mathcal{MS}$ is the set of Morse-Smale diffeomorphisms,
- $\mathcal{I}$ is the set of diffeomorphisms having transverse homoclinic intersection.

A diffeomorphism is Morse-Smale if its non-wandering set consists of finitely many hyperbolic periodic orbits.

- J. Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, **Astérisque**,Vol. 261, 2000.
- J. Palis, *On Open questions leading to a global perspective in dynamics*, **Nonlinearity**,21:, 2008.

The Weak Palis Conjecture



(Illustration from Palis, *Nonlinearity* 2008)

We are interested in the "structural stability" of the following class of examples, called "Kuperberg flows":

Theorem (K. Kuperberg, 1994) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

- K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.
- É Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, 227: 283–307, 1995.
- S. Hurder & A. Rechtman, *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377 (216), 250 pages.

What we know of the dynamics of the Kuperberg flows:

Theorem (A. Katok, 1980) *Let M be a closed, orientable 3-manifold. Then an aperiodic flow ϕ_t on M has entropy zero.*

Theorem (Ghys, Matsumoto, 1995) The Kuperberg flow has a unique minimal set $\mathfrak{M} \subset M$.

Theorem (Hurder & Rechtman, 2015) *Let Φ_t be a generic Kuperberg flow on a plug $\mathbb{K}$. There the unique minimal set $\mathfrak{M}$ for the flow is a 2-dimensional lamination "with boundary" which is equal to the non-wandering set of Φ_t.*

Moreover, the flow restricted to $\mathfrak{M}$ has non-zero "slow entropy", for exponent $\alpha = 1/2$.

So, a generic Kuperberg flow *almost* has positive entropy.

Question: Where do the Kuperberg flows sit in the scheme of the Weak Palis Conjecture?

Theorem 1: *Let Φ_t be a Kuperberg flow on a plug $\mathbb{K}$. Then there is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $-1 < \epsilon \leq 0$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ is "partially Morse-Smale" and so has entropy 0.*

Theorem 2: *Let Φ_t be a Kuperberg flow on a plug $\mathbb{K}$. Then there is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $0 \leq \epsilon < a$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ admits a "horseshoe", and so has positive entropy.*

Conclusion: *The generic Kuperberg flows lie at the boundary of chaos (entropy > 0) and the boundary of tame dynamics.*

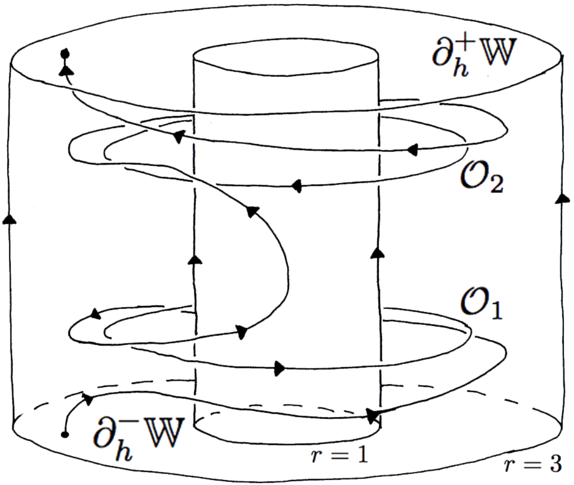
- S. Hurder & A. Rechtman, *Aperiodic flows at the boundary of chaos*, *in preparation*, available March 2016.

Definition: A plug is a 3-manifold with boundary of the form $P = D \times [-1, 1]$ with D a compact surface with boundary. P is endowed with a non-vanishing vector field $\vec{X}$, such that:

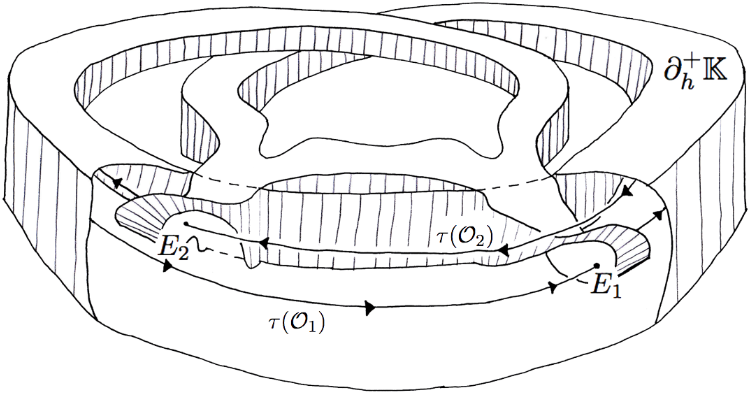
- $\vec{X}$ is vertical in a neighborhood of ∂P, that is $\vec{X} = \frac{d}{dz}$. Thus $\vec{X}$ is inward transverse along $D \times \{-1\}$ and outward transverse along $D \times \{1\}$, and parallel to the rest of ∂P.
- There is at least one point $p \in D \times \{-1\}$ whose positive orbit is trapped in P.
- If the orbit of $q \in D \times \{-1\}$ is not trapped then its orbit intersects $D \times \{1\}$ in the facing point.
- There is an embedding of P into $\mathbb{R}^3$ preserving the vertical direction.

Modified Wilson Plug $\mathbb{W}$ (sort of Morse-Smale)

Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f\,\frac{f}{d\theta}$
f is asymmetric in z and $\vec{W}_1 = g\,\frac{f}{dz}$ is vertical.

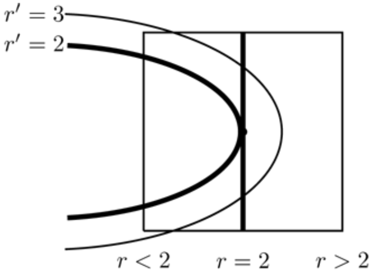


Grow horns and embed them to obtain Kuperberg Plug $\mathbb{K}$, matching the flow lines on the boundaries.



Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself.

The insertion map as it appears in the face E_1

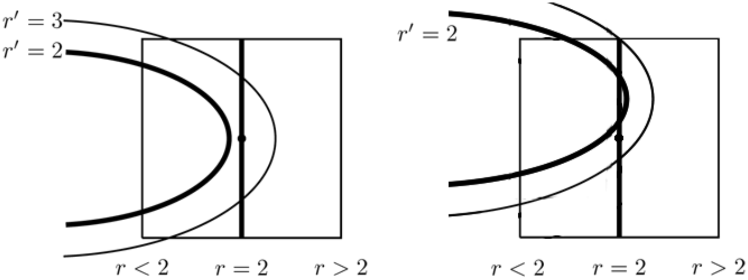


Radius Inequality:

For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r'$ unless $x' = (2, \theta_i, -2)$ and then $r = 2$.

Parametrized Radius Inequality: For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^\epsilon(r', \theta', -2) \in \mathcal{L}_i$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

The modified radius inequality for the cases $\epsilon < 0$ and $\epsilon > 0$:



Proposition: *Let Φ_t^ϵ be a Kuperberg flow for which the insertion map satisfies the Parametrized Radius Inequality with $\epsilon < 0$. Then the flow in the plug $\mathbb{K}_\epsilon$ has two periodic orbits that bound an invariant cylinder, and the flow has topological entropy zero.*

Idea of the proof: This follows from the techniques for the standard flow when $\epsilon = 0$, which imply that every flow orbit of a point x with radius $r(x) \neq 2$ entering an insertion, exits at the same radius.

Varying the radius of the insertion for $\epsilon < 0$, we obtain Theorem 1.

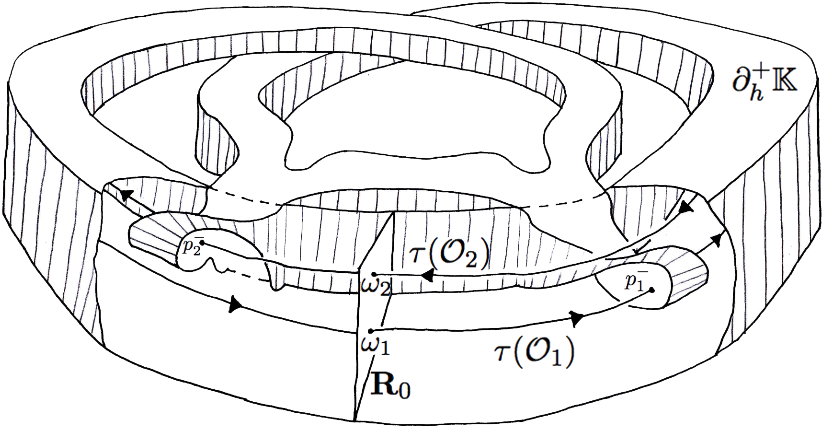
The idea of the proof of Theorem 2 is to study the dynamics of Kuperberg flows Φ_t^ϵ for $\epsilon \geq 2$.

Our approach in the Asterisque paper introduced the technique of comparing the dynamics of the flow Φ_t^0 with that of an induced map on a (partial) section to the flow.

Return map of a flow Φ_t^ϵ induces a smooth pseudogroup $\mathcal{G}_{\Phi^\epsilon}$ on $\mathbf{R}_0$

Critical difficulty: *There is not always a direct relation between the continuous dynamics of the flow Φ_t^ϵ and the discrete dynamics of the action of the pseudogroup $\mathcal{G}_{\Phi^\epsilon}$.*

The section $\mathbf{R}_0 \subset \mathbb{K}$ used to define pseudogroup $\mathcal{G}_{\Phi^\epsilon}$.



The flow of Φ_t^ϵ is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$, so the action of the pseudogroup has singularities along this line.

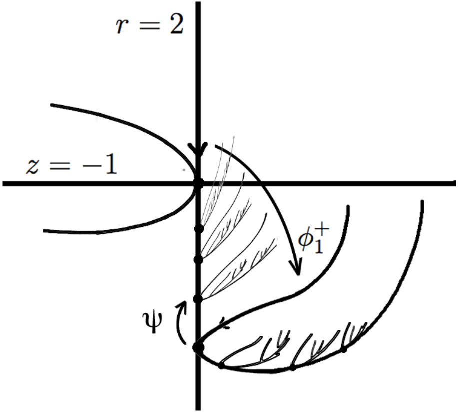
We consider two maps with domain in $\mathbf{R}_0$

- ψ which is the return map of the *Wilson flow* Ψ_t
- ϕ_1^ϵ which is the return map of the *Kuperberg flow* Φ_t^ϵ for orbits that go through the entry region E_1

Form the pseudogroup they generate $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$.

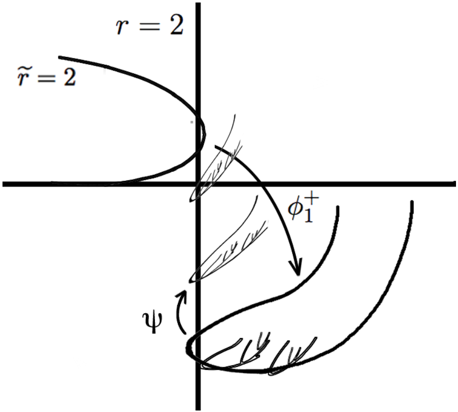
Proposition: The restriction of $\widehat{\mathcal{G}}_\epsilon$ to the region $\{r > 2\} \cap \mathbf{R}_0$ is a sub-pseudogroup of $\mathcal{G}_{\Phi^\epsilon}$

Action of $\widehat{\mathcal{G}}_0 = \langle \psi, \phi_1^\epsilon \rangle$ on the line $r = 2$ for $\epsilon = 0$.



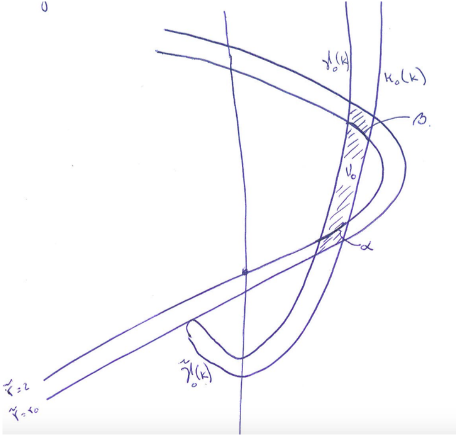
This looks like a ping-pong game, except that the play action is too slow to generate entropy.

Action of $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$ on the line $r = 2$ for $\epsilon > 0$.

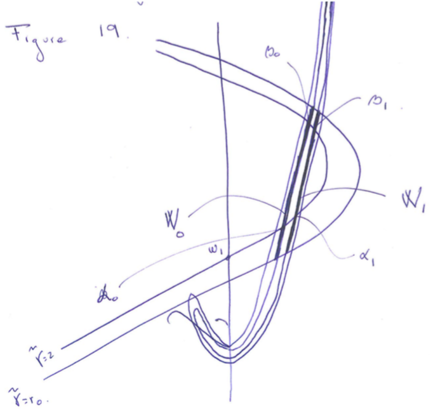


The dynamics of this action is actually too complicated to draw precisely, or calculate with.

Instead, we define a compact region $U_0 \subset \mathbf{R}_0$ which is mapped to itself by the map $\varphi = \psi^k \circ \phi_1^\epsilon$ for k sufficiently large.



The images of the powers φ^ℓ of the map the map φ form a δ-separated set for the action of the pseudogroup $\widehat{\mathcal{G}}_\epsilon$.



We then show that for $\epsilon > 0$ well-chosen with respect to the choice of k above, the restriction of the map φ to U_0 is defined by the return map of Φ_t^e and hence Φ_t^e has positive entropy.

Conclude with two remarks:

- For $\epsilon < 0$, the dynamics of the map Φ_t^ϵ is tame, and completely predictable, except that as $\epsilon \to 0$ the dynamics approaches that of the Kuperberg flow.

- For $\epsilon > 0$, the dynamics of the map Φ_t^ϵ is chaotic, but making calculations of entropy for example, is only possible for well-chosen embeddings. We have no intuition, for example, of how to describe the nonwandering set for the flows Φ_t^ϵ.

Thank you for your attention!