Arboreal Dynamics

Midwest Dynamical Systems
Northwestern University
November 13, 2021

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Fifty Years of Midwest Dynamics

First meetings were small conferences organized by

- [Spring 1970] Ken Meyer, Sheldon Newhouse, Clark Robinson, Don Saari, Carl Simon, Bob Williams
- [Spring 1971] plus John Franks, Charles Conley, Joel Robbin

Steve Batterson gives a history of these early days of MWDS in:

Steve Smale and the prehistory of the MWDS Conference, 2009.

We will give two applications of arboreal dynamics today, one from 50 years ago, and the other much more recent about Galois actions.

Recall a result from "The Beginning" by Mike Shub:

Theorem: M a compact manifold without boundary. If there exists expansive diffeomorphism $\phi\colon M \to M$, then $\Gamma = \pi_1(M, x)$ admits a self-embedding $\varphi\colon \Gamma \to \Gamma$ and Γ has polynomial growth type.

⋆ M. Shub, *Expanding maps*, **Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)**, 1970.

⋆ C. Pugh & M. Shub, *Ergodicity of Anosov actions*, **Invent. Math.**, 1972.

⋆ M. Hirsch, C. Pugh & M. Shub, *Invariant manifolds*, **Lecture Notes in Mathematics, Vol. 583**, 1977.

Definition: A finitely generated group Γ is said to be strongly scale-invariant if there exists a proper finite index inclusion $\phi\colon \Gamma \to \Gamma$, with finite intersection in the limit. That is, for the descending chain of proper subgroups $\Gamma_0 \supset \Gamma_1 \supset \cdots$

$$\Gamma_0 = \Gamma, \Gamma_1 = \phi(\Gamma_0), \ldots, \Gamma_{\ell+1} = \phi(\Gamma_\ell), \ldots$$

then also require that $\cap_{\ell>0} \Gamma_\ell$ is a finite group.

⋆ V. Nekrashevych and G. Pete, *Scale-invariant groups*, **Groups Geom. Dyn.**, 2011.

The fundamental group of a manifold in Shub's Theorem is strongly scale invariant, as the uniformly expanding condition forces $\cap_{\ell>0} \Gamma_\ell$ to be the trivial group.

We then have the celebrated result of Gromov:

Theorem: Let Γ be a finitely generated group with polynomial growth type, then Γ is *virtually nilpotent*; that is, Γ has a finite-index nilpotent subgroup.

$\star$ M. Gromov, *Groups of polynomial growth and expanding maps*, **Inst. Hautes Études Sci. Publ. Math.**, 1981.

Conjecture: [Benjamini 1996] If Γ is strongly scale-invariant, then it contains a finitely-generated nilpotent subgroup of finite index.

Is this true, without assuming the existence of an expanding map?

Arboreal dynamics & results for *profinite groups* are used to show:

Theorem: Let Γ be strongly scale-invariant, and assume that the induced map on the profinite completion is strongly scale-invariant. Then Γ is virtually nilpotent.

$\star$ S. Hurder, O. Lukina and W. Van Limbeek, *Cantor dynamics of renormalizable groups*, **Groups Geom. Dyn.**, to appear, 2021; arXiv:2002.01565.

First review some basics of arboreal actions.

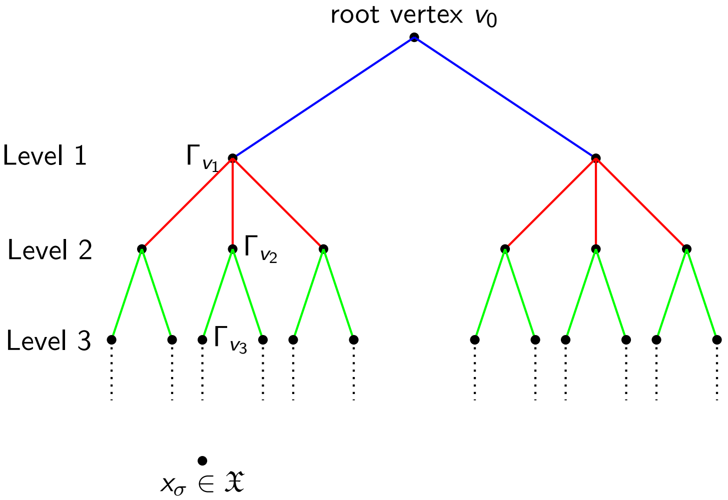
Arboreal actions:

- $\mathcal{T}$ a connected tree with root vertex v_0 and bounded valence
- Level of vertex v is number of edges in unique path to v_0
- V_n denotes the set of vertices at level n; $V_0 = \{v_0\}$

- Γ a countable group
- $\phi \colon \Gamma \times \mathcal{T} \to \mathcal{T}$ action preserving root vertex v_0
- $\phi_n \colon \Gamma \to \mathrm{Perm}(V_n), \ G_n = \mathrm{image}(\phi_n)$
- $\Gamma_{v_n} = \{\gamma \in \Gamma \mid \phi_n(\gamma)(v_n) = v_n\}$ for choice of $v_n \in V_n$

Definition: ϕ is transitive if for each n, ϕ_n acts transitively on the vertices at level n.

rooted tree and chain of isotropy groups



Boundary of $\mathcal{T}$ is totally disconnected space $\mathfrak{X}$

- Infinite path σ in $\mathcal{T} \leftrightarrow$ a sequence of vertices $\{v_0, v_1, v_2, \ldots\}$ where $v_n \in V_n$ & each pair $\{v_n, v_{n+1}\}$ connected by an edge
- σ defines point $x_\sigma \in \mathfrak{X}$ of the boundary of $\mathcal{T}$

Tree action ϕ induces boundary action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$

Proposition: Arboreal action ϕ is transitive then $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is minimal equicontinuous Cantor action.

Conversely:

- $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ an equicontinuous minimal action
- clopen set $U \subset \mathfrak{X}$ is *adapted* if set of "return times" is a group

$$\Gamma_U = \{\gamma \in \Gamma \mid \Phi(\gamma)(U) \cap U \neq \emptyset\}$$

- Collection of adapted sets $\mathcal{A}$ are partially ordered by inclusion
- Each $U \in \mathcal{A}$ corresponds to a vertex in tree $\mathcal{T}_{\mathcal{A}}$
- Action on $\mathcal{A}$ gives an arboreal action on $\mathcal{T}_{\mathcal{A}}$.

Group Chain Model

Choose a path τ in $\mathcal{T}$ with sequence of vertices $\{v_0, v_1, \ldots\}$

Get chain of finite index subgroups $\Gamma_\ell = \Gamma_{v_\ell}$ where

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

Γ acts on finite set $X_\ell = \Gamma/\Gamma_\ell \cong V_\ell$, so Γ acts on infinite product of X_ℓ and so on the Cantor space

$$\mathfrak{X}_\infty = \varprojlim \; \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell > 0} X_\ell$$

The action $\Phi_\infty \colon \Gamma \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$ is conjugate to the boundary action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ for the tree model.

Conversely:

- given group chain $\mathcal{G} = \{G_0 \supset \Gamma_1 \supset \Gamma_2 \cdots\}$
- vertices $V_n = \Gamma/\Gamma_n$ with quotient map $\Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n$
- coset inclusion defines edge from V_n to V_{n+1}
- obtain tree $\mathcal{T}_{\mathcal{G}}$ with Γ action.

Profinite Model

Let $C_\ell \subset \Gamma_\ell$ be *normal core*, largest normal subgroup:
$\Gamma/C_\ell \cong G_\ell \subset \mathrm{Perm}(V_\ell)$ is finite group with order m_ℓ

$$\Gamma_{\ell+1} \subset \Gamma_\ell \implies C_{\ell+1} \subset C_\ell \implies p_{\ell+1}\colon G_{\ell+1} \to G_\ell \text{ is onto}$$

$$\mathfrak{G}(\Phi) = \varprojlim \ \{p_{\ell+1}\colon G_{\ell+1} \to G_\ell \mid \ell \geq 0\} \subset \prod_{\ell>0} G_\ell$$

For each $\ell > 0$ have $G_\ell = \Gamma/C_\ell$ and $X_\ell = \Gamma/\Gamma_\ell$ so get action $G_\ell \times X_\ell \to X_\ell$ with isotropy group $D_\ell = \Gamma_\ell/C_\ell$.
Induces action $\widehat{\Phi}\colon \mathfrak{G}(\Phi) \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$, and restriction to image of Γ in $\mathfrak{G}(\Phi)$ is conjugate to boundary action.

The profinite action includes the action of the isotropy subgroup

$$D_\infty \cong \{\widehat{g} \in \mathfrak{G}(\Phi) \mid \widehat{\Phi}(\widehat{g})(x_\infty) = x_\infty\}$$

This is a "hidden aspect" of an arboreal action – the action can be free yet the group D_∞ is non-trivial, even be a Cantor group.

- $\mathfrak{X}_\infty \cong \mathfrak{G}(\Phi)/D_\infty$ is profinite homogeneous space
- action of D_∞ on $\mathfrak{X}_\infty$ is induced from adjoint action on $\mathfrak{G}(\Phi)$
- isotopy at $x_\infty \in \mathfrak{X}_\infty$ is $D_\infty = \varprojlim \; \{D_{\ell+1} \to D_\ell \mid \ell \geq 0\}$.

Principle: Dynamical properties of $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ are determined by adjoint action of D_∞ on profinite group $\mathfrak{G}(\Phi)$.

Many sources of examples with variety of dynamical properties:

- Towers of finite covers of compact manifold M correspond to group chains in $\Gamma = \Pi_1(M, x)$
- Solutions of polynomial equations $f^{\circ n}(z) = \alpha$ generate the tree of iterated roots, with action of its absolute Galois group, the inverse limit of Galois groups of $f^{\circ n}(z)$ for $n \geq 1$
- Actions on rooted trees, generated by finite automata; weakly branch groups & self-similar groups
- Renormalization of groups generate Cantor actions with contraction mappings

We focus on the case of renormalizable groups: Γ is finitely generated, and $\phi \colon \Gamma \to \Gamma$ is injection with finite-index image.

Abelian odometers:

$\Gamma = \mathbb{Z}^k$ and $\phi\colon \mathbb{Z}^k \to \mathbb{Z}^k$ is multiplication by a matrix $M \in GL(k, \mathbb{Z})$ with determinant $d = |M| > 1$.

For example, $k = 1$ then $M = [d]$ for $d > 1$ and so

$$\mathfrak{G}_\phi = \varprojlim_{\ell \to \infty} \mathbb{Z}/d^\ell \mathbb{Z} \cong \widehat{\mathbb{Z}}_d$$

is a classical odometer. D_ϕ is the trivial group, as $\Gamma_\ell = C_\ell$.

For $k > 1$ things get more complicated, but not much more so.

In any case, the dynamics is quite boring.

Heisenberg group: Example 1:

$$\Gamma \; = \; \left\{ [x,y,z] = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

- Let $m > 1$. Define $\phi_1[x,y,z] = [mx, my, m^2 z]$

$$\Gamma_\ell \; = \; \left\{ [x,y,z] = \begin{bmatrix} 1 & m^\ell x & m^{2\ell} z \\ 0 & 1 & m^\ell y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

Fact: Γ_ℓ is not normal in Γ, as $m^{2\ell}$ does not divide m^ℓ.

- None-the-less, the inverse limit D_{ϕ_1} is the trivial group.
- $\mathfrak{X}_{\phi_1} = \mathfrak{G}_{\phi_1}/D_{\phi_1} = \mathfrak{G}_{\phi_1}$ is a nilpotent group.

Heisenberg group: Example 2:

- $p, q > 1$ distinct primes. Define $\phi_2[x, y, z] = [px, qy, pqz]$

$$\Gamma_\ell = \left\{ [x, y, z] = \begin{bmatrix} 1 & p^\ell x & (pq)^\ell z \\ 0 & 1 & q^\ell y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

$$C_\ell = \left\{ [x, y, z] = \begin{bmatrix} 1 & (pq)^\ell x & (pq)^\ell z \\ 0 & 1 & (pq)^\ell y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

- The inverse limit $D_{\phi_2} \cong \widehat{\mathbb{Z}}_q \times \widehat{\mathbb{Z}}_p$, so action has large isotropy
- $\mathfrak{X}_{\phi_2} \cong \{[x, y, z] \mid x \in \widehat{\mathbb{Z}}_p \ , \ y \in \widehat{\mathbb{Z}}_q \ , \ z \in \widehat{\mathbb{Z}}_{pq}\}$.

Renormalizable Groups:

Given $\phi \colon \Gamma \to \Gamma$ with finite index image, define $\Gamma_\ell = \phi^\ell(\Gamma)$

$$\mathfrak{X}_\phi = \varprojlim \ \{p_{\ell+1} \colon \Gamma/\Gamma_{\ell+1} \to \Gamma/\Gamma_\ell \mid \ell \geq 0\}$$

ϕ induces the shift (or contraction) map $\tau_\phi \colon \mathfrak{X}_\phi \to \mathfrak{X}_\phi$.

For the proof of Gromov, the contraction on the universal covering of the manifold intertwines with the group action, which is used to show the action of Γ is nilpotent.

- For the renormalizable case, we want a contraction map $\widehat{\phi}\colon \mathfrak{G}(\phi) \to \mathfrak{G}(\phi)$ that intertwines with the contraction map τ_ϕ

- It is immediate that $\phi\colon \Gamma \to \Gamma$ induces a map of the *full profinite completion* $\widehat{\Gamma}$ of Γ, so we get a contraction mapping $\widehat{\phi}\colon \widehat{\Gamma} \to \widehat{\Gamma}$.

- The problem is that $\widehat{\Gamma}$ is huge, and the isotropy of the action $\widehat{\Gamma} \times \mathfrak{X}_\phi \to \mathfrak{X}_\phi$ is also huge, and may also be highly ineffective.

- The dynamics of the contraction $\widehat{\phi}$ need not be closely connected to the action of τ_ϕ. Need for image $\widehat{\Phi}(\widehat{\Gamma}) \subset \mathrm{Homeo}(\mathfrak{X}_\phi)$

- Need to show that $\widehat{\phi}\colon \widehat{\Gamma} \to \widehat{\Gamma}$ descends to a contraction $\widehat{\tau}_\phi\colon \mathfrak{G}_\phi \to \mathfrak{G}_\phi$ that intertwines the contraction τ_ϕ

- Show that $\widehat{\Phi}\colon \mathfrak{G}_\phi \times \mathfrak{X}_\phi \to \mathfrak{X}_\phi$ satisfies a *regularity property*.

Definition: An action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ is topologically free if

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\, g \in \Gamma \;,\; g \neq id \;,\; g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)$$

is meager in $\mathfrak{X}$.

Definition: For H is a topological group and $\mathfrak{X}$ is a Cantor space, an action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ is quasi-analytic if for each non-empty clopen set $U \subset \mathfrak{X}$ and $g \in H$ such that $\Phi(g)(U) = U$:

- if the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For H a countable group, the quasi-analytic and topologically free properties for an action are equivalent.

The first step in the proof of the theorem is the technical claim.

It says what happens locally, determines what happens globally.

Theorem: Let Γ be a finitely generated group and $\phi\colon \Gamma \to \Gamma$ a renormalization. Then $\widehat{\Phi}\colon \mathfrak{G}_\phi \times \mathfrak{X}_\phi \to \mathfrak{X}_\phi$ is quasi-analytic.

The proof of this result in the **GGD** paper uses ideas that appear in a variety of formats in study of renormalizable groups.

Corollary: Let Γ be a finitely generated group and $\phi\colon \Gamma \to \Gamma$ a renormalization. Then ϕ induces a contraction mapping $\widehat{\phi}\colon \mathfrak{G}_\phi \to \mathfrak{G}_\phi$ that intertwines with the contraction $\tau_\phi\colon \mathfrak{X}_\phi \to \mathfrak{X}_\phi$.

We can now take advantage of works in the literature on open contraction mappings of profinite groups by:

⋆ Baumgartner, Caprace, Glöckner, Reid, Willis:

The result we need follows from the “elementary” result

Lemma: A finite p-group is nilpotent.

See the papers:

⋆ C. Reid, *Endomorphisms of profinite groups*, **Groups Geom. Dyn.**, 2014.

⋆ W. van Limbeek, *Structure of normally and finitely non-co-Hopfian groups*, **Groups Geom. Dyn.**, 2021.

Theorem: Let $\phi\colon \Gamma \to \Gamma$ be a renormalization for the finitely generated group Γ, with associated Cantor action $(X_\phi, \Gamma, \Phi_\phi)$.

Let $\widehat{\phi}\colon \mathfrak{G}_\phi \to \mathfrak{G}_\phi$ be the embedding induced from ϕ. Then there exists a closed *pro-nilpotent normal* subgroup $\widehat{N}_\phi \subset \mathfrak{G}_\phi$ so that:

1. $\mathfrak{G}_\phi \cong \widehat{N}_\phi \rtimes D_\phi$ is a semi-direct product;
2. $\widehat{N}_\phi$ is $\widehat{\phi}$-invariant, and $\widehat{\phi}$ restricts to an open contracting embedding on $\widehat{N}_\phi$;
3. D_ϕ is $\widehat{\phi}$-invariant, and $\widehat{\phi}$ restricts to an automorphism of D_ϕ.

Moreover, let $\widehat{e} \in \mathfrak{G}_\phi$ be the identity element, then we have

$$\widehat{N}_\phi = \{g \in \mathfrak{G}_\phi \mid \lim_{n\to\infty} \widehat{\phi}^n(g) = \widehat{e}\} \quad , \quad D_\phi = \bigcap_{n>0} \widehat{\phi}^n(\mathfrak{G}_\phi) \ .$$

Here is the precise form of our main result:

Theorem: Let ϕ be a renormalization of the finitely generated group Γ. Suppose that $\bigcap_{\ell>0} \phi^{\ell}(\Gamma) \subset \Gamma$ and D_ϕ are both finite groups. Then Γ is virtually nilpotent. If both groups are trivial, then Γ is nilpotent.

Problem: Analyze the induced automorphism $\widehat{\phi}\colon D_\phi \to D_\phi$. Show that D_ϕ is always nilpotent in this case.

Problem: Suppose that ϕ is induced from a Riemannian expanding map of compact manifold M. How is the isotropy group D_ϕ related to the geometry of the manifold M?

Arboreal actions are studied in terms of their invariants:

- is the action *stable* or *wild*?
  - ⋆ Hurder, Lukina, *Wild solenoids*, **Trans. AMS**, 2019.
- is the *prime spectrum* finite?
  - ⋆ Similar to the Steiniz numbers, see

  Hurder, Lukina, *Prime spectrum of solenoidal manifolds*, preprint, 2021.
- is the action *turbulent*?
  - ⋆ Work in progress, based on the ideas in

  Gröger, Lukina, *Measures and stabilizers in group Cantor actions*, **DCDS**, 2021.

Renormalized actions are always stable, with finite prime spectrum and not turbulent.

Arboreal representations of Galois groups

• Studied in Arithmetic Dynamics starting with

⋆ R.W.K. Odoni, *The Galois theory of iterates and composites of polynomials*, **Proc. London Math. Soc. (3)**, 1985.

• Much work has been done, see the survey in

⋆ R. Jones, Galois representations from pre-image trees: an arboreal survey, "Théorie de Nombres et Applications", 2013,

and also later works by Benedetto, Ferraguti, Ingram, Jones, Juul, Looper, Pagano, Tucker, and others ...

• Applications of topological dynamics to study arboreal representations in the works of Cortez and Lukina

Let K be a number field, and let $f(x)$ be a polynomial of degree $d \geq 2$ with coefficients in the ring of integers of K.

Let $\alpha \in K$, and let $f^n(x) = f \circ f^{n-1}(x)$ be the n-th iteration.

Suppose $f^n(x) - \alpha$ is irreducible over K and has d^n distinct roots.

Adjoin these roots to K to obtain a finite extension $K(f^{-n}(\alpha))/K$.

Then the action of the Galois group

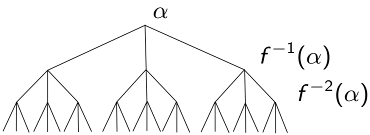
$$H_n = \mathrm{Gal}(K(f^{-n}(\alpha))/K)$$

permutes the roots of $f^n - \alpha$ and this action is transitive.

To obtain a tree $\mathcal{T}$, let

$V_n = \{ \text{ roots of } f^n(x) - \alpha \}$.

Draw an edge from b to a if $f(b) = a$.



Finite Galois groups H_n act transitively on vertices $V_n = f^{-n}(\alpha)$.

There are homomorphisms $H_n \to H_{n-1}$, which preserve $\mathcal{T}$.

Thus there is a profinite group

$$\mathcal{G} = \varprojlim \{H_n \to H_{n-1}\} \subset Aut(\mathcal{T}).$$

This group is the image of the arboreal representation

$$\rho_{f,\alpha} : \mathrm{Gal}(K^{sep}/K) \to Aut(\mathcal{T}),$$

where K^{sep} is a separable closure of K.

Question: For which polynomials is the image group $\mathcal{G}$ finite index in the full automorphism group $Aut(\mathcal{T})$?

$\star$ R.W.K. Odoni, *The Galois theory of iterates and composites of polynomials*, **Proc. London Math. Soc. (3)**, 1985.

Question: What can you say about the image group $\mathcal{G}$ for specific classes of polynomials, for instance, for PCF quadratic polynomials?

We introduce another dynamical property of arboreal actions related to this question, its local regularity.

Definition: For H is a topological group and $\mathfrak{X}$ is a Cantor space, an action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic if there exists $\epsilon > 0$, such that for each clopen set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$:

- for $g \in H$ and adapted clopen set $V \subset U$, if the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ is the identity on U.

Definition: An equicontinuous Cantor action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ is stable if the associated profinite action $\widehat{\Phi}\colon \widehat{\Gamma} \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic. The action is said to be wild otherwise.

Theorem: [Lukina 2018] If $\mathcal{G}$ has finite index in $Aut(\mathcal{T})$, then the action of $\mathcal{G}$ on the tree boundary $\mathfrak{X}$ is wild.

Theorem: [H-Lukina 2019] Suppose that $\mathcal{G} \subset Aut(\mathcal{T})$ is abelian, then the action is not wild.

Corollary: If $\mathcal{G}$ is abelian, then it must have infinite index in $Aut(\mathcal{T})$ (for any $d \geq 2$).

Remark: This statement was obtained for quadratic polynomials ($d = 2$) by different arguments in

⋆ Ferraguti, Pagano, *Constraining images of quadratic arboreal representations*, **IMRN**, 2020.

Theorem: [H-Lukina 2021] Suppose that $\mathcal{G} \subset Aut(\mathcal{T})$ is topologically finitely generated, virtually nilpotent and has finite prime spectrum, then the action is not wild.

Problem: How to relate the dynamics of the image of the arboreal representation with its number theory properties?

Thank you for your attention!

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