TYPE II INDEX THEOREMS FOR MANIFOLDS WITH BOUNDARY

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THESIS

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In memory of my parents I dedicate this thesis to them.

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SUMMARY

This thesis focuses on proving index theorems for noncompact manifolds with boundary and the study of the corresponding eta invariant defect terms. More specifically, we prove a von Neumann index theorem for Dirac operators on coverings, and for leaves of foliations, of compact manifolds with boundary. These theorems are the analogue, in the noncompact case, of a celebrated theorem of Atiyah, Patodi and Singer which relates the index of boundary value problems for compact manifolds with boundary, with Pontryagin numbers and eta invariants.

1. INTRODUCTION

This thesis focuses on proving Breuer index theorems for noncompact manifolds with boundary. More specifically we prove an index theorem for Dirac operators on coverings, and for leaves of foliations of compact manifolds with boundary. This theorem is the analogue of a celebrated theorem of Atiyah, Patodi and Singer [3], which relates the index of boundary value problems for compact manifolds with boundary, with Pontryagin numbers and eta invariants.

Cheeger and Gromov (cf. [7] and [8]) proved the existence of eta invariants for signature operators on Galois coverings of compact manifolds. They also proved a key estimate for these eta invariants. In our effort (Ramachandran, 1989, unpublished), to understand these Cheeger-Gromov estimates for Dirac operators we were led to a unified proof of existence and estimates for the eta invariants for Dirac operators on coverings and foliations of compact manifolds. (In the case of foliations, the eta invariant for Dirac operators, was independently proved to exist, by G. Peric). (We prove these results in Chapter 3 of this thesis.) The key techniques used, are estimates for the heat kernel of Dirac operators (Ramachandran, 1989, unpublished), [29] and the definition of a tempered measure associated to the spectral projections of these Dirac operators.

Before we proceed further we state our main theorems. See Chapter 6 for a more precise statement.

Theorem 1.1. Let D be a Dirac operator on a compact manifold M with boundary, acting on a graded Clifford bundle S, with grading operator ϵ. We assume that the data (D, S, ϵ) has a product structure, near the boundary, (in the sense of Definition 2.1.1). Let $\widetilde{M}$ be a Galois covering of M with Galois group Γ. Let $(\widetilde{D}, \widetilde{S}, \widetilde{\epsilon})$ be the lift of the data (D, S, ϵ) to $\widetilde{M}$. Let $\widetilde{B}$ be the Atiyah, Patodi, Singer boundary condition of Definition 2.1.3 associated to $\widetilde{D}$. Then the Γ index (in the sense of Chapter 5) of the Brauer Fredholm operator $\widetilde{D}$ with boundary condition $\widetilde{B}$ is given by

$$\text{ind}_\Gamma(\widetilde{D}) = \int_M ch(\sigma_D)Td(M) - \frac{\eta_\Gamma(0) + h}{2} \tag{1}$$

where $\eta_\Gamma(0)$ is the Γ eta invariant on the boundary $\partial\widetilde{M}$. Next we state our index theorem for foliations.

Theorem 1.2. Let $(M, \mathcal{F})$ be a compact foliated manifold with boundary, with the foliation $\mathcal{F}$ transverse to the boundary. Let $D_\mathcal{F}$ be a leafwise Dirac operator on S a Clifford bundle over $T\mathcal{F}$, the tangent bundle to the foliation $\mathcal{F}$. We assume that S is graded, with grading operator ϵ. Let ν be a holonomy invariant transverse measure for the foliation $\mathcal{F}$. Further assume that the data $(D_\mathcal{F}, S, \epsilon)$ has a product structure near the boundary (in the sense of Definition 2.3.2). Let $B_\mathcal{F}$ be the family of Atiyah, Patodi, Singer boundary conditions corresponding to the family $D_\mathcal{F}$. Then the ν-index of this family of Brauer Fredholm operators (in the sense of Chapter 5) is finite and

$$\operatorname{ind}_\nu(D_\mathcal{F}) = \langle ch(\sigma_{D_\mathcal{F}})Td(M), \nu\rangle - \frac{\eta_\nu(0) + h}{2}$$

where $\eta_\nu(0)$ is the foliation eta invariant of Chapter 2.

The proofs of Theorems 1 and 2 follow, in outline, that of [3], making use of a reformulation of the proof in [3] by (Roe, 1988, unpublished). The extension of this approach to the open manifold case requires several technical innovations. Each chapter is a step of the proof. The proof is completed in Chapter 6.

In Chapter 2, we describe Sobolev spaces for manifolds with boundary. We also state the Spectral decomposition theorem for self-adjoint elliptic operators due to Browder and Garding independently [11], [13]. We use this theorem to prove restriction theorems for these Sobolev spaces. Philosophically we think of the Browder-Garding theorem as giving us a "Fourier transform" on the boundary. This will be the key to the way the definitions are formulated.

Chapter 3 deals with type II eta invariants and is based on our work (Ramachandran, 1989, unpublished) as described earlier. The key technical chapters from the point of view of analysis, is Chapters 4 and 5. Here we prove the self-adjointness of the boundary value problem and the Sobolev regularity of the boundary value problem. In Chapter 5 we deal with the corresponding parabolic initial boundary value problem and construct a parametrix as in [3] which plays a key role in the evaluation of the index.

The techniques of proof in Chapters 4 and 5 are very much in the spirit of [3] based on the reformulation in (Roe, 1988, unpublished). The spectral transform given by the theorem of Browder and Garding, along with Fourier transform in the direction normal to the boundary, are used to reduce all estimates near the boundary to a one dimensional problem with parameters. Most of these results are very classical in spirit.

In Chapter 6 we formulate the Brauer index of these boundary values problems. The main novelty in the foliation case is that, we work with the equivalence relation given by the foliation rather than the holonomy groupoid. The methods in this thesis also applies to the holonomy groupoid and we will study this from a KK theoretic view point in a future paper. By working with the foliation equivalence relation, one finds that the index theorem applies directly to the leaves rather than to their holonomy covers. The boundary value problem for the leaves is more natural than that for the covers.

In the last chapter, which is Chapter 7, we complete the proof of our index theorem. In the computation of the Index we follow quite closely [3]. There are some modifications required because these type II eta invariants do not admit meromorphic continuation to the right-half plane. The philosophy behind the computations in this chapter is to replace the summation signs in the original work of Atiyah, Patodi and Singer (especially section 2, [3]) by an integral with respect to a tempered measure.

Finally we make a few concluding remarks. Bismut and Cheeger [5], have proved a much stronger version of Theorem 1.2 when the foliation is a fibration of compact manifolds and the boundary Dirac operators are invertible. Our main contributions are the formulation of the von Neumann boundary value problem, the use of the spectral theorem of Browder and Garding to prove the self-adjointness of the boundary value problem, and the modifications necessary to use the computations in [3] to compute the index.

2. PRELIMINARIES

This chapter introduces some terminology and basic facts about Sobolev spaces on manifolds with boundary. We state the spectral decomposition theorem of Browder and Garding for self-adjoint elliptic operators.

2.1. The Atiyah Patodi Singer boundary condition.

Our data will be the following. M will denote a C^∞ complete Riemannian manifold with C^∞ boundary N. By the data (D, S, ϵ) we mean a Dirac operator D acting on smooth sections of a graded Clifford bundle S, with grading operator ϵ. For more details see Roe [26], [28], Lawson and Michelson [19].

Definition 2.1.1. By a product structure on the given data (D, S, ϵ) in a neighborhood of the collar $[0,1] \times N$ we mean the following:

1. The Riemannian metric on M is a product in a neighborhood of $[0,1] \times N$.
2. The Dirac operator has the special form $D = \sigma(\frac{\partial}{\partial y} + Q)$ where σ is the Clifford multiplication by the unit normal to the boundary N, y is the co-ordinates normal to the boundary and Q is a Dirac operator on N and Q is independent of y.

We further assume that the data (D, S, ϵ) has a product structure as in Definition 2.1.1. We also assume that on the geometric double of M which we denote by dbM, the doubled data $(\widehat{D}, \widehat{S}, \hat{\epsilon})$ satisfy the bounded geometry hypothesis in section 2 of Roe [26]. Such examples naturally arise in the study of Dirac operators on Galois coverings of compact manifolds with boundary and leafwise Dirac operators on foliations of compact manifolds with boundary with the foliation transverse to the boundary. All manifolds with boundary considered in this thesis will be smooth.

$C_c^\infty(M; S)$ will denote compactly supported smooth sections of S smooth up to the boundary of M. Then we have the following Green's formula

$$\langle s_1, Ds_2 \rangle - \langle Ds_1, s_2 \rangle = \int_N \langle \sigma b s_1, s_2 \rangle \tag{2.1.1}$$

$\langle\ ,\ \rangle$ denotes the L^2 inner product on sections of S, s_1 and s_2 belong to $C_c^\infty(M;S)$. bs denotes the restriction of the section $s \in C_c^\infty(M;S)$ to the boundary N.

Following Roe (1988, unpublished) we make the following definition.

Definition 2.1.2. We say $B : C_c^\infty(N;S) \to C^\infty(N;S)$ defines a self-adjoint boundary condition if

1. B extends to a bounded operator on $L^2(N;S)$.
2. $B = B^*$ and $\sigma B + B\sigma = \sigma$.

If $s_1, s_2 \in C_c^\infty(M;S)$ satisfying $Bbs_1 = 0$ and $Bbs_2 = 0$ then

$$\langle s_1, Ds_2\rangle = \langle Ds_1, s_2\rangle. \tag{2.1.2}$$

Remark. If we analyze the interaction of the product structure of the data (D, S, ϵ) with the grading operator ϵ we find that Q is essentially self-adjoint on $C_c^\infty(N;S)$ and commutes with ϵ. Since ϵ is an involution, diagonalizing it splits $S = S^+ \oplus S^-$ and Q preserves this decomposition. We label Q restricted to sections of $S^\pm$ by $Q_\pm$ respectively.

Definition 2.1.3. The Atiyah, Patodi, Singer boundary condition henceforth abbreviated to A.P.S. boundary condition, is the operator B which restricts to the projection onto the nonnegative part of the spectrum of Q_+ on the $+$ part of the grading and restricts to the projection onto the positive part of the spectrum of Q_- on the $-$ part of the grading. One easily checks that B satisfies the conditions of Definition 2.1.2.

2.2. Sobolev Spaces.

We now define nonlocal Sobolev spaces for manifolds with boundary. Henceforth the assumptions of section 2.1 carry over to the rest of the thesis. The main references for this section will be Roe [26], (Roe, 1988, unpublished) and, Lions and Magenes [20].

Definition 2.2.1. Let k be a nonnegative integer. The Sobolev space $W^k(M;S)$ is the completion of $C_c^\infty(M;S)$ in the norm

$$\|s\|_k = \{\|s\|^2 + \|Ds\|^2 + \ldots + \|D^k s\|^2\}^{1/2}$$

where $\|s\| = \langle s, s\rangle$. For k a negative integer $W^k(M;S)$ is the dual of $W^{-k}(M;S)$ considered as a space of distributional sections.

Also $W^{-\infty}(M;S) = \cup\, W^k(M;S)$ and $W^{\infty}(M;S) = \cap\, W^k(M;S)$; $W^{\infty}(M;S)$ has the obvious Frechet topology and $W^{-\infty}(M;S)$ is equipped with weak topology that it inherits as the dual of $W^{\infty}(M;S)$.

For k nonnegative we observe that any element of $W^k(M;S)$ can be extended to $db(M)$ with control over the norm.

Proposition 2.2.1. There is a bounded linear operator

$$E_k : W^k(M;S) \to W^k(dB(M);\widehat{S})$$

for every integer $k \geq 0$ with the property that $E_k f$ restricted to M is f.

Proof. If $s \in W^k(M;S)$ vanishes in a neighborhood of the collar $[0,\frac{1}{2})\times N$ we define

$$E_k s = 0.$$

By using a bump function, it is enough to define E_k for sections supported in the collar $[0,1)\times N$. Let $s \in W^k(M;S)$ be supported in $[0,1)\times N$. Then define

$$E_k s(y,n) = \begin{cases} s(y,n) & \text{if } y \geq 0 \\ \sum\limits_{j=1}^{k+1} \alpha_j s(-jy,n) & \text{if } y < 0 \end{cases}$$

where α_j's are chosen so that the first k derivatives in the y direction match at $y = 0$. This implies that the α_j's satisfy the following system of equations

$$\sum_{\ell=1}^{k+1}(-1)^{\ell} j^{\ell}\alpha_{\ell} = 1 \text{ for } 0 \leq j \leq k-1.$$

The determinant of this linear equation is not zero, so the appropriate α_j's can be found.

Q.E.D.

Definition 2.2.2. Let r be a nonnegative integer. The uniform C^r space $UC^r(M;S)$ is the Banach space of C^r sections of S, C^r up to the boundary of M such that the norm

$$|||s|||_r = \sup\{|\nabla_{v_1}\ldots\nabla_{v_q}s(m)|\}$$

is finite, where the supremum is taken over all $m \in M$ and all collections $v_1,\ldots,v_q$ $(0 \leq q \leq r)$ of unit tangent vectors at m. Also $UC^{\infty}(M;S) = \bigcap\limits_r UC^r(M;S)$.

Proposition 2.2.2. The Frechet space $W^\infty(M;S)$ is continuously included in $UC^\infty(M;S)$.

Proof. By Proposition 2.2.1 $W^\infty(M;S)$ is continuously included $W^\infty(db(M);\widehat{S})$. Proposition 2.8 of Roe [26] implies the result. Q.E.D.

Proposition 2.2.3. A continuous linear operator from $W^{-\infty}(M;S)$ to $W^\infty(M;S)$ is represented by a smoothing kernel smooth up to the corners on $M \times M$. Also the kernel and all its covariant derivatives are uniformly bounded.

Proof. The strategy for the proof is the same as in Proposition 2.9 of Roe [26]. We use Proposition 2.2.2 instead of Proposition 2.8 used in Roe [27]. Q.E.D.

In the remark following Definition 2.1.2 we mentioned that Q is essentially self-adjoint on $C_c^\infty(N;S)$. This follows immediately by a minor modification of the proof in Chernoff [9]. The main difference between Chernoff [9] and our context is that N need not be connected, but the fact that N is a disjoint union of countably many complete Riemannian manifolds implies that the proof in [9] can be used here. We leave the details to the reader.

Since Q is essentially self-adjoint it has a unique closure which we denote by Q. By the spectral theorem we can define Sobolev spaces on the boundary N as follows.

Definition 2.2.3. Let k be a nonnegative half integer. Then

$$W^k(N;S) = \text{domain}(Q^k)$$

if k is a nonnegative integer. Then it coincides with the closure of $C_c^\infty(N;S)$ under the norm

$$\{\|s\|^2 + \|Qs\|^2 + \ldots + \|Q^k s\|^2\}^{1/2}.$$

We now state the generalized eigenfunction expansion theorem for Q due to Browder and Garding.

Theorem 2.2.1. (Browder and Garding). There exists a sequence of smooth sectional maps $e_j : \mathbf{R} \times N \to S$, namely e_j is measurable and for each $\lambda \in \mathbf{R}$ $e_j(\lambda, \ )$ is a smooth section of S over N, and measures μ_j on $\mathbf{R}$ such that

$$Qe_j(\lambda, n) = \lambda e_j(\lambda, n). \tag{2.2.1}$$

Further, the map

$$(Vs)_j(\lambda) = \int_N (s(n) \mid e_j(\lambda, n)) d\mathrm{vol}_N \tag{2.2.2}$$

defined on $C_c^\infty(N; S)$ extends to a isometry of Hilbert spaces

$$V : L^2(N; S) \to \bigoplus_j L^2(\mu_j) \tag{2.2.3}$$

where the sum on the right hand side of (2.2.3) is the Hilbert direct sum. Also (|) inside the integral sign of (2.2.2) is the fibre wise inner product on S. Further V intertwines the operator $f(Q)$ with multiplication by $f(\lambda)$,

$$\text{domain } f(Q) = \left\{ s \mid \sum_j \int_{\mathbf{R}} |f(\lambda)|^2 |(Vs)_j(\lambda)|^2 d\mu_j(\lambda) < \infty \right\} \tag{2.2.4}$$

and

$$\int_N |s(n)|^2 d\mathrm{vol}_N = \sum_j \int_{\mathbf{R}} |(Vs)_j(\lambda)|^2 d\mu_j(\lambda). \tag{2.2.5}$$

Proof. See pages 300 - 307 Dieudonne [11], Dunford and Schwartz [13].

We now sketch the proof of a restriction theorem for the Sobolev spaces that we defined earlier in this section.

Theorem 2.2.2. $b : C_c^\infty(M; S) \to C_c^\infty(N; S)$ extends to a bounded operator

$$b : W^k(M; S) \to W^{k-1/2}(N; S) \qquad \text{for any natural number } k.$$

Proof. Again it is enough to consider elements of $W^k(M; S)$ supported in the collar $[0, 1) \times N$. We extend them to elements of $W^k(db(M); \widehat{S})$ supported in $(-1, 1) \times N$. Note that our data $(\widehat{D}, \widehat{S}, \widehat{\epsilon})$ has product structure on $(-1, 1) \times N$. Hence we can consider them as elements of $W^k((-\infty, \infty) \times N; \widehat{S})$. Therefore it is enough to prove that

$$W^k((-\infty, \infty) \times N; \widehat{S}) \xrightarrow{r} W^{k-1/2}(0 \times N; S) \tag{2.2.6}$$

where r is the restriction map is bounded. Because our data has product structure on $(-\infty, \infty) \times N$, we have $D^2 = -\frac{\partial^2}{\partial y^2} + Q^2$. Hence $W^k((-\infty, \infty) \times N; \widehat{S})$ is the closure of $C_c^\infty((-\infty, \infty) \times N; \widehat{S})$ with respect to the following norm

$$\left\{ \langle s, s \rangle + \left\langle \left(-\frac{\partial^2}{\partial y^2} + Q^2 \right) s, s \right\rangle + \ldots + \left\langle \left(-\frac{\partial^2}{\partial y^2} + Q^2 \right)^k s, s \right\rangle \right\}^{1/2}.$$

Using the map V given in (2.2.3) in the N direction and the Fourier transform in the y direction we see that

$$(2.2.7)\quad W^k((-\infty,\infty)\times N;\widehat{S}) = \left\{ s \in L^2 \mid \sum_j \int_{\mathbf{R}} (1+\lambda^2+\xi^2)^k |(\widehat{V}s)_j(\lambda,\xi)|^2 d\mu_j d\xi < \infty \right\}$$

where $\widehat{}$ inside the integral in (2.2.7) is the Fourier transform in the y direction. We call the operator V, the spectral transform. The rest of the proof follows the proof of Theorem B.1.11 of Hormander [17]. Q.E.D.

2.3. Foliations of manifolds with boundary.

Definition 2.3.1. A C^∞ manifold with boundary with the foliation transverse to the boundary is a C^∞ manifold with boundary with a collection of open sets $\{U_\alpha\}$ covering M and homeomorphisms

$$\varphi_\alpha : U_\alpha \to V_\alpha \times W_\alpha$$

with V_α open in $\mathbf{H}^p = \{(x_1, \dots, x_p) \in \mathbf{R}^p \mid x_1 \geq 0\}$ and

W_α open in $\mathbf{R}^q$ which satisfies the following condition:

1. If we write $\varphi_\alpha = (v, w)$ the co-ordinate changes are given by C^∞ map φ and local diffeomorphism ψ namely

$$v' = \varphi(v,w) \qquad \text{and} \qquad w' = \psi(w).$$

Further the collection $\{U_\alpha\}$ is assumed maximal among all such collections. Since co-ordinate changes smoothly transform level surfaces w = constant to w' = constant, the level sets coalesce to form maximal connected sets called leaves and the manifold M is foliated by these leaves and these leaves intersect the boundary transversely to give a smooth foliation of the boundary with same codimension as the foliation of the interior of M. We denote the foliation by $\mathcal{F}$ and $(M, \mathcal{F})$ the manifold with the foliation. If we consider the tangent bundle to the leaves of $\mathcal{F}$ then we get a smooth vector bundle over M which is a sub-bundle of the tangent bundle of M. We denote this sub-bundle by $T\mathcal{F}$.

We say the foliation $\mathcal{F}$ is transversely orientable if the quotient bundle $TM/T\mathcal{F}$, is orientable.

From now on we assume that our foliation is transversely oriented.

Fact 2.3.1. $(M, \mathcal{F})$ as above then there is a collar W on N such that $\mathcal{F}|_W$ is diffeomorphic to $[0,1) \times (\mathcal{F}|_N)$.

Proof. See page 43, Hector and Hirsch [16].

Let $D_{\mathcal{F}}$ be a leafwise Dirac operator on S where S is a Clifford bundle over $T\mathcal{F}$, (cf. Roe [27]).

Definition 2.3.2. We say the data $(D_{\mathcal{F}}, S, \epsilon)$ where S is a graded Clifford bundle over $T\mathcal{F}$ with grading ϵ has a product structure in a neighborhood of the foliation collar $[0,1] \times \mathcal{F}|_N$ if

1. the Riemannian metric on M is a product in a neighborhood of $[0,1] \times N$.
2. The leafwise Dirac operators $D_{\mathcal{F}}$ has the form $D_{\mathcal{F}} = \sigma(\frac{\partial}{\partial y} + Q_{\mathcal{F}})$ where $Q_{\mathcal{F}}$ is a leafwise Dirac operator on $\mathcal{F}|_N$. σ is the Clifford multiplication of the unit normal along the leaves to the boundary foliation.

We remark that, implicitly included in our data $(D_{\mathcal{F}}, S, \epsilon)$ is the fact that M has a Riemannian metric.

3. TYPE II ETA INVARIANTS

This chapter proves the existence of eta invariants for Dirac operators on coverings of compact manifolds, and for leafwise Dirac operators on foliations of compact manifolds. Cheeger and Gromov [8] proved the existence of eta invariants for the signature operator on coverings of compact manifolds. Peric [24], proved the existence of the foliation eta invariant. In our study of Cheeger-Gromov estimates and generalizations of A.P.S. theorem to foliations we discovered a proof of existence of these eta invariants which also gave the Cheeger-Gromov estimates (cf. Ramachandran, 1989, unpublished). These Cheeger-Gromov estimates were applied in Douglas, Hurder and Kaminker [14]. This chapter presents our proof. Section 3.1 deals with the covering eta invariant and section 3.2 with the foliation eta invariant.

3.1. Eta invariant for coverings.

Let N be a compact Riemannian manifold without boundary. Let D be a Dirac operator on S a Clifford bundle. Now D is essentially self-adjoint and De^{-tD^2} defined by the spectral theorem is a smoothing operator. See Roe [28] for a proof of these statements. We assume that the pointwise trace of De^{-tD^2} is $0(t^{1/2})$. This local cancellation property was first observed by Bismut and Freed [4] for Dirac operators arising in geometric situations. Following them we call this local cancellation property the Bismut-Freed cancellation property.

Let $\widetilde{N}$ be a Γ principal bundle over N where Γ is a discrete countable group. Let $\widetilde{D}$ and $\widetilde{S}$ be the lifts of D and S respectively to $\widetilde{N}$. By Atiyah [1], $\widetilde{D}$ acting on $C_c^\infty(\widetilde{N};\widetilde{S})$ is essentially self-adjoint.

$$\begin{aligned}&\mathrm{End}_\Gamma(L^2(\widetilde{N};\widetilde{S}))\\&=\{\text{Bounded operator on } L^2(\widetilde{N};\widetilde{S}) \text{ commuting with } \Gamma\}\end{aligned}$$

If $T\in \mathrm{End}_\Gamma(L^2(\widetilde{N};\widetilde{S}))$ is an integral operator with smooth kernel k_T then we define the Γ trace of T as follows.

Definition 3.1.1. $tr_\Gamma(T) = \int_F Tr_x k_T(x,x)dx$ where F is a fundamental domain for the Γ action on $\widetilde{N}$ and Tr_x is the matrix trace on $\mathrm{End}(\widetilde{S}_x)$.

Definition 3.1.2. Let $RB(\mathbf{R}) = \{$Borel function $\mathbf{R}$ which are rapidly decreasing$\}$. By rapidly decreasing we mean

$$\sup_{x\in\mathbf{R}}(1+|x|)^k|f(x)| < C_k \text{ for every positive integer } k.$$

Remark. By the spectral theorem and Sobolev lemma $f(\widetilde{D}) \in \mathrm{End}_\Gamma(L^2(\widetilde{N};\widetilde{S}))$ and is a integral operator with smooth kernel for $f \in RB(\mathbf{R})$. See Chapter 13 of Roe [28] for more details. Hence $tr_\Gamma f(\widetilde{D})$ is finite for $f \in RB(\mathbf{R})$.

Let the kernel of the integral operator $e^{-t\widetilde{D}^2}$ be $K_t(x,y)$ and the kernel $e^{-t\Delta_g}$ be $k_t(x,y)$ where Δ_g is the Laplace-Beltrami operator on $\widetilde{N}$. We have

Lemma 3.1.1.

$$|Tr_x\widetilde{D}K_t(x,x)| < At^{1/2} \tag{3.1.2}$$

where A is constant depending on the local geometry of $\widetilde{N}$, $\widetilde{S}$, the dimension of $\widetilde{N}$ and the rank of $\widetilde{S}$.

Proof. Let $P(x,y,t)$ be a smooth parametrix for the kernel $K_t(x,y)$, supported in an ϵ neighborhood of the diagonal of $\widetilde{N}\times\widetilde{N}$ where ϵ is one-half the injectivity radius of $\widetilde{N}$. We also assume that $P(x,y,t)$ satisfies the following additional properties.

$$P(x,y,t) \in \mathrm{Hom}(\widetilde{S}_y,\widetilde{S}_x) \tag{3.1.3}$$

$$\left(\frac{\partial}{\partial t} + \widetilde{D}^2\right)P(x,y,t) \text{ is } 0(t^m) \tag{3.1.4}$$

$$\widetilde{D}\left(\frac{\partial}{\partial t} + \widetilde{D}^2\right)P(x,y,t) \text{ is } 0(t^{m-1}) \tag{3.1.5}$$

m is chosen so that

$$\int_0^t (t-s)^{-n/2}s^{m-1}ds \text{ is } 0(t^{1/2}) \text{ where } n = \dim N \tag{3.1.6}$$

$$\|P(x,y,t)\|_{x,y} \leq At^{-n/2} \qquad 0 \leq t \leq 1 \tag{3.1.7}$$

'A' in this chapter will denote constant depending on the data described in the statement of the lemma. If the constant A appears in two places repeated by a little bit of text then they are different. For a construction of such a parametrix P see Patodi [23]. By Theorem 3.5, page 294 of Rosenberg [29] we have

$$\|K_t(x,y)\|_{x,y} \leq e^{-ct}k_t(x,y) \tag{3.1.8}$$

where c depends only on the local geometry. The proof in Rosenberg of (3.1.8) uses probabilistic methods. For a proof of (3.1.8) based on the work of Dodziuk [14] see (Ramachandran, 1989, unpublished).

By Duhamel's principle (cf. Roe [28])

$$\begin{aligned} Tr_x(\widetilde{D}K_t(x,x)) &= Tr_x(\widetilde{D}P(x,x,t)) \\ &+ \int_0^t ds Tr_x\left(\int_{\widetilde{N}} K_{t-s}(x,y)\left(\frac{\partial}{\partial s}+\widetilde{D}^2\right)\widetilde{D}P(y,x,s)d\mathrm{vol}_{\widetilde{N}}(y)\right) \end{aligned} \tag{3.1.9}$$

Bismut and Freed [4], showed that

$$|Tr_x(\widetilde{D}P(x,x,t))| \leq At^{1/2}. \tag{3.1.10}$$

Therefore

$$\begin{aligned} &\left|Tr_x \int_{\widetilde{N}} K_{t-s}(x,y)\widetilde{D}\left(\frac{\partial}{\partial s}+\widetilde{D}^2\right)P(y,x,s)d\mathrm{vol}_{\widetilde{N}}(y)\right| \\ &\qquad A\int_{B(x,\epsilon)} \|K_t(x,y)\|_{x,y}s^{m-1} \leq A \ \mathrm{vol}(B(x,\epsilon))(t-s)^{-n/2}s^{m-1} \end{aligned}$$

where we have used standard estimates heat kernel for the Laplace Beltrami operator

$$0 \leq k_t(x,y) \leq At^{-n/2}$$

see Chavel [6] and (3.1.5). Here $B(x,\epsilon)$ is the metric ball of radius ϵ centered at x. The bounded geometry of $\widetilde{N}$ implies that

$$\left|\int_0^t Tr_x \int_{\widetilde{N}} K_{t-s}(x,y)\widetilde{D}\left(\frac{\partial}{\partial s}+\widetilde{D}^2\right)P(y,x,s)d\mathrm{vol}_{\widetilde{N}}(y)\right| \leq At^{1/2}. \tag{3.1.11}$$

The estimate of (3.1.2) is completed by using estimates (3.1.10) and (3.1.11).

Q.E.D.

By the remark following Definition 3.1.2 we have for an $f \in \mathcal{S}(\mathbf{R})$ ($\mathcal{S}(\mathbf{R})$ denotes the Schwartz space) $tr_\Gamma(\widetilde{D})$ is finite. Further if $f \geq 0$ then $tr_\Gamma f(\widetilde{D}) \geq 0$.

Consider the linear functional

$$I(f) = tr_\Gamma f(\widetilde{D}) \quad \text{for} \quad f \in \mathcal{S}(\mathbf{R}). \tag{3.1.12}$$

By standard methods in harmonic analysis

$$I(f) = \int_{\mathbf{R}} f \, dm_\Gamma \tag{3.1.13}$$

where m_Γ is a tempered measure on $\mathbf{R}$, namely there exists a positive integer ℓ such that

$$\int_{\mathbf{R}} \frac{1}{(1+|x|)^\ell} dm_\Gamma \text{ is finite.}$$

We now proceed to the main theorem of this section. Let

$$\eta_\Gamma(0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2}) dt. \tag{3.1.14}$$

Theorem 3.1.1.

$$|\eta_\Gamma(0)| \leq A \text{ vol}(N) \tag{3.1.15}$$

where A is a constant satisfying the properties described in Lemma 3.1.1.

Proof. As in Cheeger and Gromov [7] we split the integral in (3.1.14) into the following integrals and estimate them separately.

$$\begin{aligned} &\frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2}) dt \\ &\quad = \frac{1}{\Gamma(1/2)} \int_0^1 t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2}) dt + \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2}) dt. \end{aligned}$$

Now

$$\begin{aligned} &\left| \frac{1}{\Gamma(1/2)} \int_0^1 t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2}) dt \right| \\ &\quad \leq \int_0^1 \int_F t^{-1/2} |Tr_x(\widetilde{D}K_t(x,x))| d\text{vol}_{\widetilde{N}} dt \\ &\quad \leq A \int_0^1 t^{-1/2} t^{1/2} dt \int_F d\text{vol}_{\widetilde{N}} = A \text{ vol}(N) \end{aligned} \tag{3.1.16}$$

where the final inequality follows from (3.1.2).

We estimate

$$\begin{aligned}
(3.1.17)\qquad & \left|\frac{1}{\Gamma(1/2)}\int_1^\infty t^{-1/2} tr_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2})dt\right| \\
&\leq \left|\frac{1}{\Gamma(1/2)}\int_1^\infty t^{-1/2}\int_{\mathbf{R}} \lambda e^{-t\lambda^2} dm_\Gamma(\lambda)dt\right| \\
&\leq \frac{1}{\Gamma(1/2)}\int_1^\infty t^{-1/2}\int_{\mathbf{R}} |\lambda| e^{-t\lambda^2} dm_\Gamma(\lambda)dt \\
&= \frac{1}{\Gamma(1/2)}\int_{\mathbf{R}} |\lambda| e^{-\lambda^2}\int_1^\infty t^{-1/2} e^{-(t-1)\lambda^2} dt\; dm_\Gamma(\lambda) \\
&\leq \int_{\mathbf{R}} e^{-\lambda^2} dm_\Gamma(\lambda) = tr_\Gamma(e^{-\widetilde{D}^2}).
\end{aligned}$$

By (3.1.8), $tr(e^{-\widetilde{D}^2}) \leq A$ vol(N). Combining (3.1.16) and (3.1.17) we have (3.1.15).

Q.E.D.

3.2. Eta invariants for foliations.

Let N be a C^∞ closed Riemannian manifold and $\mathcal{F}$ a smooth foliation on N. For each leaf L of $\mathcal{F}$ we denote the volume element of the induced Riemannian metric by d vol$_L$. Let ν be a holonomy invariant transverse measure and $D_\mathcal{F}$ a leafwise Dirac operator acting on a Clifford bundle S (cf. Roe [27], Moore and Schochet [22]). Further we assume that $D_\mathcal{F}$ restricted to each leaf satisfies the Bismut-Freed cancellation property. The main references for this section are Roe [26], [27], Moore and Schochet [22] and Connes [10]. We now state the main theorem of this section.

Theorem 3.2.1. The integral

$$(3.2.1)\qquad \eta_\mathcal{F}(0) = \frac{1}{\Gamma(1/2)}\int_0^\infty t^{-1/2} tr_\nu(D_\mathcal{F} e^{-tD_\mathcal{F}^2})dt$$

exists and satisfies the following inequality

$$(3.2.2)\qquad |\eta_\mathcal{F}(0)| \leq A\ \mu(N)$$

where tr_ν is the foliation trace given by the holonomy invariant transverse measure ν, A a constant depending on the rank of the vector bundle S and the local geometry of the leaves and μ is the total measure on N given by combining the leafwise volume elements with transverse measure ν.

Proof. The Dirac operator D_L on each leaf L is essentially self-adjoint on $C_c^\infty(L; S_L)$. By the spectral theorem, $f(D_L)$ is a bounded operator, for f a bounded Borel function on $\mathbf{R}$. If $f \in \mathcal{S}(\mathbf{R})$ then $f(D_L)$ is an integral operator with smooth kernel, (cf. Roe [26]).

We can define a smooth measure on the leaf L, $Tr_{pt}f(D_L)d\text{vol}_L$ if $f \in \mathcal{S}(\mathbf{R})$. Here $Tr_{pt}f(D_L)$ is the pointwise trace of the integral kernel of $f(D_L)$. The family of measures $\{Tr_{pt}f(D_L)d\text{vol}_L\}$, by the parameterized version of the spectral theorem, is a Borel family of tangential measures on the equivalence relation corresponding to the foliation $\mathcal{F}$. By the uniform geometry of the leaves, $Tr_{pt}f(D_L)$ is uniformly bounded over all leaves. By Proposition 4.22 in Moore and Schochet [22], the integral $tr_\nu f(D_\mathcal{F}) = \int_N \lambda \, d\nu$ is well defined and finite, where $\lambda = \{\lambda_L\}_{L\in\mathcal{F}}$ denotes the tangential measures $\{Tr_{pt}f(D_L)d\text{vol}_L\}$. If $f \geq 0$ then $Tr_{pt}f(D_L)d\text{vol}_L$ is a positive measure. Therefore $tr_\nu f(D_\mathcal{F}) \geq 0$ for $f \geq 0$.

Consider the positive linear functional $I : \mathcal{S}(\mathbf{R}) \to \mathbf{C}$ defined as follows

$$I(f) = tr_\nu f(D_\mathcal{F}). \tag{3.2.3}$$

There exists a tempered measure $m_\mathcal{F}$ on $\mathbf{R}$ such that

$$I(f) = \int_\mathbf{R} f dm_\mathcal{F}. \tag{3.2.4}$$

Let

$$\eta_\mathcal{F}^\infty(0) = \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} tr_\nu(D_\mathcal{F} e^{-tD_\mathcal{F}^2})dt \tag{3.2.5}$$

From (3.2.4), replacing m_Γ by $m_\mathcal{F}$ in (3.1.17), we have $|\eta_\mathcal{F}^\infty(0)| \leq A\mu(V)$.

To deal with the integral

$$\int_0^1 t^{-1/2} tr_\nu(D_\mathcal{F} e^{-tD_\mathcal{F}^2})dt \tag{3.2.6}$$

we observe that (3.1.2) implies

$$|Tr_{pt}(D_L e^{-tD_L^2}(x,x))| < A_L t^{1/2} \qquad 0 \leq t \leq 1. \tag{3.2.7}$$

Since we have global bounds on the local geometry of the leaves in terms of the geometry of V and S, we have a uniform bound for the A_L's in (3.2.7). Hence there is a constant A such that

$$|Tr_{pt}(D_L e^{-tD_L^2}(x,x))| < At^{1/2} \qquad 0 \leq t \leq 1 \ \text{ for all leaves } L. \tag{3.2.8}$$

From (3.2.8) we get

$$|tr_\nu(D_{\mathcal{F}}e^{-tD_{\mathcal{F}}^2})| = \left| \int_N Tr_{pt}(D_L e^{-tD_L^2}(x,x)) d\text{vol}_L d\nu \right| \leq At^{1/2}\mu(N). \tag{3.2.9}$$

Therefore

$$\left| \int_0^1 t^{-1/2} tr_\nu(D_{\mathcal{F}}e^{-tD_{\mathcal{F}}^2}) dt \right| \leq A\mu(N).$$

This completes the proof of the theorem. Q.E.D.

4. SELF ADJOINTNESS OF THE BOUNDARY VALUE PROBLEM

Our assumptions are the same as in sections 2.1 and 2.2. Henceforth B will denote the boundary condition satisfying Definitions 2.1.2 and 2.1.3. This boundary condition will be known as the A.P.S. boundary condition. In this chapter, we prove that D acting on $W^\infty(M;S)_B$ is essentially self-adjoint, where

$$W^\infty(M;S)_B = \{S \in W^\infty(M;S) \mid Bbf = 0\}.$$

Our approach to the problem of essential self-adjointness is inspired by (Roe, 1988, unpublished). This will involve the construction of bounded linear operators R_1 and R_2 satisfying the following properties.

(4.1) $$R_i : W^k(M;S) \to W^{k+1}(M;S)_B$$

for $i = 1,2$ and k a nonnegative integer, is continuous.

(4.2) $$DR_1 - Id = S_1$$

$$R_2 D - Id = S_2$$

where S_1 and S_2 are smoothing operators.

(4.3) $$W^k(M;S)_B = \{s \in W^k(M;S) \mid Bbs = 0\} \text{ for } k \geq 1.$$

Assuming existence of operators R_i and their adjoints R_i^* satisfying (4.1), (4.2), and (4.3) we prove the essential self-adjointness of the Boundary Value problem which in the future will be abbreviated to B.V.P.

Theorem 4.1. The unbounded operator

$$D : L^2(M;S) \to L^2(M;S)$$

with domain $W^\infty(M;S)_B$ is essentially self-adjoint.

Proof. We first prove that the minimal domain of D is $W^1(M;S)_B$. This proof follows (Roe, 1988, unpublished). The closure of $W^\infty(M;S)_B$ in the graph norm is clearly contained in $W^1(M;S)_B$. To prove that these two spaces are the same, we show that any element in $W^1(M;S)_B$ can be approximated by elements in $W^\infty(M;S)_B$. Again it is enough to prove that elements in $W^1(M;S)_B$ supported in a collar $[0,1)\times N$ can be approximated by elements in $W^\infty(M;S)_B$. Let $s\in W^1(M;S)_B$ be supported in $[0,1)\times N$. Extending by zero, we can think of s as an element of $W^1([0,\infty)\times N;S)_B$. Regularize s in the boundary direction by the operator

$$H_t s = Be^{-tQ^2}Bs + (I-B)e^{-tQ^2}(I-B)s. \tag{4.4}$$

By reflection extend s to $W^1((-\infty,\infty)\times N;\widehat{S})$, and regularize s in the cylinder direction by

$$s_\epsilon = \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}\varphi\left(\frac{x-y}{\epsilon}\right)s\ dy \tag{4.5}$$

where ϕ is a smooth compactly supported even positive function on $[-1,1]$ with $\int\phi = 1$. Combining (4.4) and (4.5) gives the required approximation.

Next we prove that

$$\text{domain of } D^* = W^1(M;S)_B \tag{4.6}$$

where D^* is the Hilbert space adjoint of D. Now

$$\operatorname{Dom}(D^*) = \left\{ s\in L^2(M;S) \;\middle|\; \begin{array}{c} f\mapsto\langle s,Df\rangle \text{ for} \\ f\in W^1(M;S)_B \\ \text{extends to a bounded} \\ \text{linear functional on } L^2 \end{array} \right\}. \tag{4.7}$$

By definition, $W^1(M;S)_B\subseteq \operatorname{Dom}(D^*)$. We now prove that, $\operatorname{Dom}(D^*)\subseteq W^1(M;S)_B$. Let $f\in \operatorname{Dom}(D^*)$. Then there is a $g\in L^2$ so that

$$\langle f,Ds\rangle = \langle g,s\rangle \text{ for all } s\in W^1(M;S)_B. \tag{4.8}$$

Since $DR_1 s + S_1 s = s$, we have by (4.8)

$$\begin{aligned}\langle f,s\rangle &= \langle f, DR_1 S + S_1 s\rangle = \langle g,R_1 s\rangle + \langle f,S_1 s\rangle \\ &= \langle R_1^* g,s\rangle + \langle S_1^* f,s\rangle.\end{aligned}$$

Therefore $f = R_1^* g + S_1^* f \in W^1(M;S)$. This implies that bf is defined. If $s \in W^1(M;S)_B$ then

$$\langle f, Ds\rangle = \langle Df, s\rangle + \langle bf, \sigma bs\rangle = \langle g, s\rangle. \tag{4.9}$$

By choosing s compactly supported in the interior of M we see that $Df = g$. By (4.9)

$$\langle bf, \sigma bs\rangle = 0 \text{ for all } s \in W^1(M;s)_B.$$

Hence $bf \in (\sigma \ker B)^{\perp} = \ker B$. This implies that $f \in W^1(M;S)_B$. Q.E.D.

The construction of the operators R_i is based on the following procedure. We first construct an interior parametrix for D. Following the notation of section 2.1 we consider the data $(\widehat{D}, \widehat{S}, \widehat{\epsilon})$ on the complete manifold without boundary $db(M)$. By Chernoff [9], $\widehat{D}$ is essentially self-adjoint on $C_c^\infty(db(M); \widehat{S})$. We construct parametrices I_1 and I_2 with appropriate Sobolev regularity, namely

$$I_\ell : W^k(db(M) : \widehat{S}) \to W^{k+1}(db(M); \widehat{S}) \qquad \ell = 1, 2 \tag{4.10}$$

and k a nonnegative integer and

$$DI_1 - Id = S_1' \tag{4.11}$$

$$I_2 D - Id = S_2' \tag{4.12}$$

where S_1' and S_2' are smoothing operators.

The next step is to construct a boundary parametrix and then patch the two together to get operators R_i. The construction of boundary parametrix is very similar to section 2 of A.P.S. [3]. The main tools in the construction of boundary parametrix are separation of variables and Theorem 2.2.1.

Since we will be constructing the boundary parametrix in a neighborhood of the collar $[0,1) \times N$ and then truncate it by smooth bump functions, we can assume without loss of generality that the collar is $[0,\infty) \times N$, with the product metric. For the rest of this section let $M = [0,\infty) \times N$.

Diagonalizing ϵ the grading operator we have $S = S^+ \oplus S^-$ where S^+ and S^- are the $+1$ and -1 eigenspaces of ϵ respectively. Then

$$D : W^\infty(M;S^+) \oplus W^\infty(M;S^-) \to W^\infty(M;S^+) \oplus W^\infty(M;S^-). \tag{4.13}$$

Now $D = \sigma\left(\frac{\partial}{\partial y} + Q\right)$ where $\sigma^2 = -1$, σ anticommutes with ϵ and Q commutes with ϵ. Therefore we have

$$Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \tag{4.14}$$

where $\beta : S^+ \to S^-$ is an isomorphism given by the Clifford action. Identifying S^- with S^+ using β we can assume $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$Q = \begin{pmatrix} Q_+ & 0 \\ 0 & -Q_+ \end{pmatrix} : W^\infty(M;S^+) \oplus W^\infty(M;S^+). \tag{4.15}$$

Further

$$B = \begin{pmatrix} P & 0 \\ 0 & I-P \end{pmatrix} \text{ where } P = \chi_{[0,\infty)}(Q_+). \tag{4.16}$$

Further Q_+ and all its powers are essentially self-adjoint. We can therefore apply Theorem 2.2.1. Under these identifications our B.V.P. is

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{\partial}{\partial y} + \begin{pmatrix} Q_+ & 0 \\ 0 & -Q_+ \end{pmatrix} \right) \text{ and } B = \begin{pmatrix} P & 0 \\ 0 & I-P \end{pmatrix}. \tag{4.17}$$

The application of the spectral transform to prove Sobolev regularity of the boundary parametrix is based on the observation

$$f \in W^k(M;S) \Leftrightarrow f \in L^2([0,\infty); W^k(N;S)) \quad \text{and} \quad \frac{\partial^j f}{\partial y^j} \in W^j([0,\infty);\ W^{k-j}(N;S)) \quad 1 \le j \le k,\ k \ge 0,\ k \in \mathbf{Z}. \tag{4.18}$$

Where

$$W^k([0,\infty);H) = \left\{ f \,\middle|\, \frac{\partial^j f}{\partial y^j} \in L^2([0,\infty);H) \quad 0 \le j \le k \right\}$$

for H a Hilbert space.

Theorem 4.2. There exists an inverse T to D acting on $C_c^\infty([0,\infty);\; W^\infty(N;S))$ with the following properties.

$$T : C_c^\infty([0,\infty);\; W^\infty(N;S)) \to C^\infty([0,\infty);\; W^\infty(N;S))_B \tag{4.19}$$

$$D\,Tf = f \quad \text{and} \quad T\,Df = f. \tag{4.20}$$

Further T extends to a continuous operator

$$T : W^k(M;S) \to W^{k+1}_{y\text{loc}}(M;S) \quad k \geq 0 \tag{4.21}$$

where

$$W^k_{y\text{loc}}(M;S) = \left\{ \begin{matrix} f \text{ a function} \\ \text{on } M \end{matrix} \;\middle|\; \begin{matrix} \phi f \in W^k(M;S) \\ \phi \in C_c^\infty([0,\infty)) \end{matrix} \right\}$$

Proof. Our construction follows the one in A.P.S. [3], with the exception of using Theorem 2.2.1 instead of eigenfunction expansions.

We need to solve the following equations

$$\left(\frac{\partial}{\partial y} + Q_+\right) f_1 = g_1 \quad \text{with} \quad Pf_1(0) = 0$$
$$\left(\frac{\partial}{\partial y} - Q_+\right) f_2 = g_2 \quad \text{with} \quad (I-P)f_2(0) = 0.$$

Applying the spectral transform we find

$$\begin{aligned} \left(\frac{\partial}{\partial y} + \lambda\right)(Vf_1)_j(\lambda,y) &= (Vg_1)_j(\lambda,y) \text{ with } (Vf_1)_j(\lambda,0) = 0 \\ & \qquad\qquad\qquad\qquad \text{for } \lambda \geq 0 \\ \left(\frac{\partial}{\partial y} - \lambda\right)(Vf_2)_j(\lambda,y) &= (Vg_2)_j(\lambda,y) \text{ with } (Vf_2)_j(\lambda,0) = 0 \\ & \qquad\qquad\qquad\qquad \text{for } \lambda < 0. \end{aligned} \tag{4.22}$$

We denote the Fourier Laplace transform by

$$\widetilde{g}(\xi) = \int_0^\infty e^{-iy\xi} g(y)\,dy.$$

Solutions of equations (4.22) are

$$
\begin{aligned}
(Vf_1)_j(\lambda,y) &= \int_0^y e^{\lambda(x-y)}(Vg_1)_j(\lambda,x)dx && \text{if } \lambda \geq 0 \\
&= -\int_y^\infty e^{\lambda(x-y)}(Vg_1)_j(\lambda,x)dx && \text{if } \lambda < 0 \\
(Vf_2)_j(\lambda,y) &= -\int_y^\infty e^{\lambda(y-x)}(Vg_2)_j(\lambda,x)dx && \text{for } \lambda \geq 0 \\
&= \int_0^y e^{\lambda(y-x)}(Vg_2)_j(\lambda,x)dx && \text{for } \lambda < 0.
\end{aligned}
\tag{4.23}
$$

Apply the Fourier Laplace transform w.r.t. the y variable to (4.22), we get

$$
\begin{aligned}
&(\lambda+i\xi)(\widetilde{V}f_1)_j(\lambda,\xi) = (\widetilde{V}g_1)_j(\lambda,\xi) + (Vf_1)_j(\lambda,0) \\
&\qquad\qquad \text{where } (Vf_1)_j(\lambda,0) = 0 \text{ for } \lambda \geq 0 \\
&(-\lambda+i\xi)(\widetilde{V}f_2)_j(\lambda,\xi) = (\widetilde{V}g_2)_j(\lambda,\xi) + (Vf_2)_j(\lambda,0) \\
&\qquad\qquad \text{where } (Vf_2)_j(\lambda,0) = 0 \text{ for } \lambda < 0.
\end{aligned}
\tag{4.24}
$$

Also

$$
\begin{aligned}
(Vf_1)_j(\lambda,0) &= -\int_0^\infty e^{\lambda x}(Vg_1)_j(\lambda,x)dx && \text{if } \lambda < 0 \\
(Vf_2)_j(\lambda,0) &= -\int_0^\infty e^{-\lambda x}(Vg_2)_j(\lambda,x)dx && \text{if } \lambda \geq 0.
\end{aligned}
\tag{4.25}
$$

When estimating the L^2 norms we find that f_1 and f_2 are $L^2_{y\mathrm{loc}}$. We now prove that

$$
T^{\pm} : L^2 \to W^1_{y\mathrm{loc}} \tag{4.26}
$$

(where

$$
T^+ g_1 = V^{-1}(Vf_1)
$$

and

$$
T^- g_2 = V^{-1}(Vf_2)
$$

where (Vf_1) and (Vf_2) are defined by equations (4.32)) are continuous. It follows from (4.24) it follows that

$$\lambda^2 \int |(\widetilde{V} f_i)_j(\lambda,\xi)|^2 d\xi \le \int |(\widetilde{V} g_i)_j(\lambda,\xi)|^2 d\xi \qquad \text{for } \lambda \ge 0$$
$$\le 2\Big\{ \int |(\widetilde{V} g_i)_j(\lambda,\xi)|^2 d\xi + \lambda^2 |(Vf_i)_j(\lambda,0)| \int_{-\infty}^{\infty} \frac{d\xi}{\lambda^2+\xi^2} \Big\}$$
$$\le 4 \int |(Vg_i)_j(\lambda,y)|^2 dy.$$

Here $i = 1, 2$.

Using (4.22) we have

$$\frac{\partial}{\partial y}(Vf_1)_j(\lambda,y) = -\lambda (Vf_1)_j(\lambda,y) + (Vg_1)_j(\lambda,y) \tag{4.27}$$
$$\frac{\partial}{\partial y}(Vf_2)_j(\lambda,y) = \lambda (Vf_2)_j(\lambda,y) + (Vg_2)_j(\lambda,y).$$

We get

$$\int |\frac{\partial}{\partial y}(Vf_i)_j(\lambda,y)|^2 dy \le 9 \int |(Vg_i)_j(\lambda,y)|^2 dy.$$

We consider

$$\int |(Vf_1)_j(\lambda,y)|^2 dy = \int \Big| \int_0^y e^{\lambda(x-y)} (Vg_1)_j(\lambda,x) dx \Big|^2 dy \text{ for } \lambda \ge 0.$$

Applying Cauchy-Schwarz inequality

$$\le \int e^{-2\lambda y} \Big[\Big(\int_0^y e^{2\lambda x} dx \Big) \int_0^y |(Vg_1)_j(\lambda,x)|^2 dx \Big] dy \tag{4.28}$$
$$\le \int e^{-2\lambda y} \Big[\frac{e^{2\lambda y}-1}{2\lambda} \Big] dy \int_0^\infty |(Vg_1)_j(\lambda,x)|^2 dx.$$

One can estimate uniformly in λ, $\frac{1-e^{-2\lambda y}}{2\lambda}$. Hence this implies that $(Vf)_j(\lambda,y) \in L^2_{y\mathrm{loc}}$.

Putting these estimates together we have

$$T_1 = \begin{pmatrix} T^+ & 0 \\ 0 & T^- \end{pmatrix} : L^2 \to W^1_{y\mathrm{loc}}$$

where T_1 is the solution to (4.22). Higher derivative estimates are made in the same fashion as in section 2 of A.P.S. [3]. We get the inverse of D

$$T = \begin{pmatrix} T^+ & 0 \\ 0 & T^- \end{pmatrix} \quad \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}.$$ Q.E.D.

We patch T and I_ℓ together as in section 3 of A.P.S. [3]. Let $\rho(a,b)$ denote an increasing C^∞ function of the real variable y with $\rho = 0$ for $t \leq a$ and $\rho = 1$ for $y \geq b$.

Define four C^∞ functions ϕ_1, ϕ_2, ψ_1, ψ_2 by

(4.29)
$$\begin{cases} \phi_2 = \rho\left(\frac{1}{4}, \frac{1}{3}\right), & \psi_2 = \rho\left(\frac{1}{2}, \frac{3}{4}\right) \\ \phi_1 = 1 - \rho\left(\frac{5}{6}, 1\right) & \psi_1 = 1 - \psi_2 \end{cases}$$

Then $R_1 = \phi_1 T \psi_1 + \phi_2 I_\ell \psi_2$.

Putting the estimates for T and I_ℓ together we see that R_ℓ satisfies the properties (4.1), (4.2), and (4.3) and so does its adjoint.

5. PARAMETRIX FOR THE PARABOLIC INITIAL BOUNDARY VALUE PROBLEM

In Chapter 4, we showed that the densely defined unbounded operator

$$D : L^2(M;S) \to L^2(M;S)$$

with domain $W^1(M;S)_B$ is self-adjoint. Further from the regularity properties of the parametrix constructed in Chapter 4 we have $\mathrm{Dom}(D^k) \subseteq W^k(M;S)$ for k a positive integer. Therefore by duality and the spectral theorem we have

Proposition 5.1. If $f \in RB(\mathbf{R})$ then

$$f(D) : W^{-k_1}(M;S) \to W^{k_2}(M;S)$$

for all k_1 and k_2 positive integers. Hence $f(D)$ is represented by a smooth kernel.

Proof. Use functional calculus and Proposition 2.2.3. Q.E.D.

By Proposition 5.1, e^{-tD^2} is a smoothing operator. The rest of this chapter focuses on constructing a parametrix for this initial boundary value problem. As in A.P.S. [3] we construct an interior parametrix by considering the heat kernel of $\widehat{D}$ on the $db(M)$ and restricting to the interior of M. As in Chapter 4, in the construction of the boundary parametrix we can assume that our collar is $M = [0,\infty) \times N$. Under the identifications (4.13), (4.14), and (4.15) the initial boundary value problem,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + D^2\right) f(\ ,t) &= 0 \\ f(\ ,0) &= g(\ ) \end{aligned}$$

$$\text{and } Bbf = 0 \text{ and } BbDf = 0 \tag{5.1}$$

reduces, using the spectral transform to

$$\left\{ \begin{array}{l} \left(\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial y^2} + \lambda^2\right)(Vf_1)_j(\lambda,y,t) = 0 \\ (Vf_1)_j(\lambda,0,t) + \lambda(Vf_1)_j(\lambda,0,t) = 0 \text{ for } \lambda \geq 0 \\ \left(\dfrac{\partial}{\partial y}(Vf_1)_j(\lambda,0,t) + \lambda(Vf_1)_j(\lambda,0,t)\right) = 0 \text{ for } \lambda < 0 \\ (Vf_1)_j(\lambda,y,0) = (Vg_1)_j(\lambda,y) \end{array} \right. \tag{5.2}$$

and

$$(5.3)\qquad \begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} + \lambda^2\right)(Vf_2)_j(\lambda, y, t) = 0 \\ (Vf_2)_j(\lambda, 0, t) = 0 \text{ for } \lambda < 0 \\ -\frac{\partial}{\partial y}(Vf_2)_j(\lambda, 0, t) + \lambda(Vf_2)_j(\lambda, 0, t) = 0 \text{ for } \lambda \geq 0 \\ (Vf_2)_j(\lambda, y, 0) = (Vg_2)_j(\lambda, y) \end{cases}$$

where $f = \binom{f_1}{f_2}$.

The solution to (5.2) is given by

$$(5.4)\qquad \begin{aligned} (Vf_1)_j(\lambda, x, t) &= \int_0^\infty a_\lambda(x, y, t)(Vg_1)_j(\lambda, y)dt \text{ for } \lambda \geq 0 \\ &= \int_0^\infty b_\lambda(x, y, t)(Vg_1)_j(\lambda, y)dy \text{ for } \lambda < 0 \end{aligned}$$

where

$$(5.5)\qquad a_\lambda(x, y, t) = \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right]$$

and

$$(5.6)\qquad b_\lambda(x, y, t) = a_\lambda(x, y, t) + \lambda e^{-\lambda(x+y)} \operatorname{erfc}\left\{\frac{x+y}{2\sqrt{t}} - \lambda\sqrt{t}\right\}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi.$$

Similarly the solution of (5.3) is

$$(5.7)\qquad \begin{aligned} (Vf_2)_j(\lambda, x, t) &= \int_0^\infty a_\lambda(x, y, t)(Vg_2)_j(\lambda, y)dy \text{ for } \lambda < 0 \\ &= \int_0^\infty b_{-\lambda}(x, y, t)(Vg)_j(\lambda, y)dy \text{ for } \lambda \geq 0. \end{aligned}$$

These solutions can be found in A.P.S. [3].

Let

$$(5.8)\qquad \begin{aligned} E^+(t)g(x, n) = &\sum_j \int_{\lambda\geq 0}\int_0^\infty a_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)dyd\mu_j(\lambda) \\ &\sum_j \int_{\lambda<0}\int_0^\infty b_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)dyd\mu_j(\lambda) \end{aligned}$$

$$(5.9)\qquad \begin{aligned} E^-(t)g(x, n) = &\sum_j \int_{\lambda<0}\int_0^\infty a_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)dyd\mu_j(\lambda) \\ &\sum_j \int_{\lambda\geq 0}\int_0^\infty b_{-\lambda}(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)dyd\mu_j(\lambda). \end{aligned}$$

The off diagonal exponential decay, of $e^{-\lambda^2 t}$, in $a_\lambda(x,y,t)$, $b_\lambda(x,y,t)$ and its derivatives, along with the term $e^{-\lambda^2 t}$, imply that $E^+(t)$ and $E^-(t)$ are smoothing operators and are represented by smooth kernels for $t > 0$.

We define

$$E_1(t) = \begin{pmatrix} E^+(t) & 0 \\ 0 & E^-(t) \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}. \tag{5.10}$$

Then $E_1(t)$ is the fundamental solution of the initial boundary value problem (5.1) on the cylinder $[0,\infty) \times N$. Let $F(t)$ be the fundamental solution of the heat equation on the double of M for $\widehat{D}$. Then we will patch $F(t)$ and $E_1(t)$ to get a parametrix for the initial boundary value problem on M. We defined the functions ϕ_1, ϕ_2 and ψ_1, ψ_2 in Chapter 4. We use them to construct

$$E(t) = \phi_1 E_1(t)\psi_1 + \phi_2 F(t)\psi_2. \tag{5.11}$$

Note that $(\frac{\partial}{\partial t} + D^2)E(t)$ is $0(t^k)$ for all $k > 0$. This follows from the off diagonal exponential decay of $a_\lambda(x,y,t)$, $b_\lambda(x,y,t)$ and $F(t)$.

Theorem 5.2.

$$e^{-tD^2} - E(t) : W^{-k_1}(M;S) \to W^{k_2}(M;S) \tag{5.12}$$

is a bounded linear operator for all k_1 and k_2 positive integers. For k_1 and k_2 very large positive integers there exists an $\alpha > 0$ such that

$$\|e^{-tD^2} - E(t)\|_{L(W^{-k_1},W^{k_2})} \leq Ct^\alpha \qquad \text{for all } 0 \leq t \leq 1.$$

Proof. This follows from the Duhamel's formula

$$e^{-tD^2} - E(t) = \int_0^t e^{-(t-s)D^2}\left(\frac{\partial}{\partial s} + D^2\right)E(s)$$

and Sobolev estimates. The basic idea is the following

$$\left(\frac{\partial}{\partial s} + D^2\right)E(s) = \frac{\partial^2 \phi_2}{\partial y^2}F(t)\psi_2 + \frac{\partial \psi_2}{\partial y}\ \frac{\partial F(t)}{\partial y}\ \psi_2$$
$$\frac{\partial \phi_1}{\partial y}\ \frac{\partial E_1(t)}{\partial y}\ \psi_1 + \frac{\partial^2 \psi_1}{\partial y^2}E_1(t)\psi_1.$$

Now the off diagonal exponential decay of $a_\lambda(x,y,t)$, $b_\lambda(x,y,t)$ and $F(t)$ along with the fact that

$$\frac{\partial^2 \phi_2}{\partial y^2}, \psi_2;\ \frac{\partial \phi_2}{\partial y}, \psi_2;\ \frac{\partial \phi_1}{\partial y}, \psi_1;\ \frac{\partial^2 \phi_1}{\partial y^2}, \psi_1$$

have disjoint supports proves the theorem. Q.E.D.

6. BREUR INDEX FOR COVERINGS AND FOLIATIONS

This chapter formulates the Breuer index for elliptic operators on coverings of compact manifolds with boundary and for elliptic operators on leaves of foliations compact manifolds with boundary with leaves transverse to the boundary.

Heuristically, in case of Galois coverings of a compact manifold with boundary M, with Galois group Γ, we deal with the von Neumann algebra with a semifinite faithful trace

$$\mathrm{End}_\Gamma L^2(\widetilde{M}) = \{\text{All bounded linear operators commuting with the } \Gamma \text{ action}\}$$

This trace is defined on a smaller class of operators. We will work with a dense subalgebra of $\mathrm{End}_\Gamma L^2(\widetilde{M})$ to define finite-Γ dimensionality.

In the case of a foliation $\mathcal{F}$ of M we have the Borel equivalence relation

$$\mathcal{R} = \{(x,y) \mid x \text{ and } y \text{ are on the same leaf } L \in \mathcal{F}\}$$

which has the structure of a measurable groupoid. See Moore [21], Moore and Schochet [22] for definitions and more details. Now the groupoid $\mathcal{R}$ acts on the field of Hilbert spaces $H = \{L^2(L_x)\}_{x\in M}$ naturally. Following Connes [10] we study the von Neumann algebra of intertwining endomorphisms of the field H up to suitable equivalence given by a holonomy invariant transverse measure ν. Moreover the transverse measure gives rise to a natural faithful semifinite trace, which is used to define the notion of finite ν dimension. Section 6.1 deals with the case of coverings and section 6.2 the case of foliations.

6.1. Finite dimensionality of the Γ index.

Let M be a compact Riemannian manifold with boundary N. Let D be a Dirac operator on a graded Clifford bundle S with grading operator ϵ. We assume that the data (D,S,ϵ) has a product structure in a neighborhood of the collar $[0,1]\times N$ as in Definition 2.1.1. Let $\widetilde{M}$ be a Galois covering of M with Galois group Γ. We denote the lifts of D and S to $\widetilde{M}$ by $\widetilde{D}$ and $\widetilde{S}$ respectively. Let $\widetilde{B}$ be the A.P.S. boundary condition associated to $\widetilde{D}$ as in Definitions 2.1.2 and 2.1.3.

In Chapter 4 we showed that the densely defined operator $\widetilde{D} : L^2(\widetilde{M};\widetilde{S}) \to L^2(\widetilde{M};\widetilde{S})$ with $\mathrm{Dom}(\widetilde{D}) = W^1(\widetilde{M};\widetilde{S})_{\widetilde{B}}$ is self-adjoint. Further $\widetilde{D}$ commutes with the action of Γ. Following Roe [28] we introduce a dense subalgebra $\mathfrak{A}$ of $\mathrm{End}_\Gamma(L^2(\widetilde{M};\widetilde{S}))$.

Definition 6.1.1. $A \in \mathfrak{A}$ if

1. A is given by an integral kernel $k(x,y)$ with the following property. There is a constant C such that $\int |k(x,y)|^2 d\mathrm{vol}_{\widetilde{N}}(y) < C$ and $\int |k(x,y)|^2 d\mathrm{vol}_{\widetilde{N}}(x) < C$ for every x and $y \in \widetilde{M}$ respectively.

2. A is smoothing, namely

$$As(x) = \int_{\widetilde{N}} k(x,y)s(y)d\mathrm{vol}_{\widetilde{N}}(y) \quad \text{for} \quad S \in L^2(\widetilde{M};\widetilde{S})$$

   and the maps $x \to k(x,\cdot)$ and $y \to k(\cdot,y)$ are smooth maps of $\widetilde{M}$ to the Hilbert space $L^2(\widetilde{M};\widetilde{S})$.

Proposition 6.1.1. The set of operators $\mathfrak{A}$ form an algebra.

Proof. The proof for manifolds with boundary follows exactly as that of Proposition 13.5 of Roe [28]. Q.E.D.

Lemma 6.1.1. There exists a fundamental domain for the Γ action on $\widetilde{M}$, which we label F.

Proof. The proof is exactly the same as in Atiyah [1] where it is proved for manifolds without boundary. Q.E.D.

We now define a functional $\tau : \mathfrak{A} \to \mathbf{C}$, which we call a trace, as follows.

Definition 6.1.2. Let $A \in \mathfrak{A}$, then

$$\tau(A) = \int_F tr\ k(x,x)d\mathrm{vol}_{\widetilde{N}}(x) \tag{6.1.1}$$

where F is a fundamental domain for the Γ action. The fact that A commutes with the Γ action implies that the definition of τ is independent of the choice of fundamental domain. We also use the notation tr_Γ for τ.

Proposition 6.1.2. For $A_1, A_2 \in \mathfrak{A}$ we have

$$\tau(A_1 A_2) = \tau(A_1 A_1).$$

Proof. The proof for manifolds with boundary is the same as the proof for manifolds without boundary, see Proposition 13.10 of Roe [28].

Definition 6.1.3. A closed subspace H of $L^2(\widetilde{M}; \widetilde{S})$ is said to be of finite Γ-dimension if the orthogonal projection $P : L^2(\widetilde{M}; \widetilde{S}) \to H$ belongs to $\mathfrak{A}$. In this case we define

$$\dim_\Gamma(H) = \tau(P). \tag{6.1.2}$$

Proposition 6.1.3. For any $f \in RB(\mathbf{R})$, $f(\widetilde{D}) \in \mathfrak{A}$.

Proof. Follows immediately from Proposition 6.1. Q.E.D.

Theorem 6.1.1. $\widetilde{D} : L^2(\widetilde{M}; \widetilde{S} \to L^2(\widetilde{M}; \widetilde{S})$ is closed densely defined operator with $\mathrm{Dom}(\widetilde{D}) = W^1(\widetilde{M}; \widetilde{S})_{\widetilde{B}}$ has finite Γ-dimensional kernel, and the Γ index $\mathrm{ind}_\Gamma(\widetilde{D}) = \dim_\Gamma(\ker \widetilde{D}^+) - \dim_\Gamma(\ker \widetilde{D}^-)$ is finite.

Proof. The self-adjointness of $\widetilde{D}$ implies that $\widetilde{D}^+$ and $\widetilde{D}^-$ are Hilbert space adjoints of each other. By Proposition 6.1.3, the projection onto $\ker(\widetilde{D})$ belongs to $\mathfrak{A}$. Therefore $\dim_\Gamma \ker(\widetilde{D})$ is finite. This implies that $\mathrm{ind}_\Gamma(\widetilde{D})$ is finite. Q.E.D.

The next proposition is called the McKean Singer formula.

Proposition 6.1.4.

$$\mathrm{ind}_\Gamma(\widetilde{D} = \tau(\epsilon e^{-t\widetilde{D}^2}). \tag{6.1.3}$$

Proof. The proof is the same as the proof of Proposition 13.14 of Roe [28], if we observe that $\widetilde{D}^+$ and $\widetilde{D}^-$ are Hilbert space adjoints of each other. Q.E.D.

In section 7.1 we will use Proposition 6.1.4 to identify the Γ-index in terms of topological data and the correction term arising from the eta invariant of section 3.1.

6.2. Finite ν-dimensionality of the foliation index problem.

Let $(M, \mathcal{F})$ be a compact foliated manifold with boundary with the foliation transverse to the boundary as in Definition 2.3.1. Let $D_{\mathcal{F}}$ be a leaf wise Dirac operator on a graded Clifford bundle S with grading operator ϵ. We also assume that the data $(D_{\mathcal{F}}, S, \epsilon)$ has a product structure in the sense of Definition 2.3.2, near the boundary. Let $\mathcal{R}_M$ denote the measurable induced equivalence relation on M given by the leaves of the foliation $\mathcal{F}$; and R_N the equivalence relation on the foliation of the boundary N.

Let $H = \{L^2(L_x; S_{L_x})\}_{x \in M}$, where L_x is the leaf through x, denote a Borel field of Hilbert spaces. See Dixmier [12] Chapter I, part II for definitions. By Proposition 4, page 167 of Dixmier [12], to prescribe a measure structure on the field of Hilbert spaces H, it is enough to prescribe a countable sequence $\{s_j\}$ of sections of this field of Hilbert spaces with the additional property that for all $x \in M$ the countable set $\{s_j(x)\} \subset L^2(L_x; S_{L_x})$ is a complete orthonormal set. We can do this in our context with the property that each $s_j(x)$ is also a smooth section on the leaf L_x. See Appendix of (Heitsch and Lazarov, 1990, unpublished).

There is a natural representation of the equivalence relation $\mathcal{R}$ on H as follows. If $(x, y) \in \mathcal{R}$ then the unitary isomorphism from $L^2(L_x; S_{L_x})$ to $L^2(L_y; S_y)$ is just the identity map.

A Borel transversal to the foliation $\mathcal{F}$ is a Borel subset of M which intersects every leaf in at most a countable set. The Borel transversals of $\mathcal{F}$ generate a σ-ring $\mathcal{S}$. Namely it is closed under countable unions and relative complementation. Note that the holonomy pseudo group acts on the σ-ring $\mathcal{S}$. For more on this see Hector and Hirsch [16] Chapters III and X.

Definition 6.2.1. A transverse measure ν is a measure ν on the σ-ring $\mathcal{S}$ of Borel transversals such that $\nu|_T$ is σ-finite for every $T \in \mathcal{S}$.

Definition 6.2.2. A transverse measure ν is holonomy invariant if it is invariant under the action of the holonomy pseudo group on the σ-ring.

Note that the natural representation of $\mathcal{R}$ on H is "square integrable" in the sense of

Connes [10]. Denoted by

(6.2.1) $\quad \mathrm{End}_{\mathcal{R}}(H) = \{$uniformly bounded measurable field of bounded operators intertwining the natural representation of $\mathcal{R}$ on $H\}$.

Given a holonomy invariant transverse measure ν Connes [10] defines in Chapter V, a von Neumann algebra

(6.2.2) $\quad \mathrm{End}_{\nu}(H) = \{[T] \mid T \in \mathrm{End}_{\mathcal{R}}(H)$ and $T_1 \sim T_2$ if they are equal for ν almost every leaf$\}$.

He also shows that $\mathrm{End}_{\nu}(H)$ is a direct integral of type I and type II von Neumann algebras. In part it has a semifinite faithful trace tr_{ν} obtained from ν.

If the field of operators $T \in \mathrm{End}_{\mathcal{R}}(H)$ in the domain of tr_{ν}, is implemented by a family of integral operators, one for every leaf $L \in \mathcal{F}$, with the family of leafwise kernels $\{k_L(x,y)\}_{\substack{x,y\in L,\\ L\in\mathcal{F}}}$ then the von Neumann trace of T is given by

(6.2.3)
$$tr_{\nu}(T) = \int_M k_L(x,x) d\mathrm{vol}_L d\nu.$$

The right hand side integral in (6.2.3) is well defined, and the holonomy invariance of the measure implies that the modular automorphism group generated by this state is trivial.

Lemma 6.2.1. If the kernels $k_L(x,y)$ are uniformly bounded over all leaves then $tr_{\nu}(T)$ is finite.

Proof. This follows immediately from (6.2.3). Q.E.D.

Definition 6.2.3. We say a measurable field of closed subspaces of H has finite ν-dimension if the corresponding family of orthogonal projections have finite ν trace.

Let $B_{\mathcal{F}} = \{B_L\}_{L\in\mathcal{F}}$ denote the family of A.P.S. boundary conditions, for each leaf $L \in \mathcal{F}$. We thus have a family of closed densely defined unbounded operators

(6.2.4)
$$D_{L_x} : L^2(L_x; S_{L_x}) \to L^2(L_x; S_{L_x}),\ x \in M.$$

(Here L_x is the leaf through x, with $\mathrm{Dom}(D_{L_x}) = W^1(L_x; S_{L_x})_{B_{L_x}}$ and B_{L_x} is the A.P.S. boundary condition defined in Section 2.1 for the leaf L_x) which are self-adjoint. We wish to show that the family $\{D_{L_x}\}_{x\in M}$ is a measurable family of self-adjoint operators. This enables us to use the measurable spectral theorem (cf. Reed and Simon [25] Theorem XIII.85), to prove

Theorem 6.2.1. If f is a bounded Borel function, then

$$\{f(D_{L_x})\}_{x\in M} \in \mathrm{End}_{\mathcal{R}}(H). \tag{6.2.5}$$

Proof. To prove the measurability of the family of operators in (6.2.4) it is enough to show that the family $\{(D_{L_x}+i)^{-1}\}_{x\in M}$ is a measurable family of bounded operators. The family Hilbert spaces $\{W^1(L_x; S_{L_x})_{B_{L_x}}\}_{x\in M}$ has a natural measure structure given by its inclusion into H. Q.E.D.

Proposition 6.2.1. The field of bounded operators

$$\{D_{L_x}+i\}_{x\in M} : \{W^1(L_x; S_{L_x})_{B_{L_x}}\}_{x\in M} \to \{L^2(L_x; S_{L_x})\}_{x\in M} \tag{6.2.6}$$

is measurable, and the leafwise defined inverse are also a measurable family.

Proof. The self-adjointness of D_{L_x} with domain $W^1(L_x; S_{L_x})_{B_{L_x}}$ implies that

$$D_{L_x}+i : W^1(L_x; S_{L_x})_{B_{L_x}} \to L^2(L_x; S_{L_x})$$

is a Hilbert space isomorphism. Let s, t be measurable sections of the domain and range respectively. Following (Heitsch and Lazarov, 1990, unpublished) we can choose t so that $t(x)$ is smooth on L_x. Then

$$\langle (D_{L_x}+i)s(x), t(x)\rangle_{L^2(L_x;S_{L_x})} = \langle s(x), (D_{L_x}-i)t(x)\rangle_{L^2(L_x;S_{L_x})} + \int_{\partial L_x} \langle\ ,\ \rangle$$

by formula (2.1.1). The measurability of the right hand side as a function of x follows immediately. From Example 2, page 180 of Dixmier [12] the leafwise inverse is a measurable field of operators. Combining Proposition 6.2.1 with the measurable spectral theorem in Reed and Simon [25] immediately proves the theorem. Q.E.D.

Lemma 6.2.2. If $f \in RB(\mathbf{R})$ then

$$tr_\nu f(D_{\mathcal{F}}) \text{ is finite.}$$

Proof. Proposition 5.1 and Theorem 6.2.1 imply that $\{f(D_{L_x})\}_{x\in M}$ is a measurable family of integral operators with the integral kernels smooth and uniformly bounded. Therefore by Lemma 6.2.1 the proof is complete. Q.E.D.

In particular if we take $f(x) = \begin{Bmatrix} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{Bmatrix}$ then

$$\dim_\nu(\ker D_{\mathcal{F}}) = \dim_\nu\{\ker D_{L_x}\} \text{ is finite.} \tag{6.2.7}$$

This immediately implies the following theorem.

Theorem 6.2.2. The family of A.P.S. boundary value problems has finite ν dimensional kernel, and therefore

$$\text{ind}_\nu D_{\mathcal{F}} = \dim_\nu \ker(D^+_{\mathcal{F}}) - \dim_\nu \ker(D^-_{\mathcal{F}}) \text{ is finite.} \tag{6.2.8}$$

Finally we have the McKean Singer formula. Namely

Proposition 6.2.2.

$$\text{ind}_\nu(D_{\mathcal{F}}) = tr_\nu(\epsilon e^{-tD^2_{\mathcal{F}}}).$$

Proof. We first observe that on each leaf L, D^+_L and D^-_L are Hilbert space adjoints of each other. The family wise partial isometries in the polar decomposition of D^+_L implement the isomorphism of the spectral projections of $D^-_L D^+_L$ and $D^+_L D^-_L$ on Borel subsets of $(0,\infty)$. See Moore and Schochet [22], Proposition (7.38) and Connes [10], Corollary 8, page 134 for more details. Q.E.D.

In Section 7.2 we use Proposition 6.2.2 to compute $\text{ind}_\nu(D_{\mathcal{F}})$ in terms of topological data and a correction term which is a foliation eta invariant defined in Section 3.2.

7. THE INDEX THEOREM FOR COVERINGS AND FOLIATIONS

In this chapter, we complete the proofs of the main theorems stated in the Introduction. In the earlier chapters we formulated the appropriate Breuer index, for boundary value problems. This chapter is devoted to the computation of this index. We obtain a topological component in the interior and a boundary spectral component, the eta invariant of Chapter 3. The proof is in strategy very similar to the one used by Atiyah, Patodi and Singer [3]. In section 7.1 we prove the index formula in the case of coverings of compact manifolds with boundary. Section 7.2 gives the proof for the case of foliations. Since this case is very similar to the case of coverings in 7.1, we only sketch the proof, indicating where the necessary changes have to be made.

7.1. Proof of Index Theorem for Galois Coverings.

Let M be a compact Riemannian manifold with boundary N. Let D be a Dirac operator on a graded Clifford bundle S with grading ϵ. We assume that the data (D, S, ϵ) has a product structure in a neighborhood of the collar $[0,1] \times N$ as in Definition 2.1.1. Let $\widetilde{M}$ be a Galois covering of M with Galois group Γ. We denote the lifts of D and S to $\widetilde{M}$ by $\widetilde{D}$ and $\widetilde{S}$ respectively. Let $\widetilde{B}$ be the A.P.S. boundary condition associated to $\widetilde{D}$ as in Definitions 2.1.2 and 2.1.3. We state our index theorem.

Theorem 7.1.1.

$$\text{(7.1.1)} \qquad \operatorname{ind}_\Gamma(\widetilde{D}) = \int_M \operatorname{ch}(\sigma_D) Td(M) - \frac{\eta_\Gamma(0) + h}{2}$$

where the integral on the right hand side of (7.1.1) is the standard formula in the calculation of the index on a manifold without boundary.

$$\text{(7.1.2)} \qquad h = \dim_\Gamma \ker(Q_+)$$

$$\text{(7.1.3)} \qquad \eta_\Gamma(0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} tr_\Gamma(Q_+ e^{-tQ_+^2}) dt$$

where $\widetilde{D} = \sigma(\frac{\partial}{\partial y} + Q)$ in a neighborhood of the collar $[0,1] \times \partial\widetilde{M}$ and $Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix}$

$$Q_+ : L^2(\partial\widetilde{M}; \widetilde{S}^+) \to L^2(\partial\widetilde{M}; \widetilde{S}^+)$$

is a Dirac operator on the boundary satisfying the Bismut-Freed local cancellation property.

Proof. By Proposition 6.1.4 we have

$$\mathrm{ind}_\Gamma(\widetilde{D}) = tr_\Gamma(\epsilon e^{-t\widetilde{D}^2}).$$

By Theorem 5.2 we can replace $e^{-t\widetilde{D}^2}$ by the parametrix constructed in Chapter 4 denoted by $\widetilde{E}(t)$, as $t \to 0^+$. In the definition of $\widetilde{E}(t)$ we lift cut off functions ϕ_i, ψ_i from the base to M to $\widetilde{M}$. Therefore

$$\mathrm{ind}_\Gamma(\widetilde{D}) \sim tr_\Gamma(\epsilon\widetilde{E}(t))$$

$$tr_\Gamma(\epsilon\widetilde{E}(t)) = tr_\Gamma(\epsilon\phi_1\widetilde{E}(t)\psi_1) + tr_\Gamma(\epsilon\phi_2\widetilde{F}(t)\psi_2).$$

Now

$$\lim_{t\to 0+} tr_\Gamma(\epsilon\phi_2\widetilde{F}(t)\psi_2) = \lim_{t\to 0+}\int_M F(t,x)\psi_2(x)dx$$

where $F(t,x)$ is the local supertrace of the heat kernel on the double of $\widetilde{M}$ considered as a function on $db(M)$. Since we have a product structure in a neighborhood of the collar it follows that

$$\lim_{t\to 0+} F(t,x) = 0 \qquad \text{for } x \text{ in the collar.}$$

Therefore by the calculation in Atiyah, Bott and Patodi [2] we have

$$\lim_{t\to 0+} tr_\Gamma(\epsilon\phi_2\widetilde{F}(t)\psi_2) = \lim_{t\to 0+}\int_M F(t,x)d\mathrm{vol}_M = \int_M ch(\sigma_D)Td(M).$$

Next we need to evaluate

$$tr_\Gamma(\epsilon\phi_1\widetilde{E}_1(t)\psi_1). \tag{7.1.4}$$

We note that the operator

$$\phi_1\widetilde{E}(t)\psi_1 \in \mathrm{End}_\Gamma(L^2([0,1]\times\partial\widetilde{M};\widetilde{S})$$

and has an integral kernel. Now Γ acts only on $\partial\widetilde{M}$. Therefore if the integral kernel of $\phi_1\widetilde{E}\psi_1$ is $k_E(x_1,m_1;x_2,m_2)$ then by Fubini's theorem

$$tr_\Gamma(\epsilon\phi_1\widetilde{E}_1(t)\psi_1) = \int_0^1 tr_\Gamma(\epsilon k_E(x,m;x,m))dx \tag{7.1.5}$$

where tr_Γ under the integral sign is the Γ trace on $\partial\widetilde{M}$. Combining (7.1.5) with the definition of $\widetilde{E}_1(t)$ we have

$$tr_\Gamma(\epsilon\phi_1\widetilde{E}_1(t)\psi_1)$$

(7.1.6)

$$= \int_0^1 \psi_1(y) \int_{\mathbf{R}} \text{sign }\lambda\left\{\frac{e^{-\lambda^2 t}e^{-y^2/t}}{\sqrt{\pi t}} + |\lambda|e^{2|\lambda|y}\text{ erfc}\left(\frac{y}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right\}d\mu(\lambda)dy$$

where $d\mu(\lambda)$ is the measure on the real line determined by the Γ trace of the spectral measure of the self-adjoint operator Q_+. Here sign $\lambda = \begin{cases} 1 & \text{if } \lambda \geq 0 \\ -1 & \text{if } \lambda < 0 \end{cases}$

If we replace $\int_0^1$ by $\int_0^\infty$ and $\psi_1(y)$ by 1 in (7.1.6), the error is estimated by the integral

$$C\int_{\mathbf{R}} \frac{1}{2}e^{2|\lambda|}\text{ erfc}\left(\frac{1}{\sqrt{t}} + |\lambda|\sqrt{t}\right)d\mu(\lambda) \tag{7.1.7}$$

which is bounded above by

$$C\frac{e^{-1/t}}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-\lambda^2 t}d\mu(\lambda) < Ce^{-1/t}t^{-n/2}$$

which decays exponentially as $t \to 0^+$.

Therefore $tr_\Gamma(\epsilon\phi_1\widetilde{E}(t)\psi_1)$ is asymptotic to the integral

$$K(t) =^{\text{dfn}} \int_0^\infty \int_{\mathbf{R}} \text{sign }\lambda\frac{\partial}{\partial y}\left\{\frac{1}{2}e^{2|\lambda|y}\text{erfc}\left(\frac{y}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right\}d\mu(\lambda)dy. \tag{7.1.8}$$

Changing the order of integration in (7.1.8) we get

$$K(t) = -\int_{\mathbf{R}} \text{sign }\lambda\text{ erfc }(|\lambda|\sqrt{t})d\mu(\lambda). \tag{7.1.9}$$

Differentiating w.r.t. t we get

$$K'(t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbf{R}} \lambda e^{-\lambda^2 t}d\mu(\lambda). \tag{7.1.10}$$

By the normality of the Γ trace on the boundary we have

$$K(t) \to -\frac{1}{2}h \text{ as } t \to \infty$$

where $h = \dim_\Gamma \ker(Q_+)$. Therefore $K(t) + \frac{1}{2}h \to 0$ as $t \to \infty$.

We now consider the following integral. For Re(s) large

$$\int_0^T \left(K(t)+\frac{1}{2}h\right)t^{s-1}dt = \frac{(K(T)+\frac{1}{2}h)}{s}T^s - \frac{1}{s}\int_0^T t^s K'(t)dt \tag{7.1.11}$$

$$s\int_0^T \left(K(t)+\frac{1}{2}h\right)t^{s-1}dt - \left(K(T)+\frac{1}{2}h\right)T^s = -\int_0^T t^s K'(t)dt. \tag{7.1.12}$$

Now

$$-\int_0^T t^s K'(t)dt = -\frac{\Gamma(s+1/2)}{2\sqrt{\pi}}\eta_T(2s)$$

where

$$\eta_T(2s) = \frac{1}{\Gamma(s+1/2)}\int_0^T t^{s-1/2} tr_\Gamma(Q_+ e^{-tQ_+^2})dt$$

if

$$K(t) \sim \sum_{k\geq -n} a_k t^{k/2} \text{as } t \to 0^+$$

then taking limit as $s \to 0$ in (7.1.12) we get

$$-2(a_0+h/2) + 2\left(K(T)+\frac{1}{2}h\right) = \eta_T(0). \tag{7.1.13}$$

We let $T \to \infty$. By Theorem 3.1.1 we have

$$-(2a_0+h) = \lim_{T\to\infty}\eta_T(0) = \eta_\Gamma(0). \tag{7.1.14}$$

As $t \to 0^+$ we have

$$K(t) \sim \mathrm{ind}_\Gamma(\widetilde{D}) - tr_\Gamma(\epsilon\phi_2\widetilde{F}(t)\psi_2). \tag{7.1.15}$$

Now (7.1.14) and (7.1.15) together imply that

$$\mathrm{ind}_\Gamma(\widetilde{D}) = \int_M ch(\sigma_D)Td(M) - \frac{\eta_\Gamma(0)+h}{2}$$

Q.E.D.

7.2. Proof of the index theorem for foliations.

Let $(M, \mathcal{F})$ be a compact foliated manifold with boundary with the foliation $\mathcal{F}$ transverse to the boundary. Let $D_{\mathcal{F}}$ be a leafwise Dirac operator on S a graded Clifford bundle over $T\mathcal{F}$. We assume that M and $\mathcal{F}$ are oriented. Let g be a Riemannian metric on M. We assume that data $(D_{\mathcal{F}}, S, \epsilon)$ has a product structure as in Definition 2.3.2.

Theorem 7.2.1. Let ν be a holonomy invariant transverse measure. We have

$$\operatorname{ind}_\nu(D_{\mathcal{F}}) = \langle ch(\sigma_{D_{\mathcal{F}}})Td(M), \nu\rangle - \frac{\eta_\nu(0) + h}{2} \tag{7.2.1}$$

where the first term on the right hand side is the term that one gets in the computation of the index of a leafwise elliptic operator on a manifold without boundary.

$$h = \dim_\nu(\ker Q_+) \tag{7.2.2}$$

$Q_+ = \{Q^+_{\partial L_x}\}$ the family of Dirac operators on the boundary ∂L_x satisfying the Bismut-Freed cancellation property leafwise.

$\eta_\nu(0)$ is the foliation eta invariant defined in Section 3.2.

Proof. Proposition 6.2.2 implies that

$$\operatorname{ind}_\nu(D_{\mathcal{F}}) = tr_\nu(\epsilon e^{-tD^2_{\mathcal{F}}}). \tag{7.2.3}$$

On each leaf we replace the heat operator $e^{-tD^2_{L_x}}$ by the parametrix constructed in Chapter 5, $E(t)_{L_x}$. The cut off functions used in defining the parametrix are functions on M restricted to L_x. By the uniformity of Sobolev estimates given in Theorem 5.2, we have

$$\|e^{-tD^2_{L_x}} - E(t)_{L_x}\|_{L(W^{-k},W^k)} < Ct^\alpha \quad 0 \le t \le 1,\ \alpha > 0 \tag{7.2.4}$$

where C is uniform in x and k is a very large positive integer. Hence as $t \to 0^+$

$$\operatorname{ind}_\nu(D_{\mathcal{F}}) \sim tr_\nu(E(t)_{\mathcal{F}}) \tag{7.2.5}$$

where $E(t)_{\mathcal{F}}$ is the family of operators $\{E(t)_{L_x}\}_{x\in M}$. Therefore

$$\operatorname{ind}_\nu(D_{\mathcal{F}}) \sim tr_\nu(\epsilon\phi_1 E_1(t)\psi_1) + tr_\nu(\epsilon\phi_2 F(t)\psi_2). \tag{7.2.6}$$

Applying the arguments of the previous section leafwise we have

$$\lim_{t\to 0^+} tr_\nu(\epsilon\phi_2 F(t)\psi_2) = \langle ch(\sigma_{D_\mathcal{F}})Td(M), \nu\rangle. \tag{7.2.7}$$

We now study the term

$$\lim_{t\to 0^+} tr_\nu(\epsilon\phi_1 E_1(t)\psi_1)$$

where $\phi_1 E_1(t)\psi_1$ is a family of leafwise integral operators with

$$\phi_1 E_1(t)\psi_1 \in \ \mathrm{End}_{\mathcal{R}}(L^2([0,1]\times\partial L_x; S))$$

and $\mathcal{R}$ is the restriction of the equivalence relation $\mathcal{R}_M$ to $[0,1]\times N$. By applying Fubini's theorem, as in the previous section we have

$$\begin{aligned} &tr_\nu(\epsilon\phi_1 E_1(t)\psi_1) \\ &= \int_0^1 \psi_1(y)\int_{\mathbf{R}}\left\{\frac{e^{-\lambda^2 t}e^{-y^2/t}}{\sqrt{4\pi t}} + |\lambda|e^{2|\lambda|y}\ \mathrm{erfc}\left(\frac{y}{\sqrt{t}}+|\lambda|\sqrt{t}\right)\right\}d\mu_\nu(\lambda)dy \end{aligned} \tag{7.2.8}$$

where $d\mu_\nu(\lambda)$ is the tempered measure on $\mathbf{R}$ given by the foliation trace of the family of spectral projections of the family of Dirac operators $\{Q^+_{\partial L_x}\}_{x\in N}$. From here on, the proof is the same as in the previous section, with the exception of replacing $d\mu(\lambda)$ by $d\mu_\nu(\lambda)$. This completes the sketch of the proof of Theorem 7.2.1. Q.E.D.

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TYPE II INDEX THEOREMS FOR MANIFOLDS WITH BOUNDARY

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This thesis focuses on proving index theorems for noncompact manifolds with boundary and the study of the corresponding eta invariant defect terms. More specifically, we prove a von Neumann index theorem for Dirac operators on coverings, and for leaves of foliations, of compact manifolds with boundary. These theorems are the analogue, in the noncompact case, of a celebrated theorem of Atiyah, Patodi and Singer which relates the index of boundary value problems for compact manifolds with boundary, with Pontryagin numbers and eta invariants.