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A LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS

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CHAPTER 1
INTRODUCTION

An elliptic complex over a compact manifold M has a well-defined index. That is, if E^* is a finite sequence of smooth vector bundles over M with differential operators $d_i : C^\infty(E^i) \longrightarrow C^\infty(E^{i+1})$ such that the symbol sequence is exact, then the homology groups of this complex are finite dimensional and the index of the complex is defined to be $\sum(-1)^i \dim H^i$. The celebrated Atiyah-Singer index theorem tells how to compute this index in terms of topological data coming from the manifold and the symbols of the operators.

The context for the classical Lefschetz formula is an elliptic complex with the additional datum of an endomorphism of the complex. The endomorphism consists of a map $f : M \longrightarrow M$ of the base space together with vector bundle maps $f^*E^i \xrightarrow{T^i} E^i$ lying over f. These induce an endomorphism of the complex, and hence give a map on the homology of the complex. The Lefschetz number of T^* is defined to be $\mathrm{Lef}(T^*) = \sum(-1)^i \mathrm{tr}(T^{i*} \text{ on } H^i)$.

The Lefschetz number of T^* is a generalization of the index of the complex, since the Lefschetz number of the identity is the index of the complex. In spite of this, the Lefschetz formula of Atiyah and Bott [2] is actually much easier to prove than the index theorem because of the hypothesis that f is far from the identity. More precisely, they assume that f has only simple fixed points. This can be defined either as $\det(\mathrm{Id} - f_{*x}) \neq 0$ at every fixed point, or as the diagonal map of M into $M \times M$ is transverse to graph f. With this assumption, Atiyah and Bott prove the formula

$$\mathrm{Lef}(T^*) = \sum_{\text{fixed points}} \frac{\sum(-1)^i \mathrm{tr}\ T^i(x)}{|\det(\mathrm{Id} - f_{*x})|}.$$

Thus global information, the Lefschetz number, is expressed in terms of local information at the fixed points.

Various efforts have been made to prove versions of these theorems in the context of a foliated manifold $(M, \mathcal{F})$. The assumption in the foliation case is that the complex is not elliptic on M, but only along the leaves of the foliation. More precisely, this means that the symbol sequence is exact when restricted to the covectors in the leaf directions. The leaves are not necessarily compact and so, for the index theorem, one is trying to calculate the index of an elliptic complex on a non-compact manifold, which might not be finite. However, if instead of trying to calculate the index on a particular leaf, one settles for an average over leaves, the assumed compactness of M can be used to advantage. Connes [7] established an index theorem for foliated manifolds assuming the existence of a transverse invariant measure which allows one to average the indices in some sense. In the terminology of von Neumann algebras, this is a "type II" index theorem [1,22].

It is equally natural to generalize the Lefschetz theorem from compact manifolds to foliated compact manifolds. To do this, we shall assume the following data:

1. a smooth compact Riemannian manifold M;
2. a smooth foliation $\mathcal{F}$ with leaves of dimension p and codimension q;
3. a leafwise Dirac complex, e.g., the leafwise deRham complex where $E^i = \bigwedge^i T^*\mathcal{F}$ and d_i is the exterior derivative only in the leaf directions;
4. a geometric endomorphism of the complex consisting of a map $f : M \longrightarrow M$ satisfying some additional conditions and vector bundle maps $T^i : f^*E^i \longrightarrow E^i$ over f;
5. an invariant transfixed density ν; and
6. a homology theory.

Since M is a foliated manifold, it is natural to impose the condition that the map f take leaves to leaves. The first foliation Lefschetz theorem was proven by Heitsch and Lazarov [16] who required, in addition, that every leaf go to itself. In this work, we shall work with the weaker hypothesis that the image of every leaf is contained in some leaf, not necessarily itself. It will be necessary, however, to impose an additional geometric condition on f which will be called being transfixed of dimension k and is somewhat analogous to the simple fixed point assumption of Atiyah-Bott. Roughly speaking, the condition is that the space of fixed leaves has constant transverse dimension k and that f is non-degenerate in the other transverse directions.

The fifth piece of data concerns the regularity of the foliation. The foliation index theorem of Connes and the foliation Lefschetz theorem of Heitsch-Lazarov assume that the foliation has a transverse invariant measure. In this work, we assume rather, the existence of an invariant density of a certain k-dimensional bundle. These are called invariant transfixed k-densities and are related to the invariant transverse k-forms introduced by Haefliger [15]. Transfixed densities often exist even when there is no transverse invariant measure.

The sixth ingredient needed for a foliated Lefschetz formula is a theory of homology and the trace of (f, T^*) on homology. Since the homology groups along leaves may not be finite dimensional, this is a non-trivial requirement. In fact, the general theory of what homology groups to use and how to define the trace of (f, T^*) on them remains open. Nevertheless, we shall identify the appropriate homology in particular examples.

The Lefschetz formula relates a local expression to a global one. A standard approach to proving both index theorems and Lefschetz theorems is to use the heat operator to connect the local with the global. This is the method of proof we use. Following is an outline of this approach. From the given complex, use the metrics to form the leafwise adjoints d_i^* and the corresponding Laplacians $\Delta = dd^* + d^*d$. From the Laplacian construct the leafwise heat operators $e^{-t\Delta_i}$. Although these are leafwise operators, Roe [21] showed that they preserve global smoothness so that for each t with $0 < t < \infty$, they give an endomorphism of the complex. Compose this endomorphism with the geometric endomorphism of the complex coming from (f, T^*). To prove the Lefschetz formula, there are then four steps.

1. Define a notion of the trace for these compositions so that $\mathrm{Tr}(T^i \circ e^{-t\Delta_i})$ is

finite.

2. Show that $\sum(-1)^i \mathrm{Tr}(T^i \circ e^{-t\Delta_i})$ is independent of t.

3. Show that as $t \to 0^+$, this converges to the local part of the Lefschetz formula.

4. Show that as $t \to \infty$, this converges to the global part of the Lefschetz formula.

The idea behind the definition of the trace in this case is that, for operators with smooth kernels, the trace is the integral of the kernel over the diagonal. However there are some difficulties in extending this to more general operators. If we allow kernels to be distributions instead of functions, then the Schwartz kernels theorem says that a very broad class of operators do have kernels. "Integrating the kernel over the diagonal" then corresponds to pulling the kernel back to M via the diagonal map and then integrating it over M. Unfortunately, although functions pull-back, distributions do not without some additional assumptions. One example where a distribution does pull-back is a 'δ-section' supported on a submanifold which pulls back via a map that is transverse to the submanifold. In our situation, an analysis of the kernels of the heat operators $e^{-t\Delta_i}$ shows that they are δ-sections supported on the holonomy groupoid $\mathcal{G}$ of the foliation, which is a $2p+q$ dimensional, possibly non-Hausdorff, immersed submanifold of $M \times M$. The distributional kernel for $T^i \circ e^{-t\Delta_i}$ is a δ-section supported on the groupoid pulled back via the map $M \times M \xrightarrow{f \times \mathrm{id}} M \times M$. The distributional approach to traces says that we should now pull this kernel back to M via the diagonal map. It is exactly at this point that the general distributional theory fails and that the geometric assumption on f and the extra bit of data enter in.

The condition we impose on f is such that the map $M \xrightarrow{f \times \mathrm{id}} M \times M$ misses being transverse to $\mathcal{G}$ by a constant dimension k. It then follows that the pull-back $\mathcal{G}^f$ is a $p+k$ dimensional immersed submanifold of M, whose image consists of all the leaves of M that are fixed by f. In this situation, the pull-back of the heat kernel becomes something that acts in a natural way on densities on the transverse part of the tangent space to $\mathcal{G}^f$ so the extra data that we require is a section ν of this density bundle over $\mathcal{G}^f$. Using it, we can then define the trace of the operators $T^i \circ e^{-t\Delta_i}$

using distribution theory, and this turns out to be the integral of a density on $\mathcal{G}^f$ that is constructed from the leafwise heat kernels, the density ν, and the transverse action of f. It will be written as $\mathrm{Tr}_\nu(T^i \circ e^{-t\Delta_i})$. To show that this alleged trace has the properties one would expect of a trace, we need a further condition on ν, namely that ν be holonomy invariant. This, then, completes the first step of the heat operator approach. It is presented in Chapters 2 and 3 together with the necessary background on foliations, distributions, and densities.

As a technical device, instead of using the full heat kernel, we make use of the ideas of Cheeger, Gromov, and Taylor [5] and Roe [21] on the finite propagation speed of Dirac operators to replace the heat kernel with compactly supported approximations. Such approximations are easier to handle. In Chapter 4, we explain these approximations and also show that $\sum(-1)^i \mathrm{Tr}_\nu(T^i \circ e^{-t\Delta_i})$ is independent of t for $0 < t < \infty$.

The third step is also carried out in Chapter 4. There we show, using the local time zero asymptotics of the leafwise heat kernels, that there is a local index $\varphi(x)$ defined on the fixed point set Z of f, with $\varphi(x) =$ (usual Atiyah-Bott local index for T^* on the leaf) $\times$ (a factor coming from the transverse action of f), such that:

$$\int_Z \varphi(x)\nu = \sum(-1)^i \mathrm{Tr}_\nu(T^i \circ e^{-t\Delta_i}).$$

The fourth step is to analyze the time infinity limit of the trace. One problem is to commute the limit past the trace, which requires a sense of 'normality' for the trace. This problem does not have a uniform solution, so we must examine its solution case-by-case in examples. In the case where $k = 0$, the limit does commute, and the appropriate homology is determined by the time infinity limit of the leafwise heat kernel. For foliations with compact holonomy covers and for arbitrary k, the limit commutes. In this case the contribution of a fixed leaf L to the global trace is a weighted sum of the trace of f^* on the finite-dimensional homology of various covers of L with the weights depending on the transverse action of f on the cover. If a fixed leaf L is not compact, then its contribution is related to how f acts on the closure of L in M. These results are discussed in Chapter 5.

The final chapter is concerned with various examples of the theory. We present three types of foliations to illustrate the applicability of the foliation Lefschetz theorem and the variety of behavior for the $t \to \infty$ limit. The first of these is the case of a fibration in which case the end result is an integrated version of the Lefschetz formula for the fibers that are fixed. A more interesting example concerns the torus foliated by lines with irrational slope. If $f : T^2 \longrightarrow T^2$ fixes only a discrete set of leaves, then $k = 0$ and the theory of Chapter 5 can be applied. The leaves in this case are dense, and the time infinity limit of the heat kernels turns out to be projection onto the finite dimensional space of constant sections. A third class of examples concerns the foliations on T^2 coming from the suspension of a diffeomorphism of S^1. In general, some of the leaves will be compact and some non-compact. If we again consider a map $f : T^2 \longrightarrow T^2$ that has only a discrete set of fixed leaves, then k is again 0. If there are some compact leaves, the time infinity limit concentrates on these.

In [13], Guillemin proved a formula concerning the horocycle foliation of $SL_2(R)/\Gamma$ where Γ is a discrete cocompact subgroup. In this formula, a local trace of an endomorphism is shown to be equal to a trace on the "homology of the complex" by a calculation using the Selberg trace formula and representation theory. This example falls within the framework of the foliated Lefschetz theorem and will be treated in a future paper.

CHAPTER 2

FOLIATED MANIFOLDS

The main idea of this chapter is to adapt the concept of holonomy groupoid of Reeb [19] and Winkelnkemper [24] to the context of foliation endomorphisms and their fix-points. The geometry of the fix-point set for the holonomy groupoid is a critical datum for understanding the foliation Lefschetz theorem, so we shall explore it in detail.

2.1. Preliminaries on Foliations and Holonomy

We begin with some review of the theory of foliated manifolds to establish notation. A p dimensional foliation of a $p+q$ dimensional smooth manifold M is a covering of M by smooth coordinate charts $\varphi_\alpha : U_\alpha \longrightarrow R^{p+q}$ such that the coordinate transformations $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ have the form $\varphi_\beta\varphi_\alpha^{-1}(x', x'') = (\varphi'_{\beta\alpha}(x', x''), \varphi''_{\beta\alpha}(x''))$ where R^{p+q} is being written as the product of R^p and R^q. The point of this definition is that the coordinate transformations unambiguously define transformations from (subsets of) R^q to (subsets of) R^q.

The foliation is written as $\mathcal{F}$, and charts as above are called foliation charts. The inverse images of points under the map $U_\alpha \xrightarrow{\varphi_\alpha} R^{p+q} \longrightarrow R^q$ are called the plaques of the chart φ_α. Thus the plaques are parametrized by the points of R^q. Because of the form of the coordinate transformations, on the overlap $U_\alpha \cap U_\beta$, there is an induced map $\varphi''_{\beta\alpha}$ from {plaques of U_α that intersect U_β} to {plaques of U_β that intersect U_α}. A foliation $\mathcal{F}$ of M determines a p dimensional integrable subbundle of TM, namely the bundle of tangent vectors to the plaques. This subbundle is written as $T\mathcal{F}$, and the quotient bundle $TM/T\mathcal{F}$ is written as $N\mathcal{F}$, the normal bundle of the foliation.

An injected immersed connected submanifold $i : L \longrightarrow M$ that is tangent to $\mathcal{F}$, i.e., $i_*(TL) \subset T\mathcal{F}$, and that is maximal with respect to these conditions is called a leaf of the foliation. A leaf is a p dimensional manifold and is a union of plaques. Of course the intersection of a leaf and a foliation chart may consist of infinitely many plaques, but, since all manifolds are assumed to be paracompact, the number of plaques in this intersection is at most countable. Starting with one plaque, one can inductively generate a leaf containing it by repeatedly adjoining plaques of other coordinate charts that have nonempty intersection with the previous ones. Every point x in M has a unique leaf containing it which we denote by L_x.

A smooth transversal Σ of $\mathcal{F}$ is a smooth immersed submanifold of M of dimension q, possibly with boundary, such that Σ is transverse to $T\mathcal{F}$. A transversal intersects each leaf at most countably many times. A complete transversal is a transversal that intersects each leaf at least once. If a point y is in the interior of a transversal Σ, then a neighborhood of y in Σ gives a parametrization of the plaques in a neighborhood of y in M. Indeed, the transversality of $\Sigma \xrightarrow{i} M$ implies that for every foliation chart U_α containing y, the composition

$$\Sigma \cap U_\alpha \xrightarrow{i} U_\alpha \xrightarrow{\varphi_\alpha} R^{p+q} \longrightarrow R^q$$

is a diffeomorphism of a neighborhood of y in Σ to some open set in R^q, and this last set parametrizes the plaques of U_α. Given two foliation charts U_α and U_β containing y, $\varphi''_{\beta\alpha}$ identifies the plaques of U_α with the plaques of U_β in a neighborhood of y. Taking this identification into account, the parametrization of plaques near y coming from a transversal Σ through y is independent of the foliation chart chosen. Given a foliation chart $U_\alpha \xrightarrow{\varphi_\alpha} R^{p+q}$ around x, there is a standard transversal through x coming from φ_α, namely $R^q \xrightarrow{\varphi_\alpha^{-1}(x',\cdot)} U_\alpha \subset M$. This transversal shall be denoted Σ_α^x or Σ_α.

A path in M is said to be leafwise if its image is in a leaf. A leafwise path $[0,1] \xrightarrow{\gamma} M$ from x to y induces a map from a neighborhood of a transveral at x to a neighborhood of a transveral at y. This map is called the holonomy along the path γ and is denoted by h_γ. It is defined as follows. Choose foliation charts $U_1, \ldots, U_n$ of $(M, \mathcal{F})$ that cover the path γ in the sense that there is a partition $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ of the interval $[0,1]$ such that $\gamma|_{[a_{i-1},a_i]}$ is contained in the chart U_i. Since $\gamma(a_i)$ is in $U_i \cap U_{i+1}$, $\varphi''_{i+1,i}$ identifies the plaques of U_i near $\gamma(a_i)$ with the plaques of U_{i+1} near $\gamma(a_i)$. The composition $\varphi''_{n,n-1} \circ \cdots \circ \varphi''_{2,1}$ maps some neighborhood of the plaques of U_1 near x to some neighborhood of the plaques of U_n near y. Since a transversal at x gives a parametrization of the plaques near x and similarly for y, we do get a map h_γ as claimed. Making a different choice of foliation charts covering γ does not change the map in some small neighborhood of x. The derivative of h_γ maps $N_x\mathcal{F}$ to $N_y\mathcal{F}$ and is called the linear holonomy.

Two leafwise paths γ_1 and γ_2 from x to y are defined to be holonomy equivalent if $h_{\gamma_1} = h_{\gamma_2}$ in a neighborhood of x. If γ_1 and γ_2 are leafwise homotopic, then they are holonomy equivalent [4]. Thus the germ of the holonomy along a path depends only on its homotopy class in the leaf, and there is a map from $\Pi^x_y =$ {homotopy classes of leafwise paths from x to y} to $G^x_y =$ {holonomy equivalence classes of leafwise paths from x to y}. We will write $[\gamma]$ or $[x \xrightarrow{\gamma} y]$ for the holonomy equivalence class of γ.

Definition 2.1.1 ([19,24]) *The* holonomy groupoid *of $(M, \mathcal{F})$ is the collection of all holonomy equivalence classes of paths in M. It is written as $\mathcal{G}$.*

There are various canonical maps associated to the holonomy groupoid:

1. source $\mathcal{G} \xrightarrow{s} M$ defined by $s([x \xrightarrow{\gamma} y]) = x$;
2. range $\mathcal{G} \xrightarrow{r} M$ defined by $r([x \xrightarrow{\gamma} y] = y$;
3. diagonal $M \xrightarrow{\Delta} \mathcal{G}$ defined by $\Delta(x) = [x \xrightarrow{0} x]$, the constant path; and
4. involution $\mathcal{G} \xrightarrow{i} \mathcal{G}$ defined by $i([x \xrightarrow{\gamma} y]) = [y \xrightarrow{\gamma^{-1}} x]$.

If $r([\gamma_1]) = s([\gamma_2])$ then $[\gamma_1]$ and $[\gamma_2]$ can be composed to get $[\gamma_1 * \gamma_2]$ with $s([\gamma_1 * \gamma_2]) = s([\gamma_1])$ and $r([\gamma_1 * \gamma_2]) = r([\gamma_2])$. $\mathcal{G}$ is a groupoid with this operation.

In variance with the standard usage, we will call the image of $\mathcal{G}$ in $M \times M$ via the map $s \times r$, the graph of the foliation $\mathcal{F}$. It consists of all (x, y) in $M \times M$ such that x and y lie of the same leaf of $\mathcal{F}$, or in other words it is the graph of the equivalence relation where $x \sim y$ if $L_x = L_y$. This can be a rather ugly space, and part of the reason for considering the holonomy groupoid is that it has a manifold structure so that we can think of the graph of $\mathcal{F}$ as an immersed submanifold of $M \times M$.

To make this precise, first put a topology on $\mathcal{G}$. A neighborhood of $[x \xrightarrow{\gamma} y]$ is obtained by choosing a path γ representing $[\gamma]$, then taking all leafwise paths β in M that are uniformly close to γ, and finally taking the equivalence classes $[\beta]$. Unfortunately this topology on $\mathcal{G}$ is not necessarily Hausdorff.

An example is the following. Take a diffeomorphism f of $S^1 = R^1/Z^1$ that fixes a closed interval $[a, b]$ and is increasing on $(b, b+\epsilon)$. Suspend this diffeomorphism, that is, form $S^1 \times R^1/\sim$ where $(f^n(\theta), r) \sim (\theta, r+n)$. The quotient is a torus. $S^1 \times R^1$ can be foliated by the R^1 factors, and this foliation descends to the torus. The holonomy along a leaf from $r = 0$ to $r = 1$ is given by the diffeomorphism f. Then, since f is the identity on $[a, b]$, the holonomy is trivial for any point in the open interval (a, b). However the holonomy at b is not trivial. Consider two distinct holonomy equivalence classes in $G_{(b,0)}^{(b,0)}$, the first being the trivial path at $(b, 0)$ and the second being the path $t \mapsto (b, t); 0 \leq t \leq 1$. These are distinct points of $\mathcal{G}$ because the holonomy at b is not trivial, but for each of these classes, any open set containing it, will contain classes of the form $[(b-\epsilon, 0) \xrightarrow{0} (b-\epsilon, 0)]$ since the path $t \mapsto (b-\epsilon, t); 0 \leq t \leq 1$ is holonomy equivalent to the trivial path. Thus two distinct points in $\mathcal{G}$ do not have disjoint neighborhoods.

Nevertheless, $\mathcal{G}$ is a smooth manifold of dimension $2p + q$ in the sense that every $[\gamma]$ has a neighborhood homeomorphic to R^{2p+q}. A coordinate chart around $[x_0 \xrightarrow{\gamma_0} y_0]$ is obtained by taking a representative γ_0 of $[\gamma_0]$ and foliation charts $U_\alpha \xrightarrow{\varphi_\alpha} R^{p+q}$ and $U_\beta \xrightarrow{\varphi_\beta} R^{p+q}$ around x_0 and y_0 respectively and then reducing the size of these if necessary so that the holonomy h_{γ_0} maps the transversal Σ_α diffeomorphically to the transversal Σ_β. Define the set $U_\alpha \overset{\gamma_0}{\times} U_\beta = \{(x, y) \in U_\alpha \times U_\beta \mid h_{\gamma_0}(x'') = y''\}$ (As before, x'' represents the transverse coordinates of x, x' represents the leafwise coordinates of x, and similarly for y.) Then for every $(x, y) \in U_\alpha \overset{\gamma_0}{\times} U_\beta$, there is a leafwise path γ in M from x to y that parallels γ_0. The set $\{[x \xrightarrow{\gamma} y] \mid (x, y) \in U_\alpha \overset{\gamma_0}{\times} U_\beta \text{ and } \gamma \text{ parallels } \gamma_0\}$ is in one-to-one correspondence with $U_\alpha \overset{\gamma_0}{\times} U_\beta$ and will also be denoted by $U_\alpha \overset{\gamma_0}{\times} U_\beta$. This is the desired neighborhood of $[x_0 \xrightarrow{\gamma_0} y_0]$ and is mapped via $\varphi_\alpha \times \varphi_\beta$ to $R^{p+q} \overset{h_{\gamma_0}}{\times} R^{p+q} = \{(x', x'', y', y'') \mid h_{\gamma_0}(x'') = y''\}$. Finally there are two natural ways of mapping $R^{p+q} \overset{h_{\gamma_0}}{\times} R^{p+q}$ invertibly to R^{2p+q}, namely $(x', x'', y', y'') \mapsto (x', x'', y')$ or $(x', x'', y', y'') \mapsto (x', y', y'')$ so that the coordinate chart ends up either as $U_\alpha \overset{\gamma_0}{\times} U_\beta \xrightarrow{\varphi_\alpha \times \varphi'_\beta} R^{2p+q}$ or $U_\alpha \overset{\gamma_0}{\times} U_\beta \xrightarrow{\varphi'_\alpha \times \varphi_\beta} R^{2p+q}$. Roughly speaking, then, these coordinates are obtained by taking leafwise coordinates around both x_0 and y_0 and transverse coordinates around only one or the other. The coordinate transformations between these charts are smooth so that $\mathcal{G}$ is a smooth manifold.

With this smooth structure, the map $\mathcal{G} \xrightarrow{s \times r} M \times M$ is an immersion. Of course it is not necessarily injective. In fact, $(s \times r)^{-1}(x, y) = G_y^x = \{$holonomy equivalence classes of paths from x to $y\}$. If x and y are on the same leaf so that there is at least one leafwise path $x \xrightarrow{\gamma} y$, then $G_x^x \cong G_y^x$ as sets since composition with $[\gamma]$ is a bijection of G_x^x with G_y^x. The group G_x^x is conjugate to G_y^y whenever $L_x = L_y$, and the leaf is said to have trivial holonomy if these groups are all the identity. Thus $\mathcal{G} \xrightarrow{s \times r} M \times M$ is injective if and only if all the leaves of $\mathcal{F}$ have trivial holonomy.

We shall henceforth assume that the manifold M is compact and that is has a Riemannian metric $(\cdot, \cdot)$ on its tangent bundle TM. The Riemannian metric on M then induces Riemannian metrics on all the leaves of $\mathcal{F}$. Although the leaves of $\mathcal{F}$ are not necessarily compact, the compactness of M does imply that the leaves are complete and have bounded geometry. Bounded geometry means that the injectivity radius is positive and that the curvature tensor is uniformly bounded, as are its covariant derivatives.

2.2. Morphisms and the Transfixed Condition

For the Lefschetz formula on foliated manifolds, we need to have a map $f : M \to M$. These maps should be related to the foliated structure $\mathcal{F}$ of M, and thus they will be required to take leaves to leaves.

Definition 2.2.1 *A map $f : M \to M$ will be called a morphism of $(M, \mathcal{F})$ if:*

1. *f is smooth, and*
2. *for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf L' of $\mathcal{F}$.*

It is not required that each leaf go to itself, but only that each leaf is carried into some other leaf. Although the space of leaves of M, denoted by $M/\mathcal{F}$, is not a reasonable topological space in general, a smooth map $f : M \to M$ that take leaves to leaves can be considered as a smooth map on this quotient object.

The Lefschetz formula of Atiyah and Bott assumes that the map f on M has simple fixed points. (A fixed point x of f is simple if $\det(\mathrm{Id} - T_x f) \neq 0$, or, equivalently, $T_x M \xrightarrow{T_x f} T_x M$ does not have 1 as an eigenvalue.) In the foliated context, this kind of condition can be considered in both the transverse direction and the leaf direction. The transverse direction shall be considered first, but first some concepts relating to morphisms of $(M, \mathcal{F})$ will be developed.

Given a morphism f of $(M, \mathcal{F})$, consider the map $M \xrightarrow{f \times \mathrm{id}} M \times M$ given by $y \mapsto (f(y), y)$. The holonomy groupoid $\mathcal{G}$ projects to $M \times M$ via $s \times r$, and thus we can form the pullback or fibered product of these two maps. This shall be denoted by $\mathcal{G}^f$.

$$\begin{array}{ccc} \mathcal{G}^f & \longrightarrow & \mathcal{G} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s \times r} \\ M & \xrightarrow{f \times \mathrm{id}} & M \times M \end{array}$$

Since $f \times \mathrm{id}$ is a bijective map of M to a closed subset of $M \times M$, the map $\mathcal{G}^f \to \mathcal{G}$ takes $\mathcal{G}^f$ bijectively to a closed subset of $\mathcal{G}$, and we will henceforth identify $\mathcal{G}^f$ with this closed subset. For any y in M, the fiber ${\mathcal{G}^f}_y = \{$holonomy classes $[\gamma]$ such that γ is a leafwise path from $f(y)$ to $y\}$. Clearly ${\mathcal{G}^f}_y = \phi$ if and only if $f(y)$ is not in L_y so that the image of $\mathcal{G}^f$ in M consists precisely of all points lying in leaves fixed by f.

For every element $[f(y) \xrightarrow{\gamma} y]$ of $\mathcal{G}^f$, there is a well-defined map $h_\gamma \circ f$ on a neighborhood of any transversal through y. The map f takes a transversal at y to a subset of a transversal at $f(y)$ and then the holonomy h_γ along the path γ carries this back to the transversal at y. Thus in the foliation chart U_α with transversal Σ_α,

$$\Sigma_\alpha \xrightarrow{f} f(\Sigma_\alpha) \xrightarrow{h_\gamma} \Sigma_\alpha.$$

Of course $h_\gamma \circ f$ will not necessarily be defined on all of Σ_α, but we can always shrink the foliation chart so that that is the case. Using a different chart U_β around y in place of U_α, merely conjugates $h_\gamma \circ f$ by the diffeomorphism $\varphi''_{\beta\alpha}$.

Just as the groupoid $\mathcal{G}$ is, roughly speaking, an unwrapped model of the set of all pairs of points in $M \times M$ that are on the same leaf, so $\mathcal{G}^f$ is, roughly speaking, an unwrapped model (unwrapped in the sense of holonomy, not homotopy) of the fixed leaves of f. This space $\mathcal{G}^f$ has a foliated structure. Given a particular element $[f(y) \xrightarrow{\gamma} y]$ of $\mathcal{G}^f$, and a leafwise path $\gamma_{yy'}$ from y to some other point y', there is associated an element γ' lying over y', namely $\gamma' = f(\gamma_{yy'})^{-1} * \gamma * \gamma_{yy'}$ or

$$f(y') \xrightarrow{f(\gamma_{yy'})^{-1}} f(y) \xrightarrow{\gamma} y \xrightarrow{\gamma_{yy'}} y'.$$

This process of passing from one element in $\mathcal{G}^f$ to others using leafwise paths is informally described as "flowing out from $[\gamma]$". The effect of flowing out on the maps $h_\gamma \circ f$ is merely to conjugate them by a holonomy map.

Lemma 2.2.2 *If $x \xrightarrow{\gamma} y$ is a leafwise path and f is a morphism of $(M, \mathcal{F})$, then $h_{f(\gamma)} \circ f = f \circ h_\gamma$ where both sides are considered to be maps of a sufficiently small neighborhood of a transversal at x to a transversal at $f(y)$.*

Proof. Parametrize γ so that $\gamma(0) = x$ and $\gamma(1) = y$ and let $h_{\gamma(t)}$ be the holonomy along γ from x to $\gamma(t)$. Similarly let $h_{f(\gamma)(t)}$ be the holonomy along $f(\gamma)$ from $f(x)$ to $f(\gamma(t))$. Then $h_{f(\gamma)(t)} \circ f$ and $f \circ h_{\gamma(t)}$ are both maps from a neighborhood of a transversal Σ^x through x to a transversal $\Sigma^{f(\gamma(t))}$ through $f(\gamma(t))$. The set of t such that $h_{f(\gamma)(t)} \circ f = f \circ h_{\gamma(t)}$ is certainly closed since the dependence of both sides on t is continuous. It is also open because if we take a foliation chart U_α containing $f(\gamma(t))$ for $a < t < b$ and identify the transversals $\Sigma_\alpha^{f(\gamma(t))}$ in the obvious way, then both sides are constant in t. Finally, when $t = 0$ the equality is trivially satisfied. Thus $h_{f(\gamma)(t)} \circ f = f \circ h_{\gamma(t)}$ for all t in $[0, 1]$, and, in particular, $t = 1$ gives $h_{f(\gamma)} \circ f = f \circ h_{\gamma}$. ■

Lemma 2.2.3 *If* $[f(y) \xrightarrow{\gamma} y]$ *is an element of* $\mathcal{G}^f$ *and* $\gamma_{yy'}$ *is a leafwise path from* y *to* y', *then* $\gamma' = f(\gamma_{yy'}^{-1}) * \gamma * \gamma_{yy'}$ *is a leafwise path from* $f(y')$ *to* y' *and* $h_{\gamma'} \circ f$ *is conjugate to* $h_\gamma \circ f$.

Proof. That γ' is a leafwise path from $f(y')$ to y' is clear.

$$h_{\gamma'} \circ f = h_{\gamma_{yy'}} \circ h_\gamma \circ h_{f(\gamma_{yy'}^{-1})} \circ f = h_{\gamma_{yy'}} \circ h_\gamma \circ f \circ h_{\gamma_{yy'}^{-1}}$$

by the previous lemma. Thus conjugating $h_\gamma \circ f$ by $h_{\gamma_{yy'}}$ gives $h_{\gamma'} \circ f$. ■

As a closed subset of $\mathcal{G}$, $\mathcal{G}^f$ inherits a topology. The interest in the maps $h_\gamma \circ f$ comes from the following proposition.

Proposition 2.2.4 *Given an element* $[f(y_0) \xrightarrow{\gamma} y_0]$ *of* $\mathcal{G}^f$, *let* U_α *and* U_β *be foliation charts around* $f(y)$ *and* y *respectively such that* $f(U_\beta) \subset U_\alpha$ *and* $h_\gamma \circ f$ *is defined on all of* Σ_β. *Then the neighborhood* $(U_\alpha \overset{\gamma}{\times} U_\beta) \cap \mathcal{G}^f$ *of* $[\gamma]$ *in* $\mathcal{G}^f$ *is homeomorphic to* $R^p \times \Sigma_\beta^{h_\gamma \circ f}$ *where* $\Sigma_\beta^{h_\gamma \circ f} = \{\tilde{y} \in \Sigma_\beta \mid (h_\gamma \circ f)(\tilde{y}) = \tilde{y}\}$.

Proof. Let $U_\beta^\gamma = (U_\alpha \overset{\gamma}{\times} U_\beta) \cap \mathcal{G}^f = \{[x \xrightarrow{\delta} y] \mid x \in U_\alpha, y \in U_\beta,\ \delta$ parallels γ, and $f(y) = x\}$. Parallel here means uniformly close. The claim is that this set corresponds (via the projection r) to the points in the plaques of U_β through $\Sigma_\beta^{h_\gamma \circ f}$. Indeed, for any $\tilde{y} \in \Sigma_\beta^{h_\gamma \circ f}$, there is a leafwise path $\tilde{\gamma}$ parallel to γ from $f(\tilde{y})$ to $\tilde{y}$. Flowing out from such a $[\tilde{\gamma}]$ to any point y in the plaque of $\tilde{y}$ via a plaque path as in the previous lemma gives another element of U_β^γ. So for every y that is in one of the plaques of $\Sigma_\beta^{h_\gamma \circ f}$, there is a path from $f(y)$ to y that parallels γ.

Conversely, suppose $[f(y) \xrightarrow{\delta} y] \in U_\beta^\gamma$. Then γ and δ go through the same coordinate charts so that we could almost say $h_\delta = h_\gamma$ except that they are not defined on the same transversals. But there is $\tilde{y} \in \Sigma_\beta$ on the same plaque as y. From δ and the plaque path $\gamma_{y\tilde{y}}$, we can get a path $\tilde{\gamma} = f(\gamma_{y\tilde{y}}^{-1}) * \delta * \gamma_{y\tilde{y}}$ from $f(\tilde{y})$ to

$\tilde{y}$ that parallels γ. Then $h_{\tilde{\gamma}} \circ f = h_\gamma \circ f$, both being defined on Σ_β. Since $\tilde{y}$ is fixed by $h_{\tilde{\gamma}} \circ f$, it is fixed by $h_\gamma \circ f$, or, in other words, $\tilde{y} \in \Sigma_\beta^{h_\gamma \circ f}$ and y is in a plaque through $\Sigma_\beta^{h_\gamma \circ f}$. ■

This proposition shows that, in order to make the space $\mathcal{G}^f$ into a manifold, we need to make an assumption about the transverse fixed sets, $\Sigma^{h_\gamma \circ f}$.

Definition 2.2.5 *Given a morphism f of $(M, \mathcal{F})$ and an element $[\gamma]$ of $\mathcal{G}^f$, f has* dimension k transverse fixed set at $[\gamma]$, *if the fixed set $\Sigma^{h_\gamma \circ f}$ is an k-dimensional smooth submanifold of the transversal Σ near $y = r([\gamma])$.*

Suppose that for every $[\gamma]$ in $\mathcal{G}^f$, there is an integer k such that f has dimension k transverse fixed set at $[\gamma]$. In this case, $\mathcal{G}^f$ is a (possibly non-Hausdorff) smooth manifold. Each component of $\mathcal{G}^f$ has a fixed dimension $p+k$ since the set of elements with neighborhoods homeomorphic to R^{p+k} for a fixed k is both open and closed, but different components may have different dimensions. However if one k does work for all $[\gamma]$ in $\mathcal{G}^f$, then $\mathcal{G}^f$ is a $p+k$ dimensional manifold.

Proposition 2.2.6 *If there is an integer k such that for every element $[\gamma]$ in $\mathcal{G}^f$, f has dimension k transverse fixed set at $[\gamma]$, then $\mathcal{G}^f$ is a $p+k$ dimensional smooth (possibly non-Hausdorff) manifold. $\mathcal{G}^f$ is a closed submanifold of $\mathcal{G}$ and the map $\mathcal{G}^f \longrightarrow M$ is an immersion.*

Proof. The coordinate charts for $\mathcal{G}^f$ come from intersecting the coordinate charts for $\mathcal{G}$ with $\mathcal{G}^f$ as in the previous proposition. This makes $\mathcal{G}^f$ into a smooth submanifold of $\mathcal{G}$ and we already knew it was closed. The proof of the previous proposition also shows that the projection map $r : \mathcal{G} \to M$ identifies the chart U_β^γ for $\mathcal{G}^f$ with the smooth subchart $\varphi_\beta^{-1}(R^p \times \Sigma_\beta^{h_\gamma \circ f})$ of U_β for M so that $\mathcal{G}^f \longrightarrow M$ is an immersion. ■

Notation. Given a covering of M by foliation charts U_α, a morphism f which has transverse dimension k everywhere, and an element $[\gamma]$ of $\mathcal{G}^f$ with $r(\gamma) \in U_\alpha$, we let U_α^γ denote the chart of $\mathcal{G}^f$ around $[\gamma]$ whose domain consists of the plaques of U_α through the neighborhood of y in $\Sigma_\alpha^{h_\gamma \circ f}$ that is a k-dimensional submanifold. This holds even if $h_\gamma \circ f$ is not defined on all of the transversal Σ_α.

In analogy with the Atiyah-Bott simple fixed point assumption, we will also impose a nondegeneracy condition on the space of fixed leaves, or more precisely on $\mathcal{G}^f$. For any $[f(y) \xrightarrow{\gamma} y]$ in $\mathcal{G}^f$, the derivative of $h_\gamma \circ f$ maps the linear transverse space $T_y\Sigma$ to itself. The space $T_y\Sigma$ is naturally identified with the fiber of $N\mathcal{F}$ at y. Another way of saying this is that there is a canonical endomorphism of the pulled-back vector bundle $r^*(N\mathcal{F})$. The canonical endomorphism of $r^*(N\mathcal{F})$ will be written as $(h \circ f)_*$ and is equal to $(h_\gamma \circ f)_*$ on $r^*(N\mathcal{F})_{[\gamma]}$. Abusing notation, we shall write $r^*(N\mathcal{F})$ as $N\mathcal{F}$ so that $N\mathcal{F}_{[\gamma]} = N\mathcal{F}_y$ if $r([\gamma]) = y$. If f has dimension k transverse fixed set everywhere then $(h \circ f)_*$ fixes a k-dimensional subbundle of $N\mathcal{F}$. Namely, at $[\gamma]$, $(h_\gamma \circ f)_*$ fixes $T_y\Sigma^{h_\gamma \circ f} \subset T_y\Sigma \cong N\mathcal{F}_{[\gamma]}$. This subbundle of $r^*(N\mathcal{F})$ will be written as $N\mathcal{F}^{hf}$. The nondegeneracy condition is then that the quotient map of $(h_\gamma \circ f)_*$ on $N\mathcal{F}/N\mathcal{F}^{hf}$ should fix nothing.

Definition 2.2.7 *A morphism f of $(M, \mathcal{F})$ is* transfixed of dimension k *if for every $[\gamma] \in \mathcal{G}^f$:*

1. *f has dimension k transverse fixed set at $[\gamma]$, and*
2. *the quotient map $(h_\gamma \circ f)_*$ on $N\mathcal{F}/N\mathcal{F}^{hf}$ has* $\det(\mathrm{Id} - (h_\gamma \circ f)_*) \neq 0$.

As already mentioned, for any morphism f of $(M, \mathcal{F})$, the space $\mathcal{G}^f$ has a foliated structure obtained by flowing out from $[\gamma]$ to elements $[f(\gamma')^{-1} * \gamma * \gamma']$. Since the maps $h_\gamma \circ f$ and $h_{f(\gamma')^{-1}*\gamma*\gamma'} \circ f$ are conjugates, knowing that f is transfixed of dimension k at $[\gamma]$ implies that this is also true at $[f(\gamma')^{-1} * \gamma * \gamma']$. Note however that over a single leaf L of M, $\mathcal{G}^f$ may have several components and $h \circ f$ may have different behavior on the different components. Indeed, given two elements $[f(x) \xrightarrow{\gamma_1} x]$ and $[f(x) \xrightarrow{\gamma_2} x]$ of $\mathcal{G}^f$ lying over x in L, $[\gamma_2]$ can be obtained from flowing out from $[\gamma_1]$ if and only if there is an element $[\gamma]$ in the holonomy group G_x^x such that $[f(\gamma)^{-1} * \gamma_1 * \gamma] = [\gamma_2]$. In other words, G_x^x acts on the fiber of $\mathcal{G}^f$ over x, and two elements in the fiber are in the same leaf of $\mathcal{G}^f$ if and only if they are in the same orbits of the G_x^x action. If they are not in the same leaf of $\mathcal{G}^f$, the transfixed condition could hold for one but not for the other.

Assume now that f is transfixed of dimension k.

Proposition 2.2.8 *There is a canonical exact sequence of vector bundles on $\mathcal{G}^f$:*

$$0 \longrightarrow r^*(T\mathcal{F}) \longrightarrow T\mathcal{G}^f \longrightarrow N\mathcal{F}^{hf} \longrightarrow 0.$$

Proof. Given an element $[\gamma]$ of $\mathcal{G}^f$ with $r(\gamma) = y$, consider the exact sequence

$$0 \longrightarrow T\mathcal{F}_y \longrightarrow TM_y \xrightarrow{\pi} N\mathcal{F}_y \longrightarrow 0$$

over y in M. The map r_* is an isomorphism from $T\mathcal{G}^f{}_{[\gamma]}$ to $\pi^{-1}(N\mathcal{F}^{h_\gamma \circ f})_y$ in TM_y. Lifting this up to $\mathcal{G}^f$ gives

$$0 \longrightarrow r^*(T\mathcal{F})_{[\gamma]} \longrightarrow r^*(TM)_{[\gamma]} \xrightarrow{\pi} r^*(N\mathcal{F})_{[\gamma]} \longrightarrow 0$$

Now r_* identifies $T\mathcal{G}^f{}_{[\gamma]}$ with the subspace $\pi^{-1}(r^*(N\mathcal{F}^{h_\gamma \circ f}))_{[\gamma]}$ of $r^*(TM)_{[\gamma]}$ so that there is a short exact sequence

$$0 \longrightarrow r^*(T\mathcal{F})_{[\gamma]} \longrightarrow T\mathcal{G}^f{}_{[\gamma]} \xrightarrow{\pi \circ r_*} r^*(N\mathcal{F}^{h_\gamma \circ f})_{[\gamma]} \longrightarrow 0.$$

But $r^*(N\mathcal{F}^{h_\gamma \circ f})$ is what is denoted by $N\mathcal{F}^{hf}$. ■

The subbundle $r^*(T\mathcal{F})$ is the tangent bundle to the p-dimensional foliation of $\mathcal{G}^f$.

So far we have only considered the transverse character of the morphism f. Now we shall also impose a leafwise assumption, and this one is much simpler, namely that if L is a leaf that is fixed by f, then $f|_L$ has only simple fixed points.

Definition 2.2.9 *A morphism f of $(M, \mathcal{F})$ will be called a* Lefschetz morphism *of dimension k if:*

1. *f is transfixed of dimension k, and*

2. $f|_L$ has only simple fixed points on any leaf L that is fixed by f.

A natural question to ask is how the space of fixed points of f compares with the space of fixed leaves. If f is a Lefschetz morphism of dimension k, then the space of fixed leaves of f is k-dimensional in some sense. Must the space $M^f = \{x \in M \mid f(x) = x\}$ also be k-dimensional? The answer is yes.

Proposition 2.2.10 *Suppose f is a Lefschetz morphism of dimension k. Then M^f is a k-dimensional embedded submanifold of M transverse to the foliation $\mathcal{F}$.*

Proof. Suppose $x \in M^f$. Then $[x \xrightarrow{0} x]$ is in $\mathcal{G}^f$. Since f has dimension k transverse fixed set at $[x \xrightarrow{0} x]$, $h_0 \circ f = f$ must fix a k-dimensional submanifold of a transversal through x. Pick a foliation chart U_α around x so that x has coordinates $\vec{0}$ in R^{p+q} and $f|_{R^q}$ fixes $R^k \times \{0\} \subset R^q$. Clearly then, any element of $U_\alpha \cap M^f$ will have to have coordinates in R^{p+k}. In fact, working in the coordinate chart given by U_α, $f : R^{p+k} \to R^{p+k}$ has the form $f(x', x'') = (f_1(x', x''), x'')$. The point (x', x'') is fixed if and only if $f_1(x', x'') = x'$. The function $g(x', x'') = f_1(x', x'') - x'$ from R^{p+k} to R^p has $\det[Dg/Dx'](\vec{0}) \neq 0$ because f has only simple fixed points when restricted to the plaque through $\vec{0}$. Then the implicit function theorem says there is a unique smooth function $x' = h(x'')$ defined near 0 such that $g(h(x''), x'') = 0$. Thus $U_\alpha \cap M^f$ corresponds to the k-dimensional smooth submanifold $\{(h(x''), x'', 0) \mid x'' \in R^k\}$ of R^{p+q}. ■

Proposition 2.2.11 *If f is a Lefschetz morphism of $\mathcal{G}^f$, then the flow-out of M^f is an open subset of $\mathcal{G}^f$.*

Proof. Proposition 2.2.10 shows that for any $x \in M^f$, there is a foliation chart U_α around x such that if we let $\Sigma_x^f = \{y \in \Sigma_x \mid f''(y) = y\}$, then every plaque through Σ_x^f contains a unique point fixed by f. Now suppose $[f(x) \xrightarrow{\gamma_x} x]$ is an arbitrary element of the flow-out of M^f. This means that there is a leafwise path $x \xrightarrow{\gamma} x_0$ where $x_0 \in M^f$, such that $[\gamma_x] = [f(\gamma) * \gamma^{-1}]$. The definition of the topology on $\mathcal{G}^f$ implies that for $[f(y) \xrightarrow{\gamma_y} y]$ sufficiently near $[\gamma_x]$, there is a representative γ_y of

$[\gamma_y]$ that is uniformly close to $f(\gamma) * \gamma^{-1}$. Write γ_y as $\gamma_1 * \gamma_2^{-1}$ where γ_1 and γ_2 are uniformly close to $f(\gamma)$ and γ respectively, and let $z = r(\gamma_1) = r(\gamma_2)$. It is not necessarily true that $f(z) = z$, but, since both $[f(y) \overset{f(\gamma_2)}{\to} f(z)]$ and $[f(y) \overset{\gamma_1}{\to} z]$ are uniformly close to $[f(x) \overset{f(\gamma)}{\to} x_0]$, we must have that $f(z)$ and z are in the same plaque near x_0. Thus this plaque is fixed by f so Proposition 2.2.10 shows that there is a y_0 in the plaque of z with $f(y_0) = y_0$. Composing γ_2 with the plaque path from z to y_0 gives a path γ' from y to M^f such that $[\gamma_y] = [f(\gamma') * \gamma'^{-1}]$. Hence $[\gamma_y]$ is in the flow-out of M^f. ■

For $[\gamma] \in \mathcal{G}$, the length of $[\gamma]$ is defined to be the minimum of the lengths of representatives γ of $[\gamma]$.

Proposition 2.2.12 *If f is a Lefschetz morphism of $(M, \mathcal{F})$, then there is an $\epsilon > 0$ such that if length $[\gamma] < \epsilon$ and $[\gamma] \in \mathcal{G}^f$, then $[\gamma]$ is in the flow-out of M^f.*

Proof. Suppose not. Then there is a sequence of leafwise paths $f(x_n) \overset{\gamma_n}{\longrightarrow} x_n$ with length$(\gamma_n) \to 0$ such that none of them are in the flow-out of M^f. Since M is compact, a subsequence of the x_n converges to x in M and then $f(x_n) \to f(x)$. Since $d(f(x_n), x_n) \to 0$, $d(f(x), x) = 0$ and thus x is in M^f. Since $[f(x_n) \overset{\gamma_n}{\to} x_n]$ converges to $[x \overset{0}{\to} x]$ in the topology of $\mathcal{G}^f$ and the flow-out of M^f is open, for n large enough $[\gamma_n]$ is in the flow-out of M^f which is a contradiction. ■

For any $R > 0$, let $\mathcal{G}^f{}_R = \{[\gamma] \in \mathcal{G}^f \mid \text{length } [\gamma] \leq R\}$. Let U_β be a family of foliation charts covering M, and for any leafwise path γ from $f(U_\beta)$ to U_β let U_β^γ be the corresponding chart for $\mathcal{G}^f$.

Proposition 2.2.13 *$\mathcal{G}^f{}_R$ can be covered by a finite number of charts of the form U_β^γ. There are constants a and b depending only on the covering $\{U_\beta\}$ such that the number of charts needed to cover $\mathcal{G}^f{}_R$ is $< ae^{bR}$.*

Proof. Since M is compact, we can assume that the covering $\{U_\beta\}$ is finite. For each U_β consider $r^{-1}(U_\beta)\cap \mathcal{G}^f{}_R = \{[\gamma]\in \mathcal{G}^f \mid \gamma$ is a leafwise path from $f(U_\beta)$ to U_β of length $\leq R\}$. A holonomy map h_γ on Σ_β is determined by the sequence of charts through which γ passes. Let ϵ be the Lebesgue number for the covering $\{U_\beta\}$ and let $d = \max_\alpha |\{\alpha' \mid U_\alpha \cap U_{\alpha'} \neq \phi\}|$. Then for any path γ of length $\leq R$, there is a covering of this path by a finite sequence of U_α's where the length of the sequence is at most R/ϵ. Then the number of such possible sequences is at most $d^{R/\epsilon}$ so that $r^{-1}(U_\beta)\cap \mathcal{G}^f{}_R$ is covered by at most $d^{R/\epsilon}$ charts of the form U^γ_β. Since there are a finite number of U_β, the result follows. ∎

Finally we discuss what the transfixed hypothesis means in the case where k has the maximal value q. In this case, every leaf of $\mathcal{F}$ must be fixed and furthermore $\mathcal{F}$ can have no nontrivial holonomy. For if f has dimension q transverse fixed set at $[f(y) \xrightarrow{\gamma} y]$ then $h_\gamma \circ f$ fixes an entire transversal Σ through y. If γ_1 is a nontrivial element of G^y_y, then $[f(y) \xrightarrow{\gamma * \gamma_1} y]$ is also in $\mathcal{G}^f$ and $h_{\gamma*\gamma_1}\circ f = h_{\gamma_1}\circ h_\gamma \circ f$ does not fix all of Σ since h_{γ_1} is nontrivial. Thus the theory to be developed does not apply well in the case where $k = q$.

2.3. Densities

We review some facts about densities [14]. A density on a finite dimensional vector space A is a map g from {ordered bases of A} to the complex numbers such that for any change of basis T and any basis α, $g(T\alpha) = |\det T| g(\alpha)$. Obviously, the image of such a map is a real ray in $\mathbf{C}$ (or 0), and the set of densities on A is a complex line. We will write $|A|$ for this complex line. If $(u_1, \ldots, u_n)$ is a basis for A, then the density that maps $(u_1, \ldots, u_n)$ to 1 will be denoted by $|u_1^* \wedge \ldots \wedge u_n^*|$.

An isomorphism $A \xrightarrow{f} B$ determines an element of $|A| \otimes |B^*|$. If $B = A$, then $|A| \otimes |A^*|$ is canonically isomorphic to $\mathbf{C}$ and under this isomorphism, the above element of $|A| \otimes |A^*|$ corresponds to $|\det f|$. An exact sequence $0 \to A \to B \to C \to 0$ induces an isomorphism of $|B|$ with $|A| \otimes |C|$.

If E is a vector bundle over a manifold M, then we can form the associated density bundle $|E|$ over M. It is a complex line bundle. In particular we can do this for the tangent bundle to get $|TM|$. Sections of this are called densities on M, and the nice thing about them is that they can be integrated over M, i.e., if $s \in C_c^\infty(|TM|)$, then $\int_M s$ is well-defined, regardless of whether M is oriented. This follows from the local formula for changing coordinates in integrals. Densities on M correspond to deRham's top degree forms of odd type [8].

Suppose that f is a transfixed morphism of dimension k of $(M, \mathcal{F})$. Then $\mathcal{G}^f$ is a manifold, but, as it is not Hausdorff, integration is somewhat delicate. Smooth partitions of unity subordinate to an atlas of charts do not necessarily exist. When a function which is smooth with compact support in a chart is extended by zero outside the chart, it is no longer necessarily continuous. Although such a function is not continuous on $\mathcal{G}^f$, we shall nevertheless refer to it as a smooth function with compact support on $\mathcal{G}^f$. Finite sums of such functions are also considered to be smooth with compact support. In other words make the definition:

Definition 2.3.1 $C_c^\infty(\mathcal{G}^f) = \{$*functions f on $\mathcal{G}^f$ | f can be written as a finite sum of f_i where for each f_i, there is a chart U_i for $\mathcal{G}^f$ with $f_i \in C_c^\infty(U_i)$*$\}$. *Define $C_c^\infty(E)$ similarly where E is any vector bundle over $\mathcal{G}^f$.*

With these definitions, integration of an element s of $C_c^\infty(|T\mathcal{G}^f|)$ is straightforward because such an s comes already decomposed into coordinate pieces. The existence of partitions of unity on coordinate charts implies that the value of $\int_{\mathcal{G}^f} s$ is independent of the decomposition of s used.

What does a section of $|T\mathcal{G}^f|$ look like locally? From Proposition 2.2.8, $|T\mathcal{G}^f| \cong |r^*(T\mathcal{F})| \otimes |N\mathcal{F}^{hf}|$. Thus a density on $\mathcal{G}^f$ is a leafwise density tensored with a section of $|N\mathcal{F}^{hf}|$. A section ν of the density bundle $|N\mathcal{F}^{hf}|$ over $\mathcal{G}^f$ will be called a transfixed density. What does this look like locally? At a particular element $[f(y) \xrightarrow{\gamma} y]$ of $\mathcal{G}^f$, choose a foliation chart $\varphi_\alpha : U_\alpha \to R^{p+q}$ around y so that $h_\gamma \circ f$ on R^q fixes $R^k \times \{0\}$. Recall that then $U_\alpha^\gamma = (f(U_\alpha) \overset{\gamma}{\times} U_\alpha) \cap \mathcal{G}^f$ is a chart for $\mathcal{G}^f$, mapping to R^{p+k} by $\varphi_\alpha \circ r$. $N\mathcal{F}^{hf}$ is locally spanned by $\partial/\partial y^{p+1}, \ldots, \partial/\partial y^{p+k}$ so that any section of $|N\mathcal{F}^{hf}|$ will be written locally as

$$\nu^\gamma(y', y^{p+1}, \ldots, y^{p+k})|dy^{p+1} \wedge \ldots \wedge dy^{p+k}|.$$

Holonomy along leafwise paths in $\mathcal{G}^f$ acts on the bundle $N\mathcal{F}^{hf}$. A leafwise path in $\mathcal{G}^f$ from $[f(y) \xrightarrow{\gamma} y]$ to $[f(x) \xrightarrow{\tilde{\gamma}} x]$ is given by a leafwise path $y \xrightarrow{\gamma'} x$ in M such that $[\tilde{\gamma}] = [f(\gamma')^{-1} * \gamma * \gamma']$. Since $N\mathcal{F}^{hf}{}_{[\gamma]} = \{v \in N\mathcal{F}_y \mid (h_\gamma \circ f)v = v\}$ and $N\mathcal{F}^{hf}{}_{[\tilde{\gamma}]} = \{v \in N\mathcal{F}_x \mid (h_{\tilde{\gamma}} \circ f)v = v\}$ and $h_{\tilde{\gamma}} \circ f$ is $h_\gamma \circ f$ conjugated by $h_{\gamma'}$, $h_{\gamma'}$ takes $N\mathcal{F}^{hf}{}_{[\gamma]}$ to $N\mathcal{F}^{hf}{}_{[\tilde{\gamma}]}$.

Definition 2.3.2 *A smooth section of $|N\mathcal{F}^{hf}|$ over $\mathcal{G}^f$ that is invariant under holonomy along leafwise paths is $\mathcal{G}^f$ will be called an* invariant transfixed k-density.

The morphism $f : M \to M$ naturally induces a map $f : \mathcal{G}^f \to \mathcal{G}^f$, namely $f([f(x) \xrightarrow{\gamma} x]) = [f^2(x) \xrightarrow{f(\gamma)} f(x)]$. The tangent map f_* takes $N\mathcal{F}^{hf}{}_{[\gamma]}$ to $N\mathcal{F}^{hf}{}_{[f(\gamma)]}$ and the dual to this pulls back $|N\mathcal{F}^{hf}|_{[f(\gamma)]} \xrightarrow{f^*} |N\mathcal{F}^{hf}|_{[\gamma]}$.

Proposition 2.3.3 *If ν is an invariant transfixed k-density, then $f^*\nu = \nu$.*

Proof. The composition $h_\gamma \circ f$ is the identity on $N\mathcal{F}^{hf}{}_{[\gamma]}$ and hence

$$\nu = (h_\gamma \circ f)^*\nu = f^*(h_\gamma^*\nu).$$

Since ν is holonomy invariant, $h_\gamma^*\nu = \nu$. Thus $\nu = f^*\nu$. ■

Note that ν lives on $\mathcal{G}^f$ and not on M. It may be the case, though, that ν comes from something on M. Haefliger [15] introduced the notion of invariant transverse k-forms for any k with $0 \leq k \leq q$. If α is a smooth transverse k-form on M, (i.e., a smooth section of $\bigwedge^k N^*\mathcal{F}$), then $|r^*\alpha|$ is a smooth section of $|N\mathcal{F}^{hf}|$. The definition of $|r^*\alpha|$ is the obvious one: given an ordered basis $(v_1, \ldots, v_k)$ of $N\mathcal{F}^{hf}{}_{[\gamma]}$, $|r^*\alpha|$ assigns to it the value $|\alpha(r_*v_1 \wedge \ldots \wedge r_*v_k)|$. Clearly, in order that $|r^*\alpha|$ be an invariant transfixed k-form, it is sufficient that α be holonomy invariant, so that Haefliger's invariant transverse k-forms induce invariant transfixed k-densities. We now investigate some conditions that almost give a converse to this.

First, there is the algebraic lemma:

Lemma 2.3.4 *If T is an endomorphism of the finite-dimensional vector space V, then $V/\ker T \xrightarrow{T} V/\ker T$ is invertible if and only if the natural map $\ker T^* \subset V^* \longrightarrow (\ker T)^*$ is a bijection.*

Proof. The kernel of the map $\ker T^* \longrightarrow (\ker T)^*$ is $\{v^* \in V^* \mid v^*(\text{im } T) = 0$ and $v^*(\ker T) = 0\}$. Thus this map is a bijection $\Longleftrightarrow$

$\text{im } T + \ker T = V \Longleftrightarrow \text{im } T/\ker T = V/\ker T \Longleftrightarrow V/\ker T \xrightarrow{T} V/\ker T$ is bijective.

■

Assuming f is a transfixed morphism, apply this lemma to $V = N\mathcal{F}_x$ and $T = \text{Id} - (h_\gamma \circ f)_*$ at any point $[f(x) \xrightarrow{\gamma} x]$ in $\mathcal{G}^f$. Then $\ker T = N\mathcal{F}^{hf}{}_{[\gamma]}$ and $\ker T^* = \ker(\text{Id} - (h_\gamma \circ f)^*) = \{\alpha \in N^*\mathcal{F}_x \mid (h_\gamma \circ f)^*\alpha = \alpha\}$. The lemma says that in each fiber of $N^*\mathcal{F}_x \longrightarrow (N\mathcal{F}^{hf}{}_{[\gamma]})^*$ there is a unique covector that is invariant under $(h_\gamma \circ f)^*$.

Proposition 2.3.5 *If f is transfixed of dimension k and α is a transverse k-form on M such that*

1. *for every $[f(x) \xrightarrow{\gamma} x] \in \mathcal{G}^f, (h_\gamma \circ f)^*\alpha_x = \alpha_x$, and*

2. *the transfixed k-density $|r^*\alpha|$ is holonomy invariant on $\mathcal{G}^f$*

then $\alpha|_{r(\mathcal{G}^f)}$ is holonomy invariant.

Proof. Consider the holonomy along $y \xrightarrow{\gamma'} x$ from $[f(y) \xrightarrow{\gamma} y]$ in $\mathcal{G}^f$ to $[f(x) \xrightarrow{\tilde{\gamma}} x] = [f(\gamma')^{-1} * \gamma * \gamma']$ in $\mathcal{G}^f$. The linear holonomy $h_{\gamma' *}$ from $N\mathcal{F}_y$ to $N\mathcal{F}_x$ takes $N\mathcal{F}^{hf}{}_{[\gamma]}$ to $N\mathcal{F}^{hf}{}_{[\tilde{\gamma}]}$. Taking the duals to this gives a commutative diagram:

$$\begin{array}{ccc} N^*\mathcal{F}_y & \longrightarrow & (N\mathcal{F}^{hf}{}_{[\gamma]})^* \\ {\scriptstyle h^*_{\gamma'}}\uparrow & & \uparrow{\scriptstyle h^*_{\gamma'}} \\ N^*\mathcal{F}_x & \longrightarrow & (N\mathcal{F}^{hf}{}_{[\tilde{\gamma}]})^* \end{array}$$

and then:

$$\begin{array}{ccc} \bigwedge^k N^*\mathcal{F}_y & \longrightarrow & \bigwedge^k (N\mathcal{F}^{hf}{}_{[\gamma]})^* \\ {\scriptstyle h^*_{\gamma'}}\uparrow & & \uparrow{\scriptstyle h^*_{\gamma'}} \\ \bigwedge^k N^*\mathcal{F}_x & \longrightarrow & \bigwedge^k (N\mathcal{F}^{hf}{}_{[\tilde{\gamma}]})^* \end{array}$$

The assumption about the holonomy invariance of $|r^*\alpha|$ says that α is invariant under the holonomy on the right side of the diagram. We want to show that it is invariant on the left side. The lemma and succeeding discussion imply that in each horizontal fiber there is a unique element satisfying the first assumption. Thus if we can show that $h^*_{\gamma'}\alpha$ satisfies 1 at $[\gamma]$, then we must have $h^*_{\gamma'}\alpha = \alpha$. But this is once again the conjugation argument: since α satisfies 1 at $[\tilde{\gamma}]$,

$$\alpha = (h_{\tilde{\gamma}} \circ f)^*\alpha = (h_{\gamma'} \circ h_\gamma \circ h_{f(\gamma')^{-1}} \circ f)^*\alpha = (h_{\gamma'} \circ h_\gamma \circ f \circ h_{\gamma'^{-1}})^*\alpha.$$

Thus $h^*_{\gamma'}\alpha = (h_\gamma \circ f)^* h^*_{\gamma'}\alpha$. ■

CHAPTER 3

KERNELS AND TRACES

The definition of trace we use in the foliated Lefschetz theorem is motivated by the formalism of distribution theory. In this chapter, we investigate how this theory applies to leafwise operators, give the definition of trace, and prove a key property of the trace.

3.1. Generalized Sections of Bundles

A smooth section of a vector bundle E over a manifold X gives a continuous linear functional on the topological vector space $C_c^\infty(E^* \otimes |TX|)$ by pairing E and E^* to get a smooth function and then integrating the product of this with the density over X. Thus, in analogy with distribution theory, we define:

Definition 3.1.1 ([14]) *A* generalized section *of the vector bundle E over X is a continuous linear map from $C_c^\infty(E^* \otimes |TX|)$ to* C. *The collection of these is written as $C^{-\infty}(E)$.*

The support of a generalized section is defined as in distribution theory, and the space of compactly supported distributions, $C_c^{-\infty}(E)$, turns out to then be the dual of $C^\infty(E^* \otimes |TX|)$.

Notation. To make the base space explicit, we shall sometimes write $C^\infty(E \to X)$, $C^{-\infty}(E \to X)$, $C_c^\infty(E \to X)$, and so forth.

One example of a generalized section of E over X is a δ-section of E with support on Z where Z is a closed submanifold of X. Let N^*Z be the conormal bundle of Z in X. Then there is a natural inclusion $C^\infty(E \otimes |N^*Z| \to Z) \xrightarrow{i} C^{-\infty}(E \to X)$. This is obtained as follows. Given $\sigma \in C^\infty(E \otimes |N^*Z| \to Z)$ and $s \in C_c^\infty(E^* \otimes |TX| \to X)$, first pull s back to Z. Over each $z \in Z$, the E and

E^* factors pair to give a number. What about $|N^*Z|$ and $|TX|$? From the exact sequence

$$0 \longrightarrow T_zZ \longrightarrow T_zX \longrightarrow N_zZ \longrightarrow 0$$

we get $|T_zX| \cong |T_zZ| \otimes |N_zZ|$ or $|T_zX| \otimes |N_z^*Z| \cong |T_zZ|$. So σ and s pair to give an element of $C_c^\infty(|TZ| \to Z)$ and this is then integrated to give a number.

Definition 3.1.2 *Such a generalized section u of E is called a δ-section of E with support in Z and the section $\sigma \in C^\infty(E \otimes |N^*Z| \to Z)$ from which it came is called the symbol of u, $\sigma(u)$. The set of all such δ-sections is written $C_Z^\delta(E)$.*

In the same way there is an inclusion of $C_c^\infty(E \otimes |N^*Z| \to Z)$ into $C_c^{-\infty}(E \to X)$. For this inclusion, instead of being a closed submanifold of X, it is enough that $Z \xrightarrow{i} X$ be an immersion. (We shall call such a manifold Z an immersed submanifold, even if i is not injective.) In this case N^*Z is still a vector bundle over Z, and $C^\infty(E^* \otimes |TX| \to X)$ pulls back to $C^\infty(E^* \otimes |TX| \to Z)$. Pairing with an element of $C_c^\infty(E \otimes |N^*Z|)$ gives a compactly supported section of $|TZ|$ which is integrated to get a number. The set of such δ-sections will be denoted by $C_{cZ}^\delta(E)$.

Actually the inclusion $C_c^\infty(E \otimes |N^*Z| \to Z) \longrightarrow C_c^{-\infty}(E \to X)$ can be extended to $C_c^{-\infty}(E \otimes |N^*Z| \to Z) \longrightarrow C_c^{-\infty}(E \to X)$. This map is just the dual map to the restriction map $C^\infty(E^* \otimes |TX| \to X) \longrightarrow C^\infty(E^* \otimes |NZ| \otimes |TZ|)$.

How do generalized sections transform under morphisms of vector bundles? A morphism (f, T) from a vector bundle $(E \to X)$ to a vector bundle $(F \to Y)$ is a smooth map $f : X \to Y$ together with a map $T : f^*F \to E$ of bundles over X. Obviously this gives a map $(f,T)^* : C^\infty(F) \to C^\infty(E)$ and hence , by duality, a map $(f,T)_* : C_c^{-\infty}(E^* \otimes |TX|) \to C_c^{-\infty}(F^* \otimes |TY|)$. We shall usually suppress the T and write only f^* for pull-backs and f_* for push-forwards. If f is proper, we also get $f^* : C_c^\infty(F) \to C_c^\infty(E)$ and by duality $f_* : C^{-\infty}(E^* \otimes |TX|) \to C^{-\infty}(F^* \otimes |TY|)$. It is not always the case that f^* can be extended continuously to $C^{-\infty}(F) \to C^{-\infty}(E)$, or, in the proper case, to $C_c^{-\infty}(F) \to C_c^{-\infty}(E)$. However if $f : X \to Y$ is a submersion, f^* does extend as discussed in [14, Chapter 6] or [17, Chapter 6]. We can describe this extension by the dual mapping: $C_c^\infty(E^* \otimes |TX|) \longrightarrow C_c^\infty(F^* \otimes |TY|)$. Starting with σ from $C_c^\infty(E^* \otimes |TX|)$, the section $T^*\sigma$ is in $C_c^\infty(f^*F^* \otimes |TX| \to X)$. Given $y \in Y$, the inverse image $f^{-1}(y)$ is a submanifold of X since f is a submersion, and for each $x \in f^{-1}(y)$,

$$0 \longrightarrow T_x(f^{-1}(y)) \longrightarrow T_xX \xrightarrow{f_*} T_yY \longrightarrow 0$$

is exact so $|T_xX| \cong |T_xf^{-1}(y)| \otimes |T_yY|$. Thus $T^*\sigma(x)$ is an element of $F_y^* \otimes |T_xf^{-1}(y)| \otimes |T_yY|$ and $T^*\sigma \in C_c^\infty(f^*F^* \otimes f^*|TY| \otimes |T(\text{fiber})|)$. We can now integrate along the fibers to get $f_*(\sigma)$ in $C_c^\infty(F^* \otimes |TY|)$.

We summarize this discussion:

Proposition 3.1.3 *If (f, T) is a vector bundle morphism from $(E \to X)$ to $(F \to Y)$, then under the following conditions, the stated maps are continuous:*

1. *no additional conditions on f*

   $f^* : C^\infty(F) \to C^\infty(E)$ $\qquad$ $f_* : C_c^{-\infty}(E^* \otimes |TX|) \to C_c^{-\infty}(F^* \otimes |TY|)$

2. *f proper*

   $f^* : C_c^\infty(F) \to C_c^\infty(E)$ $\qquad$ $f_* : C^{-\infty}(E^* \otimes |TX|) \to C^{-\infty}(F^* \otimes |TY|)$

3. *f a submersion*

   $f^* : C^{-\infty}(F) \to C^{-\infty}(E)$ $\qquad$ $f_* : C_c^\infty(E^* \otimes |TX|) \to C_c^\infty(F^* \otimes |TY|)$

4. *f a submersion and proper*

   $f^* : C_c^{-\infty}(F) \to C_c^{-\infty}(E)$ $\qquad$ $f_* : C^\infty(E^* \otimes |TX|) \to C^\infty(F^* \otimes |TY|)$.

More generally, the question of when the pull-back of functions can be extended continuously to distributions can be answered in terms of wave front sets. If $f : X \to Y$ is a smooth map, let:

$$N^*f = \{\eta \in T^*Y \mid \exists x \in X \text{ with } \eta \in T^*_{f(x)}Y \text{ and } f^*\eta = 0\}.$$

Then the pull-back f^* extends continuously to all generalized functions u on Y whose wave front sets $\mathrm{WF}(u)$ are disjoint from N^*f [17, Thm. 8.2.4]. If $u \in C_Z^\delta(F)$ where Z is a closed submanifold of Y, then $\mathrm{WF}(u) \subset (N^*Z \backslash 0)$ so a δ-section of F supported on Z can be pulled back to X provided that no nonzero conormal vector of Z is mapped to zero by f^*. But this is exactly the condition that $f : X \to Y$ should be transverse to the submanifold Z. Thus we have the proposition:

Proposition 3.1.4 *If Z is a closed submanifold of Y and $f : X \to Y$ is transverse to it, then $f^* : C^\infty(F) \to C^\infty(E)$ extends continuously to a map $f^* : C^\delta_Z(F) \to C^\delta_W(E)$ where W is the closed submanifold $f^{-1}(Z)$ of X. If $Z \xrightarrow{i} Y$ is an immersion and $f : X \to Y$ is transverse to it and proper, then $f^* : C^\infty_c(F) \to C^\infty_c(E)$ extends continuously to a map $f^* : C^\delta_{cZ}(F) \to C^\delta_{cW}(E)$ where W is the immersed submanifold $X \overset{Y}{\times} Z = \{(x,z) \mid f(x) = i(z)\}$ of X.*

The pull-back maps on δ-sections given by the proposition are easy to describe in terms of symbols of the δ-sections. An element u of $C^\delta_Z(F)$ has a symbol $\sigma(u) \in C^\infty(F \otimes |N^*Z| \to Z)$. Since f is transverse to Z, f^* is an isomorphism of the conormal bundle N^*Z of Z in Y with the conormal bundle N^*W of W in X. The fiber map T takes the fiber of F to the fiber of E so we get a section $f^*\sigma(u) \in C^\infty(E \otimes |N^*W| \to W)$, and this is the symbol of the pulled back δ-section f^*u, i.e., $f^*\sigma(u) = \sigma(f^*u)$.

We shall need to consider a variation of the situation in the proposition. Suppose that Z is a closed submanifold of Y with codimension q and $f : X \to Y$ is a smooth map such that

1. $W = f^{-1}(Z)$ is a closed submanifold of X of codimension $q - r$, and
2. $f^*N^*Z \xrightarrow{f^*} N^*W$ is surjective as a bundle map over W.

In this case, the kernel of f^* is a smooth vector bundle over W of dimension r, and there is an exact sequence of vector bundles

$$0 \longrightarrow \ker f^* \longrightarrow f^*N^*Z \longrightarrow N^*W \longrightarrow 0$$

over W. Given $\nu \in C^\infty(|\ker f^*|^* \to W)$, we can form a map $f^*_\nu : C^\delta_Z(F) \to C^\delta_W(E)$. This is obtained as follows. Any u in $C^\delta_Z(F)$ has a symbol $\sigma(u)$ in $C^\infty(F \otimes |N^*Z| \to Z)$. Over a point x in W, the fiber map T takes $F_{f(x)}$ to E_x and f^* takes $f^*N^*Z_x = N^*Z_{f(x)}$ to N^*W_x. From the exact sequence above $|f^*N^*Z_x| \cong |N^*W_x| \otimes |\ker f^*_x|$ so the pull-back of the symbol, $f^*(\sigma(u))$, naturally lies in $C^\infty(E \otimes |N^*W| \otimes |\ker f^*| \to W)$. Since $|\ker f^*| \otimes |\ker f^*|^* \cong \mathbf{C}$, the section ν pairs with $f^*(\sigma(u))$ to give an element of $C^\infty(E \otimes |N^*W| \to W)$ which is the symbol of an element of $C^\delta_W(E)$. Thus $f^*_\nu : C^\delta_Z(F) \to C^\delta_W(E)$ is characterized by $\sigma(f^*_\nu(u)) = f^*(\sigma(u)) \otimes \nu$.

This map of δ-sections over Z to δ-sections over W is not a continuous extension of $C^\infty(F \to Y) \longrightarrow C^\infty(E \to X)$. If $r > 0$ so that $f : X \to Y$ is not transverse to Z, then there is a δ-section u in $C^\delta_Z(F)$ and a sequence of smooth sections $u_n \in C^\infty(F)$ such that $u_n \to u$ as generalized sections but $f^*u_n = u_n \circ f$ fails to converge in $C^{-\infty}(E)$ [17]. Thus there is no way of extending $C^\infty(F) \xrightarrow{f^*} C^\infty(E)$ in such a way as to both include $C^\delta_Z(F)$ in the domain and to be continuous in the topology induced from $C^{-\infty}(F)$. Nevertheless, the map $f^*_\nu : C^\delta_Z(F) \to C^\delta_W(E)$ is certainly continuous by itself and is the best we can do in this situation.

If $Z \xrightarrow{i} Y$ is an immersed submanifold instead of a closed submanifold, and $f : X \to Y$ is proper, we can make the analogous assumptions that $W = X \overset{Y}{\times} Z \to X$ is an immersion of codimension $q - r$ and that $f^*(N^*Z) \to f^*W$ is surjective. Then using a smooth section ν of $|\ker f^*|^*$ over W, a map on the symbols is defined in the same way and this induces a map $f^*_\nu : C^\delta_{cZ}(F) \to C^\delta_{cW}(E)$.

In some situations, the push-forward of a δ-section is again a δ-section. For example, if there is a commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where W and Z are immersed submanifolds of X and Y respectively, and if g is a submersion, then the push-forward $f_* : C_c^{-\infty}(E^* \otimes |TX|) \longrightarrow C_c^{-\infty}(F^* \otimes |TY|)$ actually carries $C^\delta_{cW}(E^* \otimes |TX|)$ into $C^\delta_{cZ}(F^* \otimes |TY|)$. Once again the map on δ-sections is readily described in terms of the symbols. Any u in $C^\delta_{cW}(E^* \otimes |TX|)$ has a symbol $\sigma(u)$ in $C_c^\infty(E^* \otimes |TX| \otimes |N^*W| \to W)$. But $|TX| \otimes |N^*W| \cong |TW|$ and $g : W \to Z$ is a submersion so we can push forward to get $g_*\sigma(u) \in C_c^\infty(F^* \otimes |TZ| \to Z)$ which is the same as $C_c^\infty(F^* \otimes |TY| \otimes |N^*Z| \to Z)$. Then one can check that the corresponding element of $C^\delta_{cZ}(F^* \otimes |TY|)$ is actually f_*u.

In the context of generalized sections of bundles, the Schwartz kernel theorem takes the following form:

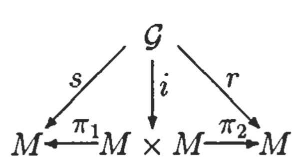
Theorem 3.1.5 ([17, Thm. 5.2.1]) *Suppose E and F are bundles over X and Y respectively. Then any element k of* $C^{-\infty}(\mathrm{Hom}(\pi_Y^* F, \pi_X^* E) \otimes \pi_Y^* |TY| \to X \times Y)$ *defines a continuous linear map $\tilde{k} : C_c^\infty(F) \to C^{-\infty}(E)$ by $g \mapsto \pi_{X*}(k \cdot \pi_Y^* g)$. Conversely, for every such continuous linear map T, there is a unique such element k with $T = \tilde{k}$. If dual spaces are given the topology of uniform convergence on bounded subsets, then the space $L(C_c^\infty(F), C^{-\infty}(E))$ of bounded linear maps is topologically isomorphic to* $C^{-\infty}(\mathrm{Hom}(\pi_Y^* F, \pi_X^* E) \otimes \pi_Y^* |TY|)$.

3.2. Kernels of Leafwise Operators

We shall now study leafwise operators on a foliated manifold $(M, \mathcal{F})$. Roughly speaking, a leafwise operator is an operator D on M that restricts to operators on each of the leaves of M. That is, for any leaf L, there is an operator D_L such that $(Ds)|_L = D_L(s|_L)$ for every section s of the appropriate bundle. One can impose various conditions on the transverse regularity of the operators and in keeping with the smooth theme of this work, the condition will be that the operators vary smoothly in the transverse direction. More precisely, we will make the following definition.

Definition 3.2.1 *A* smooth leafwise operator *from $C^\infty(F \to M)$ to $C^{-\infty}(E \to M)$ is an operator whose Schwarz kernel is in* $C^\delta_{c\mathcal{G}}(\mathrm{Hom}(F, E) \otimes \pi_2^* |TM| \to M \times M)$.

The picture is:



Note that $\mathrm{Hom}(F, E)$ is an abuse of notation for $\mathrm{Hom}(\pi_2^* F, \pi_1^* E)$.

According to the definition of δ-sections, any smooth leafwise operator comes from a symbol in $C_c^\infty(\mathrm{Hom}(F, E) \otimes |\pi_2^* TM| \otimes |N^* \mathcal{G}| \to \mathcal{G})$. Thus the definition of a smooth leafwise operator actually requires that the kernel be smooth and compactly supported along leaves as well as being smooth transversely. The reason for making the compactness of support requirement is twofold. First, just as for $\mathcal{G}^f$, the C_c^∞ theory on the non-Hausdorff manifold $\mathcal{G}$ is well-behaved while the C^∞ theory is not, as discussed in Section 2.3. Second, the theory of δ-sections supported on immersed submanifolds requires compactly supported symbols as discussed in Section 3.1.

The wavefront set of a smooth leafwise operator is contained in $N^*\mathcal{G}$.

Proposition 3.2.2 *Over $\mathcal{G}$ there is a canonical isomorphism $r^*|TM| \otimes |N^*\mathcal{G}| \cong r^*|T\mathcal{F}|$.*

Proof. Over a point $[\gamma]$ in $\mathcal{G}$, there is an exact sequence of vector bundles

$$0 \longrightarrow T\mathcal{G} \longrightarrow (s \times r)^* T(M \times M) \longrightarrow N\mathcal{G} \longrightarrow 0$$

and thus an isomorphism $s^*|TM| \otimes r^*|TM| \cong |T\mathcal{G}| \otimes |N\mathcal{G}|$ so $r^*|TM| \otimes |N^*\mathcal{G}| \cong |T\mathcal{G}| \otimes s^*|TM|^*$. Using coordinates around $[\gamma]$ on $\mathcal{G}$ coming from the leafwise and transverse coordinates of the source and the leafwise coordinates for the range, we see that there is also an exact sequence

$$0 \longrightarrow s^*TM \longrightarrow T\mathcal{G} \longrightarrow r^*\mathcal{F} \longrightarrow 0$$

and thus an isomorphism $|T\mathcal{G}| \cong s^*|TM| \otimes r^*|T\mathcal{F}|$. Combining these isomorphisms gives

$$r^*|TM| \otimes |N^*\mathcal{G}| \cong r^*|T\mathcal{F}| \otimes s^*|TM| \otimes s^*|TM|^* \cong r^*|T\mathcal{F}|.$$

■

This proposition shows that the symbol of the kernel of a smooth leafwise operator is an element of $C_c^\infty(\mathrm{Hom}(F,E) \otimes r^*|T\mathcal{F}| \longrightarrow \mathcal{G})$. By the definition of C_c^∞ over $\mathcal{G}$, this symbol is actually a finite sum of smooth sections with support inside coordinate charts. If $k \in C_c^\infty(\mathrm{Hom}(F,E) \otimes r^*|T\mathcal{F}|)$ is supported in the coordinate chart $U_\alpha \overset{\gamma}{\times} U_\beta \overset{\varphi_\alpha \times \varphi'_\beta}{\longrightarrow} R^{2p+q}$ of $\mathcal{G}$, then in this coordinate chart k can be expressed as $k^\gamma(x', x'', y')|dy^1 \wedge \ldots dy^p|$ where k^γ takes values in $\mathrm{Hom}(F,E)$. In other words we see that k really is a leafwise operator parametrized transversely. The coordinates x'' are the transverse parameters while $k^\gamma(x', y')|dy'|$ represents the kernel of a leafwise operator. As an operator

$$\mathrm{Op}(k^\gamma)\phi(x', x'') = \int_{P_{h_\gamma(x'')}} k^\gamma(x', x'', y')\phi(y', h_\gamma(x''))|dy'|$$

where $P_{h_\gamma(x'')}$ is the plaque of U_β through $h_\gamma(x'')$ and ϕ is a smooth section of F over U_β.

Proposition 3.2.3 *Any smooth leafwise operator takes $C^\infty(F)$ to $C^\infty(E)$.*

Proof. This follows immediately from the local description of the operator just given. An alternate proof uses the formalism of δ-sections. The Schwartz Theorem says that $(\mathrm{Op}k)\phi = s_*(k \cdot r^*\phi)$. The distribution $k \cdot r^*g$ is a δ-section supported on $\mathcal{G}$ whose symbol is in $C_c^\infty(E \otimes r^*|T\mathcal{F}|)$. The map $\mathcal{G} \xrightarrow{s} M$ is a submersion with $T(\text{fibers}) = r^*T\mathcal{F}$ so that the push-forward $s_*(k \cdot r^*\phi)$ is actually a smooth section of E over M. ■

As a distributional section of $\mathrm{Hom}(F, E) \otimes \pi_2^*|TM|$, any k in $C_{c\mathcal{G}}^\delta$ is an element of the dual of $C^\infty((\mathrm{Hom}(F, E) \otimes \pi_2^*|TM|)^* \otimes |T(M \times M)|)$. But

$$\pi_2^*|TM|^* \otimes |T(M \times M)| \cong \pi_1^*|TM|$$

and $\mathrm{Hom}(F, E)^* \cong F \otimes E^*$. Thus k pairs with a smooth section of $F \otimes E^* \otimes \pi_1^*|TM|$ over $M \times M$ to give a number.

Proposition 3.2.4 *The pairing between $k \in C_{c\mathcal{G}}^\delta(\mathrm{Hom}(F, E) \otimes \pi_2^*|TM|)$ and $\phi \in C^\infty(F \otimes E^* \otimes \pi_1^*|TM|)$ is*

$$\langle k, \phi\rangle = \int_{x \in M} \left(\int_{\gamma \in s^{-1}(x)} k(x \xrightarrow{\gamma} y)\phi(x, y)|dy'| \right) |dx|.$$

Proof. The definition of the δ-section k is that the pairing is given by pulling ϕ up to $\mathcal{G}$ and then integrating $k\phi$ over $\mathcal{G}$. But for any coordinate chart $U_\alpha \overset{\gamma}{\times} U_\beta$ for $\mathcal{G}$, the Fubini theorem says the integral over $U_\alpha \overset{\gamma}{\times} U_\beta$ can be obtained by first integrating over the fibers of $U_\alpha \overset{\gamma}{\times} U_\beta \to U_\alpha$ and then integrating over U_α in M. This remains valid after summing over a finite number of charts, and hence integration over $\mathcal{G}$ may be obtained by integrating over the fibers of $\mathcal{G} \xrightarrow{s} M$ and then integrating over M. This is precisely what the equation asserts. ∎

Next we shall consider the composition of two smooth leafwise operators. To make the notation less cumbersome, we shall drop the bundles F and E and only deal with functions and densities. With the functorial approach, convolving the distributional kernels of two operators B_1 and B_2 involves:

1. taking the tensor product $k_{B_1} \otimes k_{B_2}$ as a generalized section of $\pi_2^*|TM| \otimes \pi_4^*|TM|$ over $M \times M \times M \times M$;
2. pulling this back to $M \times M \times M$ via the diagonal map $\Delta(x, z, y) = (x, z, z, y)$ so that $\Delta^*(k_{B_1} \otimes k_{B_2}) \in C^{-\infty}(\pi_2^*|TM| \otimes \pi_3^*|TM| \longrightarrow M \times M \times M)$; and
3. pushing this down to $M \times M$ via the projection $\pi(x, z, y) = (x, y)$.

Thus $k_{B_1 \circ B_2} = \pi_*(\Delta^*(k_{B_1} \otimes k_{B_2})) \in C^{-\infty}(\pi_2^*|TM| \longrightarrow M \times M)$ [9, page 33].

Proposition 3.2.5 *Suppose B_1 and B_2 are smooth leafwise operators with symbols k^{γ_1} and k^{γ_2} supported in the coordinate charts $U_1 \overset{\gamma_1}{\times} U_2$ and $U_3 \overset{\gamma_2}{\times} U_4$ respectively. Then if $U_2 \cap U_3 = \phi$, the composition $B_1 \circ B_2 = 0$. If $U_2 \cap U_3 \neq \phi$, then the kernel of $B_1 \circ B_2$ is supported in $U_1 \overset{\gamma_1 * \gamma_2}{\times} U_4$ and its symbol*

$$k^{\gamma_1 * \gamma_2}(x', x'', y')|dy'| = \left(\int k^{\gamma_1}(x', x'', z') k^{\gamma_2}(z', h_{\gamma_1}(x''), y')|dz'| \right) |dy'|$$

where the integral is taken over all $z' \in$ (Plaque of U_2 through $h_{\gamma_1}(x'')) \cap U_3$

Proof. $k_{B_1} \otimes k_{B_2}$ is a δ-section with support in the immersed submanifold $\mathcal{G} \times \mathcal{G} \longrightarrow M \times M \times M \times M$. The diagonal Δ is transverse to this so the pull-back $\Delta^*(k_{B_1} \otimes k_{B_2})$ is a δ-section with support in the immersed submanifold

$$\mathcal{G}^2 \stackrel{\text{def}}{=} \{[x \xrightarrow{\gamma} z \xrightarrow{\gamma'} y] \mid [\gamma], [\gamma'] \in \mathcal{G}\} \longrightarrow M \times M \times M.$$

The symbol of $\Delta^*(k_{B_1} \otimes k_{B_2})$ at $[x \xrightarrow{\gamma} z \xrightarrow{\gamma'} y]$ is the symbol of k_{B_1} at $[x \xrightarrow{\gamma} z]$ times the symbol of k_{B_2} at $[z \xrightarrow{\gamma'} y]$ so if $U_2 \cap U_3 = \phi$, this is clearly 0 everywhere. Otherwise the symbol of $\Delta^*(k_{B_1} \otimes k_{B_2})$ becomes

$$k^{\gamma_1}(x', x'', z')|dz'| \cdot k^{\gamma_2}(z', h_{\gamma_1}(x''), y')|dy'|$$

on $\{[x \xrightarrow{\gamma} z \xrightarrow{\gamma'} y] \mid [\gamma] \in U_1 \overset{\gamma_1}{\times} U_2, [\gamma'] \in U_3 \overset{\gamma_2}{\times} U_4\}$. The projection $\pi : M \times M \times M \longrightarrow M \times M$ is of constant rank $2p+q$ on $\mathcal{G}^2$ so that the push-down of the δ-section $\Delta^*(k_{B_1} \otimes k_{B_2})$ will be a δ-section on $M \times M$. In fact, the map $\mathcal{G}^2 \longrightarrow \mathcal{G}$ given by $[x \xrightarrow{\gamma} z \xrightarrow{\gamma'} y] \mapsto [x \xrightarrow{\gamma * \gamma'} y]$ makes the following diagram commute.

$$\begin{array}{ccc} \mathcal{G}^2 & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ M \times M \times M & \longrightarrow & M \times M \end{array}$$

Pushing down the symbol is simply a matter of integrating over the fibers of $\mathcal{G}^2 \longrightarrow \mathcal{G}$, which is to say, integrating over z'. ■

Suppose f is a morphism of $(M, \mathcal{F})$ and B is a smooth leafwise operator on M. What is the kernel for the composition $f^* \circ B$? The kernel of f^* is a δ-section of $\pi_2^*|TM|$ supported on the graph of f [13] so $k_{f^*} \otimes k_B$ is a δ-section with support in (graph f) $\times \mathcal{G} \longrightarrow M \times M \times M \times M$. The diagonal Δ is transverse to this so $\Delta^*(k_{f^*} \otimes k_B)$ is a δ-section supported on

$$\{(x, f(x), [\gamma]) \in M \times M \times \mathcal{G} \mid f(x) = s(\gamma)\} \longrightarrow M \times M \times M.$$

This is a $2p+q$ dimensional manifold with coordinates coming from coordinates for x together with leafwise coordinates for $r(\gamma)$. The projection $\pi : M \times M \times M \longrightarrow M \times M$

has constant rank $2p+q$ on this immersed submanifold, and, in fact, induces a diffeomorphism of it with the manifold $\{(x,[\gamma]) \in M \times \mathcal{G} \mid f(x) = s(\gamma)\}$ which immerses in $M \times M$. We will write $f \circ \mathcal{G}$ for this manifold and $U_\alpha \overset{f\gamma}{\times} U_\beta$ for the coordinate chart $\{(x,[\gamma']) \in f \circ \mathcal{G} \mid x \in U_\alpha, [\gamma'] \in f(U_\alpha) \overset{\gamma}{\times} U_\beta\}$. What the above discussion shows is that the kernel for $f^* \circ B$ is a δ-section supported on $f \circ \mathcal{G}$ which is obtained by pulling the kernel for B back via the map $M \times M \xrightarrow{f \times \mathrm{id}} M \times M$.

$$\begin{array}{ccc} f \circ \mathcal{G} & \longrightarrow & \mathcal{G} \\ \Big\downarrow{\scriptstyle \sigma \times r} & & \Big\downarrow{\scriptstyle s \times r} \\ M \times M & \xrightarrow{f \times \mathrm{id}} & M \times M \end{array}$$

Proposition 3.2.6 *Suppose E^0, E^1, E^2 are vector bundles over M, (f,T) is a vector bundle morphism of E^1 to E^0, and B is a smooth leafwise operator $C^\infty(E^2) \to C^\infty(E^1)$ with symbol k^γ supported in the coordinate chart $U_\alpha \overset{\gamma}{\times} U_\beta$. Let $\{U_1, \ldots, U_n\}$ be a covering of $f^{-1}(U_\alpha)$ and $\psi_1, \ldots, \psi_n$ a partition of unity subordinate to it. Then the kernel of $f^* \circ B$ is a δ-section of $\mathrm{Hom}(E^2, E^0) \otimes \pi_2^*|TM|$ supported on $\sum_{i=1}^n U_i \overset{f\gamma}{\times} U_\beta$ with symbol $\psi_i(x) T_x \circ k^\gamma(f'(x', x''), f''(x''), y')|dy'|$ on $U_i \overset{f\gamma}{\times} U_\beta$.*

Proof. Recall that there is a canonical isomorphism $r^*|TM| \otimes |N^*\mathcal{G}| \cong r^*|T\mathcal{F}|$ over $\mathcal{G}$. In the same way there is a canonical isomorphism $r^*|TM| \otimes |N^*(f \circ \mathcal{G})| \cong r^*|T\mathcal{F}|$ over $f \circ \mathcal{G}$. The pull-back via $f \times \mathrm{id}$ identifies $|N^*\mathcal{G}|$ with $|N^*(f \circ \mathcal{G})|$, and this identification commutes with the isomorphisms. Thus replacing x by $f(x)$ in k^γ and then composing with the fiber map $T_x : E^1_{f(x)} \longrightarrow E^0_x$ gives the value of the symbol at the point $(x, [f(x) \xrightarrow{\gamma'} y])$ of $f \circ \mathcal{G}$, namely $T_x \circ k^\gamma([f(x) \xrightarrow{\gamma'} y])|dy'|$. The partition of unity allows this to be presented as a finite sum of terms with support on coordinate charts. ∎

The final situation we need to consider is the composition $B_1 \circ f^* \circ B_2$ where B_1 and B_2 are smooth leafwise operators on M. The ideas are no more difficult than in the previous two propositions although the notation gets more cumbersome.

Proposition 3.2.7 *If $E^i, i = 0,1,2,3$ are vector bundles over M, and:*

1. *B_1 is a smooth leafwise operator $C^\infty(E^1) \longrightarrow C^\infty(E^0)$ with symbol k_1^γ supported in $U_1 \overset{\gamma_1}{\times} U_2$,*

2. *(f, T) is a vector bundle morphism of E^2 to E^1,*

3. *B_2 is a smooth leafwise operator $C^\infty(E^3) \longrightarrow C^\infty(E^2)$ with symbol k^{γ_2} supported in $U_3 \overset{\gamma_2}{\times} U_4$,*

then the kernel of $B_1 \circ f^* \circ B_2$ is a δ-section of $\mathrm{Hom}(E^3, E^0) \otimes \pi_2^*|TM|$ supported on $f \circ \mathcal{G}$ with symbol k supported in $U_1 \overset{\gamma}{\times} U_4$ where $\gamma = f(\gamma_1) * \gamma_2$ and given by

$$k(x', x'', y')|dy'| =$$
$$\left(\int k^{\gamma_1}(x', x'', z') \circ T_{(z', h_{\gamma_1}(x''))} \circ k^{\gamma_2}(f'(z', h_{\gamma_1}(x'')), f'' h_{\gamma_1}(x''), y')|dz'| \right) |dy'|$$

the integral being taken over all

$$z' \in (\textit{Plaque of } U_2 \textit{ through } h_{\gamma_1}(x'')) \cap f^{-1}(U_3).$$

Proof. The proof is similar to Proposition 3.2.5. From the previous proposition, $k_{f^* \circ B_2}$ is a δ-section supported in $f \circ \mathcal{G}$. Thus $k_{B_1} \otimes k_{f^* \circ B_2}$ is a δ-section with support in the immersed submanifold $\mathcal{G} \times f \circ \mathcal{G} \longrightarrow M \times M \times M \times M$. The diagonal Δ is transverse to this so the pull-back $\Delta^*(k_{B_1} \otimes k_{f^* \circ B_2})$ is a δ-section with support in the submanifold $\mathcal{G} f \mathcal{G} \overset{\text{def}}{=} \{([\gamma], [\gamma']) \in \mathcal{G} \times \mathcal{G} \mid f r(\gamma) = s(\gamma')\}$. This $2p + q$ dimensional manifold immerses in $M \times M \times M$ via $([\gamma], [\gamma']) \mapsto (s(\gamma), r(\gamma), r(\gamma'))$. The symbol of $\Delta^*(k_{B_1} \otimes k_{f^* \circ B_2})$ at $([\gamma], [\gamma'])$ is $k^{\gamma_1}([\gamma])|dz'| \circ T_{r(\gamma)} \circ k^{\gamma_2}([\gamma'])|dy'|$ where z' = leafwise coordinates of $r(\gamma)$ and y' = leafwise coordinates of $r(\gamma')$.

The projection $\pi : M \times M \times M \longrightarrow M \times M$ is of constant rank $2p + q$ on $\mathcal{G} f \mathcal{G}$, and if we define a map $\mathcal{G} f \mathcal{G} \longrightarrow f \circ \mathcal{G}$ by $([\gamma], [\gamma']) \mapsto (s(\gamma), [f(\gamma) * \gamma'])$ then the diagram

$$\begin{array}{ccc} \mathcal{G} f \mathcal{G} & \longrightarrow & f \circ \mathcal{G} \\ \downarrow & & \downarrow \\ M \times M \times M & \xrightarrow{\pi} & M \times M \end{array}$$

commutes. The fibers of $\mathcal{G}f\mathcal{G} \longrightarrow f \circ \mathcal{G}$ are q-dimensional and parametrized by the leafwise coordinates of $r(\gamma)$, i.e., by z'. Integrating over the fibers then gives the result. ■

3.3. Distributional Traces and Tr_ν

For an operator T with a smooth kernel k_T on a compact space M, the trace of T is $\int_M k_T(x,x)dx$ [2]. In other words, given the kernel $k_T(x,y)dy$ of T, taking the trace involves first restricting it to the diagonal (setting $x = y$) and then integrating over M. Any operator B on M, $B : C^\infty(M) \longrightarrow C^{-\infty}(M)$ has a distributional kernel k_B in $C^{-\infty}(\pi_2^*|TM|)$ from the Schwartz kernel theorem, and one can try to take its trace in the same way. Let $\Delta : M \longrightarrow M \times M$ be the diagonal map and $\pi : M \longrightarrow \{\cdot\}$ be projection to a point. Then we define the distributional trace $\mathrm{Tr}(B) \stackrel{\text{def}}{=} \pi_* \Delta^*(k_B)$. Since distributions on a singleton space are none other than complex numbers, $\mathrm{Tr}(B) \in \mathbf{C}$. Unfortunately, of course, this procedure for taking the trace of an operator does not always work. Pushing down via π_* is never a problem, at least if M is compact, since $\pi_* : C^{-\infty}(|TM|) \longrightarrow C^{-\infty}(\{\cdot\})$ is always defined. The problem is with Δ^*, that is, with restricting to the diagonal. As discussed in Section 3.1, distributions pull back only if their wave-front set is disjoint from the conormal bundle of the map. In the case of the map $\Delta : M \longrightarrow M \times M$, the conormal bundle $N^*\Delta = \{(x, x, \xi, -\xi) \mid \xi \in T_x^*M \backslash 0\}$. Thus the above definition for $\mathrm{Tr}(B)$ is only valid if $\mathrm{WF}(k_B) \cap N^*\Delta = \phi$.

In the case of operators $f^* \circ B$ or $B_1 \circ f^* \circ B_2$ on a foliated manifold as discussed in the previous section, the Schwartz kernel is a δ-section supported on the immersed submanifold $f \circ \mathcal{G} \longrightarrow M \times M$. δ-sections pull back via maps that are transverse to their support, but $\Delta : M \longrightarrow M \times M$ is, in general, not transverse to $f \circ \mathcal{G} \longrightarrow M \times M$. The pull-back

$$M \overset{\Delta}{\times} (f \circ \mathcal{G}) = \{(x, [\gamma]) \in f \circ \mathcal{G} \mid s(\gamma) = f(x), r(\gamma) = x\} = \{[f(x) \overset{\gamma}{\rightarrow} x] \mid [\gamma] \in \mathcal{G}\}$$

which is precisely $\mathcal{G}^f$. In other words, we have the commutative diagram:

$$\begin{array}{ccccc} \mathcal{G}^f & \longrightarrow & f\circ\mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\Delta} & M\times M & \xrightarrow{f\times \mathrm{id}} & M\times M \end{array}$$

Proposition 3.3.1 *If the morphism f is transfixed of dimension k, then $\mathcal{G}^f$ is a smooth immersed $(p+k)$ dimensional submanifold of M, $\Delta^*(N^*(f\circ\mathcal{G}))\xrightarrow{\Delta^*}N^*(\mathcal{G}^f)$ is surjective, and over the point $[f(x)\xrightarrow{\gamma}x]$ of $\mathcal{G}^f$,*

$$\ker\Delta^* = \{(-\xi,\xi)\mid \xi\in N_x^*\mathcal{F} \textit{ and } \xi=(h_\gamma\circ f)^*\xi\}.$$

Proof. The first part has already been shown (Proposition 2.2.6). At a point $(x,[f(y)\xrightarrow{\gamma}y])$ of $f\circ\mathcal{G}$, $T(f\circ\mathcal{G})=\{(v',v'',w',w'')\in T_{(x,y)}(M\times M)\mid (h_\gamma\circ f)_*v''=w''\}$ and hence $N^*(f\circ\mathcal{G}) = \{(\eta,\xi)\mid \eta\in N^*{}_x\mathcal{F},\xi\in N^*{}_y\mathcal{F},\eta=-(h_\gamma\circ f)^*\xi\}$. Pulling back via Δ, $\Delta^*N^*(f\circ\mathcal{G})$ at a point $[f(x)\xrightarrow{\gamma}x]$ of $\mathcal{G}^f$ is $\{(-(h_\gamma\circ f)^*\xi,\xi)\mid \xi\in N^*{}_x\mathcal{F}\}$. Then, since the tangent map Δ_* is $v\mapsto(v,v)$, the cotangent map $\Delta^*:\Delta^*N^*(f\circ\mathcal{G})\longrightarrow N^*\mathcal{G}^f$ is $(-(h_\gamma\circ f)^*\xi,\xi)\longmapsto \xi-(h_\gamma\circ f)^*\xi$.

Note that $\ker\Delta^*\cong\ker(\mathrm{Id}-(h_\gamma\circ f)^*)$ on $N^*\mathcal{F}$ so that

$$\dim(\ker\Delta^*)=\dim(\ker(\mathrm{Id}-(h_\gamma\circ f)_*)\text{ on }N\mathcal{F}).$$

But the transfixed assumption implies that no transverse vectors are fixed by $(h_\gamma\circ f)_*$ except for those in $N\mathcal{F}^{hf}$, so $\ker(\mathrm{Id}-(h_\gamma\circ f)_*)=N\mathcal{F}^{hf}{}_{[\gamma]}$ and this has dimension k. Since $\dim N^*(f\circ\mathcal{G})=q$ and $\dim N^*\mathcal{G}^f=q-k$, Δ^* is surjective. ∎

This proposition shows that if f is transfixed of dimension k, then, although $\Delta:M\longrightarrow M\times M$ is not transverse to $f\circ\mathcal{G}$, it has constant transverse rank $q-k$. Provided that a section $\nu\in C^\infty(|\ker\Delta^*|^*\to\mathcal{G}^f)$ is given, one can then proceed as in Section 3.1 to obtain a pull-back map $\Delta^*_\nu: C^\delta_{f\circ\mathcal{G}}\longrightarrow C^\delta_{\mathcal{G}^f}$. To analyze $|\ker\Delta^*|^*$, we can apply the discussion in Section 2.3. With $V=N\mathcal{F}_x$ and $T=\mathrm{Id}-(h_\gamma\circ f)_*$ at a point $[f(x)\xrightarrow{\gamma}x]$ of $\mathcal{G}^f$, $\ker T=N\mathcal{F}^{hf}$ and

$\ker T^* = \ker \Delta^*$. Lemma 2.3.4 says that in the presence of the transfixed assumption, these spaces are naturally dual to each other: $\ker T^* \cong (\ker T)^*$. Thus there is a natural identification of $|\ker T|$ with $|\ker T^*|^*$ or $|N\mathcal{F}^{hf}| \cong |\ker \Delta^*|^*$. In coordinates this takes the following form. Choose transverse coordinates $y^{p+1}, \ldots, y^q$ such that $\partial/\partial y^{p+1}, \ldots, \partial/\partial y^{p+k}$ are fixed by $(h_\gamma \circ f)_*$ and the subspace spanned by $\partial/\partial y^{p+k+1}, \ldots, \partial/\partial y^q$ is mapped to itself by $(h_\gamma \circ f)_*$. Then $dy^{p+1}, \ldots, dy^{p+k}$ are fixed by $(h_\gamma \circ f)^*$ and $|dy^{p+1} \wedge \ldots \wedge dy^{p+k}|$ can be considered, on the one hand, as a density on the space spanned by $\partial/\partial y^{p+1}, \ldots, \partial/\partial y^{p+k}$, and, on the other hand, as an (unordered) basis of $\ker \Delta^*$ which gives an element of $|\ker \Delta^*|^*$. The identification $|N\mathcal{F}^{hf}| \cong |\ker \Delta^*|^*$ is simply this identification of $|dy^{p+1} \wedge \ldots \wedge dy^{p+k}|$ as an element of $|N\mathcal{F}^{hf}|$ with itself as an element of $|\ker \Delta^*|^*$.

We are now ready for the key definition.

Definition 3.3.2 *Let f be a morphism of $(M, \mathcal{F})$ that is transfixed of dimension k. Let ν be an invariant transfixed k-form on $\mathcal{G}^f$. Let A be an operator on a vector bundle E over M whose Schwartz kernel is in $C^\delta_{c(f\circ\mathcal{G})}(\mathrm{Hom}(E,E) \otimes \pi_2^*|TM|)$. Then $\mathrm{Tr}_\nu(A)$ is defined to be $\pi_*(\mathrm{tr}\Delta_\nu^*(k_A))$.*

Remarks.

1. π_2 represents the projection of $M \times M$ to the second factor. When pulled back via the diagonal map Δ^*, $\pi_2^*|TM|$ becomes $|TM|$ since $\pi_2 \circ \Delta = \mathrm{Id}$.

2. $\Delta_\nu^*(k_A)$ is in $C^\delta_{c\mathcal{G}^f}(\mathrm{Hom}(E,E) \otimes |TM|)$. Over a point $(x, [\gamma])$ in $\mathcal{G}^f$, $\mathrm{Hom}(E,E) = \mathrm{Hom}(E_x, E_x)$ and 'tr' means the trace of this endomorphism of the finite dimensional vector space E_x. Thus $\mathrm{tr}\Delta_\nu^*(k_A) \in C^\delta_{c\mathcal{G}^f}(|TM|)$.

3. π represents the projection of M to a point so that $\pi_*(\mathrm{tr}\Delta_\nu^*(k_a))$ is a complex number.

4. The operators to which we shall apply this definition are operators of the form $(f,T)^* \circ B$ or $B_1 \circ (f,T)^* \circ B_2$ where B, B_1, and B_2 are smooth leafwise operators on a bundle E over M and (f,T) is a vector bundle endomorphism of E as in Propositions 3.2.6 and 3.2.7.

By abuse of notation, we will use k_A both for the kernel of A and for its symbol, which is a smooth section of $\mathrm{Hom}(E,E)\otimes r^*|T\mathcal{F}|$ over $f\circ\mathcal{G}$. This symbol restricts to $\mathcal{G}^f$, and, after taking the fiber trace, gives an element of $C_c^\infty(r^*|T\mathcal{F}|)$ over $\mathcal{G}^f$. The trace $\mathrm{Tr}_\nu(A)$ has a nice description in terms of this.

Proposition 3.3.3 *Let A be an operator as in the previous definition. Then:*

$$\mathrm{Tr}_\nu(A) = \int_{\mathcal{G}^f} \frac{\mathrm{tr}(k_A)\otimes\nu}{|\det(\mathrm{Id}-(h_\gamma\circ f)_*)|}.$$

Proof. Note that $\mathrm{tr}(k_A)\otimes\nu$ is a density on $\mathcal{G}^f$ since $r^*|T\mathcal{F}|\otimes|N\mathcal{F}^{hf}|\cong|T\mathcal{G}^f|$. The factor in the denominator is the determinant of the map restricted to $N\mathcal{F}/N\mathcal{F}^{hf}$ and is guaranteed not to vanish by the transfixed hypothesis.

The claim is that at any point $(a,[\gamma])$ in $\mathcal{G}^f$,

$$\Delta_\nu^*(k_A) = \frac{k_A\otimes\nu}{|\det(\mathrm{Id}-(h_\gamma\circ f)_*)|}.$$

This identity only involves the various vector bundle maps and identifications over the point $(a,[\gamma])$. To establish it, choose transverse coordinates $y^{p+1},\ldots,y^q$ as before, i.e., such that

$$\begin{aligned}\ker(\mathrm{Id}-(h_\gamma\circ f)_*) &= \mathrm{span}(\partial/\partial y^{p+1},\ldots,\partial/\partial y^{p+k}) \text{ and}\\ \mathrm{im}(\mathrm{Id}-(h_\gamma\circ f)_*) &= \mathrm{span}(\partial/\partial y^{p+k+1},\ldots,\partial/\partial y^{p+q}).\end{aligned}$$

In $f\circ\mathcal{G}$, $k_A(a,[\gamma])$ has a value $k_A(a',a'',a')|dy'|$ in $\mathrm{Hom}(E,E)\otimes r^*|T\mathcal{F}|$. In coordinates the identification of $r^*|T\mathcal{F}|$ with $r^*|TM|\otimes|N^*(f\circ\mathcal{G})|$ is $|dy'|=|dy|\otimes|\partial/\partial y''|$. This identification is compatible with the isomorphism of $N^*(f\circ\mathcal{G}) = \{(-(h_\gamma\circ f)^*\xi,\xi)\mid\xi\in N^*\mathcal{F}\}$ with $N^*\mathcal{F}$. But then the pull-back map Δ^* from $N^*(f\circ\mathcal{G})$ to $N^*\mathcal{G}^f$ is $\xi\mapsto\xi-(h_\gamma\circ f)^*\xi$ and its kernel is $\mathrm{span}(dy^{p+1},\ldots,dy^{p+k})$. Thus $\Delta_\nu^*(k_A)$ at $(a,[\gamma])$ is

$$k_A(a',a'',a')|dy|\cdot\nu(\partial/\partial y^{p+1},\ldots,\partial/\partial y^{p+k})\cdot\Delta^*|\partial/\partial y^{p+k+1}\wedge\ldots\wedge\partial/\partial y^{p+q}|.$$

The map $\Delta^*:\mathrm{span}(dy^{p+k+1},\ldots,dy^{p+q})\longrightarrow N^*\mathcal{G}^f=\mathrm{span}(dy^{p+k+1},\ldots,dy^{p+q})$ is an isomorphism with determinant

$$\det(\mathrm{Id}-(h_\gamma\circ f)^*)=\det\left(\mathrm{Id}-(h_\gamma\circ f)_* \text{ on } N\mathcal{F}/N\mathcal{F}^{hf}\right).$$

Now an isomorphism $A \xrightarrow{f} B$ determines a map $|B| \longrightarrow |A|$. If $B = A$, then this second map is multiplication by $|\det f|$ so that the inverse map $|A| \longrightarrow |B|$ is multiplication by $1/|\det f|$. This shows that

$$\Delta^* |\partial/\partial y^{p+k+1} \wedge \ldots \wedge \partial/\partial y^{p+q}| = \frac{|\partial/\partial y^{p+k+1} \wedge \ldots \wedge \partial/\partial y^{p+q}|}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|}$$

so that

$$\begin{aligned} \Delta^*_\nu(k_A) &= k_A(a', a'', a')|dy| \cdot \nu(\partial/\partial y^{p+1}, \ldots, \partial/\partial y^{p+k}) \frac{|\partial/\partial y^{p+k+1} \wedge \ldots \wedge \partial/\partial y^{p+q}|}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|} \\ &= \frac{k_A(a', a'', a')|dy'| \otimes \nu}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|} |dy''| \otimes |\partial/\partial y''| \\ &= \frac{k_A \otimes \nu}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|}. \end{aligned}$$

This proves the claim. The final step in the definition of Tr_ν is pushing down via π_* which just means integrating over $\mathcal{G}^f$ and this gives the result. ■

What does $\mathrm{Tr}_\nu(A)$ look like using coordinates? Suppose k_A is supported in the coordinate chart $U_\alpha \overset{f\gamma}{\times} U_\beta = \{(x', x'', y', y'') \in U_\alpha \times U_\beta \mid (h_\gamma \circ f)x'' = y''\}$ for $f \circ \mathcal{G}$ with symbol $k_A(x', x'', y')|dy'|$. Pulling this back with the diagonal map Δ yields something supported on $\{(x', x'', y', y'') \in U_\alpha \times U_\beta \mid x' = y', x'' = y'' = (h_\gamma \circ f)x''\}$ so that the pull-back is non-zero only if $U_\alpha \cap U_\beta \neq \phi$ and $(h_\gamma \circ f)$ fixes a subset of the transversal through $U_\alpha \cap U_\beta$. If this is the case, let U_α^γ be the corresponding chart for $\mathcal{G}^f$ and let $\Sigma_\alpha^{h_\gamma \circ f} = \{x'' \in \Sigma_\alpha \mid (h_\gamma \circ f)x'' = x''\}$. Recall that

$$U_\alpha^\gamma = \{x \in U_\alpha \mid \text{ the plaque } P_{x''} \text{ of } x \text{ is fixed by } h_\gamma \circ f\} \cong \Sigma_\alpha^{h_\gamma \circ f} \times R^p.$$

Then the previous proposition shows that

$$\begin{aligned} \mathrm{Tr}_\nu(A) &= \int_{U_\alpha^\gamma} \frac{\mathrm{tr}\ k_A(x', x'', x')|\frac{dy'}{dx'}||dx'| \otimes \nu}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|_{(x'')}} \\ &= \int_{x'' \in \Sigma_\alpha^{h_\gamma \circ f}} \frac{\nu(x'')}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|_{(x'')}} \int_{x' \in P^\alpha_{x''} \cap P^\beta_{x''}} \mathrm{tr}\ k_A(x', x'', x')|\frac{dy'}{dx'}||dx'|. \end{aligned}$$

Note that dy'/dx' is the factor coming from leafwise change of coordinates between U_α and U_β. If $U_\alpha = U_\beta$, this factor disappears.

The factor in the denominator is constant along leaves in $\mathcal{G}^f$ since, by Lemma 2.2.3, the $h_\gamma \circ f$ maps are conjugate to one another. However, it is not necessarily constant along leaves in M. As was pointed out in the discussion following Definition 2.2.7, over a single leaf in M, there may be several leaves in $\mathcal{G}^f$, and the denominator factors may differ between these. The denominator factor is only defined on $\mathcal{G}^f$, the holonomy coverings of the fixed leaves, and not on the fixed leaves themselves. On the other hand ν will usually be taken to be a Haefliger invariant k-form so that ν is defined everywhere on M. This is why the denominator factor is not simply absorbed in the density ν.

3.4. The Key Property of Tr_ν

Theorem 3.4.1 *If*

1. *B_1 and B_2 are smooth leafwise operators on a vector bundle E over $(M, \mathcal{F})$, and*

2. *(f, T) is a vector bundle morphism ofE where f is transfixed of dimension k,*

then $\mathrm{Tr}_\nu(f^* \circ B_2 \circ B_1) = \mathrm{Tr}_\nu(B_1 \circ f^* \circ B_2)$.

Proof. The proof is a calculation using the Fubini theorem. We may assume that the Schwartz kernels of B_1 and B_2 are each supported in a single chart for $\mathcal{G}$. So we suppose that B_1 has symbol k^{γ_1} supported in $U_1 \overset{\gamma_1}{\times} U_2$ and B_2 has symbol k^{γ_2} supported in $U_3 \overset{\gamma_2}{\times} U_4$. If $f(U_2) \cap U_3 = \phi$ or $U_4 \cap U_1 = \phi$, then both sides are zero so we assume these are not the case. We shall use the variable x for U_1 coordinates, y for U_2, z for U_3, and w for U_4.

Work first with $\mathrm{Tr}_\nu(f^* \circ B_2 \circ B_1)$. Proposition 3.2.5 gives the symbol $k^{\gamma_2 * \gamma_1}$ of $B_2 \circ B_1$ on $U_3 \overset{\gamma_2\gamma_1}{\times} U_2$ as

$$k^{\gamma_2 * \gamma_1}(z', z'', y')|dy'| = |dy'| \cdot \int_{w' \in P^4_{h_{\gamma_2} x''}} k^{\gamma_2}(z', z'', w') \circ k^{\gamma_1}(w', h_{\gamma_2} z'', y')|dw'|.$$

The kernel of $f^* \circ B_2 \circ B_1$ is supported on $f^{-1}(U_3) \overset{f\gamma_2\gamma_1}{\times} U_2$, and it may take several coordinate charts to cover $f^{-1}(U_3)$, but since, in taking Tr_ν, the only part that contributes is $f^{-1}(U_3) \cap U_2$, we only need to consider the kernel of $f^* \circ B_2 \circ B_1$ on $U_2 \overset{f\gamma_2\gamma_1}{\times} U_2$. Proposition 3.2.6 shows that on this chart, the kernel has the value $T_y \circ k^{\gamma_2 * \gamma_1}(f'(y), f''(y''), y')|dy'|$. Then the expression for Tr_ν in coordinates gives

$$\mathrm{Tr}_\nu(f^* \circ B_2 \circ B_1) = \tag{3.1}$$
$$\int_{y'' \in \Sigma_2^{h_{\gamma_1} h_{\gamma_2} f}} \frac{\nu(y'')}{|(\mathrm{Id} - h_{\gamma_1} h_{\gamma_2} f)_*|_{(y'')}} \cdot \int_{y' \in P^2_{y''}} |dy'| \cdot$$
$$\int_{w' \in P^4_{h_{\gamma_2} f''(y'')}} \mathrm{tr}(T_y \circ k^{\gamma_2}(f'(y), f''(y''), w') \circ k^{\gamma_1}(w', h_{\gamma_2} f''(y''), y'))|dw'|.$$

Now work with $\mathrm{Tr}_\nu(B_1 \circ f^* \circ B_2)$. Proposition 3.2.7 shows that the kernel of $B_1 \circ f^* \circ B_2$ is supported on $U_1 \overset{ff(\gamma_1)\gamma_2}{\times} U_4$ with value

$$k(x', x'', w')|dw'| =$$
$$\left(\int_{y' \in P^2_{h_{\gamma_1} x''}} k^{\gamma_1}(x', x'', y') \circ T_{(y', h_{\gamma_1} x'')} \circ k^{\gamma_2}(f'(y', h_{\gamma_1} x''), f'' h_{\gamma_1} x'', w')|dy'| \right) |dw'|.$$

Thus

$$\mathrm{Tr}_\nu(B_1 \circ f^* \circ B_2) = \tag{3.2}$$
$$\int_{x'' \in \Sigma_1^{h_{\gamma_2} h_{f(\gamma_1)} f}} \frac{\nu(x'')}{|(\mathrm{Id} - h_{\gamma_2} h_{f(\gamma_1)} f)_*|_{(x'')}} \cdot \int_{x' \in P^1_{x''} \cap P^4_{x''}} |dw'/dx'|(x)|dx'| \cdot$$
$$\int_{y' \in P^2_{h_{\gamma_1} x''}} \mathrm{tr}(k^{\gamma_1}(x', x'', y') \circ T_{(y', h_{\gamma_1} x'')} \circ k^{\gamma_2}(f'(y', h_{\gamma_1} x''), f'' h_{\gamma_1} x'', x'))|dy'|.$$

Now transform this integral so that it becomes equal to the one in Equation 3.1. First, make a change of variable in the second integral from x' to w'. Second, use Fubini's Theorem to switch the order of the two innermost integrals. Third, replace $\mathrm{tr}(k^{\gamma_1} \circ T \circ k^{\gamma_2})$ with $\mathrm{tr}(T \circ k^{\gamma_2} \circ k^{\gamma_1})$. This step is justified because these are traces on the fibers which are finite-dimensional vector spaces. After these three steps, $\mathrm{Tr}_\nu(B_1 \circ f^* \circ B_2)$ has become

$$\int_{x'' \in \Sigma_1^{h_{\gamma_2} h_{f(\gamma_1)} f}} \frac{\nu(x'')}{|(\mathrm{Id} - h_{\gamma_2} h_{f(\gamma_1)} f)_*|_{(x'')}} \cdot \int_{y' \in P^2_{h_{\gamma_1} x''}} |dy'| \cdot$$
$$\int_{w' \in P^4_{x''}} \mathrm{tr}(T_{(y', h_{\gamma_1} x'')} \circ k^{\gamma_2}(f'(y', h_{\gamma_1} x''), f'' h_{\gamma_1} x'', w') \circ k^{\gamma_1}(w', x'', y'))|dw'|.$$

The final step is to make a change of variables on the outermost integral. The holonomy h_{γ_1} takes $\Sigma_1^{h_{\gamma_2} h_{f(\gamma_1)} f}$ to $\Sigma_2^{h_{\gamma_1} h_{\gamma_2} f}$ so we can make the change of variable

$h_{\gamma_1} x'' = y''$ or, equivalently, $x'' = h_{\gamma_2} f y''$. Since ν is holonomy invariant, $\nu(x'') = \nu(y'')$ and the same applies to the determinant factor in the denominator. Thus the integral is transformed into the integral that occurs in Equation 3.1. ■

CHAPTER 4

INVARIANCE AND TIME ZERO

4.1. Leafwise Dirac Complexes

The following three definitions are adapted from Roe [20].

Definition 4.1.1 *A* leafwise Clifford bundle *over a foliated Riemannian manifold* $(M, \mathcal{F})$ *is a smooth bundle* S *over* M *such that each fiber* S_x *is a module over the Clifford algebra* $\mathrm{Cl}(T_x\mathcal{F} \otimes \mathbf{C})$ *and* S *has a Hermitian metric* $(\cdot, \cdot)$ *and compatible connection* ∇ *such that*

1. *the Clifford action of a vector* $\xi \in T_x\mathcal{F}$ *on* S_x *is skew-adjoint*

$$(\xi s_1, s_2) + (s_1, \xi s_2) = 0;$$

2. *the connection on* S *is compatible with the Levi-Civita connection restricted to* $T\mathcal{F}$, *in the sense that for vector fields* $X, Y \in C^\infty(T\mathcal{F})$ *and* $s \in C^\infty(S)$,

$$\nabla_X(Ys) = (\nabla_X Y)s + Y\nabla_X s.$$

Definition 4.1.2 *The* leafwise Dirac operator D *of a leafwise Clifford bundle* S *is the first order differential operator on* $C^\infty(S)$ *that is defined by the following composition:*

$$C^\infty(S) \longrightarrow C^\infty(T^*M \otimes S) \longrightarrow C^\infty(T^*\mathcal{F} \otimes S) \longrightarrow C^\infty(T\mathcal{F} \otimes S) \longrightarrow C^\infty(S)$$

where the first arrow is given by the connection, the second by the projection, the third by the metric, and the fourth by the Clifford action.

Definition 4.1.3 *A* leafwise Dirac complex *on* $(M, \mathcal{F})$ *is*

1. *a sequence of smooth finite dimensional Hermitian vector bundles* $E^0, \ldots, E^k$ *over* M *with connections over* $T\mathcal{F}$ *and with first order differential operators* $d_i : C^\infty(E^i) \to C^\infty(E^{i+1})$; *and*

2. *a leafwise Clifford bundle S over M together with a Hermitian vector bundle isomorphism of S with $\bigoplus E^i$*

such that

1. $d_{i+1} \circ d_i = 0$;
2. *the d_i differentiate only in the leafwise directions;*
3. *when the leafwise formal adjoints $d_i^* : C^\infty(E^{i+1}) \to C^\infty(E^i)$ are formed using the metrics, the operator $\bigoplus(d_i + d_{i-1}^*)$ on $C^\infty(\bigoplus E^i)$ corresponds to the Dirac operator D under the isomorphism of S with $\bigoplus E^i$.*

The (non-foliated) deRham complex is a Dirac complex [20, pages 30–31]. This example can be adapted to the leafwise context. Let $E^i = \wedge^i T^*\mathcal{F} \otimes \mathbf{C}$ be the bundle of complex-valued leafwise i-forms on M and let d_i be the exterior derivative in the leafwise directions. Then $\bigoplus E^i$ is isomorphic as a vector bundle to $\mathrm{Cl}(T\mathcal{F} \otimes \mathbf{C})$ which we can consider as a bundle of Clifford modules over itself. Under this isomorphism, the Dirac operator D of $\mathrm{Cl}(T\mathcal{F} \otimes \mathbf{C})$ corresponds to the operator $d_i + d_{i-1}^*$ on $C^\infty(E^i)$.

If $p = 2m$ is even and the leaves are Kähler manifolds, then let $(E^*, \bar{\partial}_*)$ be the leafwise Dolbeault complex. If S is the Clifford bundle of leafwise spinors, then S is isomorphic as a vector bundle to $\bigoplus_{i=0}^m E^i$ and the Dirac operator of S corresponds to $\sqrt{2}(\bar{\partial}_i + \bar{\partial}_{i-1}^*)$ [20, pages 34–35]. This gives a second example of a leafwise Dirac complex.

The other two classical examples of Dirac complexes are the signature complex and the spin complex. Both of these can also be adapted to the leafwise context in a straightforward fashion.

Note that when restricted to a leaf, a leafwise Clifford bundle becomes a Clifford bundle and a leafwise Dirac complex becomes a Dirac complex. The Dirac operator D of a leafwise Dirac complex restricts to a Dirac operator D_L on the leaf L of M. L is a complete Riemannian manifold. Chernoff [6] established two important facts about Dirac operators on complete Riemannian manifolds. The first is that they are essentially self-adjoint, and the second is that the corresponding wave operators have unit propagation speed:

Proposition 4.1.4 (Chernoff [6,21]) *If D is a Dirac operator on a Clifford bundle S over a complete Riemannian manifold, then the operator e^{itD} maps $C_c^\infty(S)$ to $C_c^\infty(S)$ and the support of $e^{itD}s$ is contained within a $|t|$-neighborhood of the support of s.*

Definition 4.1.5 *An* endomorphism *of a leafwise Dirac complex (E^*, d_*) is a collection of maps $C^\infty(E^i) \xrightarrow{\phi^i} C^\infty(E^i)$ that commute with the differential operators d_i. A* geometric endomorphism *of a leafwise Dirac complex is a morphism f of $(M, \mathcal{F})$ together with vector bundle maps $f^*E^i \xrightarrow{T^i} E^i$ lying over f such that $d_iT^i = T^{i+1}d_i$ as maps from $C^\infty(E^i)$ to $C^\infty(E^{i+1})$. (Note that the endomorphisms of a complex do not necessarily commute with the adjoints d_i^*.)*

For the four classical complexes, the following are the requirements on a morphism f of $(M, \mathcal{F})$ so that it will induce a geometric endomorphism of the complex:

complex	*condition on f restricted to a leaf*
deRham	
Kähler	holomorphic isometry
Signature	isometry and orientation-preserving
Dirac	isometry and spin

A Clifford bundle is said to have bounded geometry if its curvature tensor is uniformly bounded as are all its covariant derivatives. The compactness of M implies that a leafwise Clifford bundle over M has bounded geometry when considered as a Clifford bundle over the disjoint union of the leaves.

4.2. The Heat Kernel and Approximations

Suppose E^* is a leafwise Dirac complex on $(M, \mathcal{F})$. Let $D = \bigoplus(d_i + d_{i-1}^*)$ be the corresponding Dirac operator on $C^\infty(S) = C^\infty(\bigoplus E^i)$. On each leaf L, D_L is a first-order elliptic operator whose closure in L^2 of the leaf is self-adjoint. This closure will also be written D_L. By the spectral theorem, for any bounded Borel function f on the real line, $f(D_L)$ is a bounded operator on L^2 of L. When we write $f(D)$, we shall mean the operator that is $f(D_L)$ on the leaf L. The leafwise Laplacian on S is $\Delta = D^2$.

We now want to analyze the leafwise heat operator e^{-tD^2} and explain why its kernel is a δ-section supported on the holonomy groupoid $\mathcal{G} \xrightarrow{s \times r} M \times M$. Unfortunately, this kernel is not compactly supported so that it is not in $C^\delta_{c\mathcal{G}}(\mathrm{Hom}(S, S) \otimes \pi_2^*|TM|)$, but using the finite propagation speed property, we shall show that it can be approximated by elements of this space.

To begin, consider a leaf L of M. What is the inverse image of $L \times L$ in $\mathcal{G}$? Let $\tilde{L} \xrightarrow{\pi} L$ be the holonomy covering of L, that is, $\tilde{L} = \bar{L}/N$ where $\bar{L}$ is the universal cover of L and N is the kernel of the map of $\Pi_1(L, x)$ to the holonomy group G_x^x for any x in L. This is a Galois covering since N is normal. Let G be the group of deck transformations of $\tilde{L}$.

Proposition 4.2.1 *There is a canonical isomorphism*

$$(\tilde{L} \times \tilde{L})/G \cong (s \times r)^{-1}(L \times L).$$

Proof. Define a map $\tilde{L} \times \tilde{L} \xrightarrow{\alpha} (s \times r)^{-1}(L \times L)$ by $\alpha(\tilde{x}, \tilde{y}) = [\pi(\tilde{\gamma})]$ where $\tilde{\gamma}$ is any path in $\tilde{L}$ from $\tilde{x}$ to $\tilde{y}$. The equivalence class of $\pi(\tilde{\gamma})$ is independent of the path $\tilde{\gamma}$ chosen. Clearly α is surjective since any path in L lifts to a path in $\tilde{L}$. Note that G acts on $\tilde{L} \times \tilde{L}$ by $(\tilde{x}, \tilde{y}) \xmapsto{g} (\tilde{x}g, \tilde{y}g)$ and that the map α is invariant under this action.

We claim that the fibers of the map α are precisely the orbits of G. For suppose $\alpha(\tilde{x}_1, \tilde{y}_1) = \alpha(\tilde{x}_2, \tilde{y}_2)$ and let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be the corresponding paths in $\tilde{L}$. Since $\pi(\tilde{x}_1) = s[\pi(\tilde{\gamma}_1)] = s[\pi(\tilde{\gamma}_2)] = \pi(\tilde{x}_2)$, $\tilde{x}_1$ and $\tilde{x}_2$ are in the same fiber over L and so there is a deck transformation g with $\tilde{x}_1 g = \tilde{x}_2$. The path $\tilde{\gamma}_1 g$ has the same image in L that $\tilde{\gamma}_1$ does, and hence $\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1 g$ is a path in $\tilde{L}$ whose image in L is a loop based at $\pi(\tilde{y}_2) = \pi(\tilde{y}_1)$ with trivial holonomy. This implies that $\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1 g$ is a loop in $\tilde{L}$, and since it starts at $\tilde{y}_2$ and ends at $\tilde{y}_1 g$, we must have $\tilde{y}_1 g = \tilde{y}_2$. Thus $(\tilde{x}_2, \tilde{y}_2) = (\tilde{x}_1, \tilde{y}_1)g$ which proves the claim. But then the claim immediately implies the proposition. ■

Now we want to compare the heat kernel on L with the heat kernel on $\tilde{L}$. Since D_L is a local operator, it lifts to an operator $D_{\tilde{L}}$ on $\tilde{L}$ with the same properties—$D_{\tilde{L}}$ is first-order, elliptic, and essentially self-adjoint. The lifted operator commutes with the deck transformations $g \in G$ and also satisfies $D_{\tilde{L}}(\pi^*\varphi) = \pi^*(D_L\varphi)$ where φ is any function on L in the domain of D_L. The usual elliptic methods show that $e^{-tD_L^2}$ has a smooth kernel, k_t^L, on $L \times L$ and that $e^{-tD_{\tilde{L}}^2}$ has a smooth kernel, $k_t^{\tilde{L}}$, on $\tilde{L} \times \tilde{L}$. Since $D_{\tilde{L}}$ commutes with the isometries $g \in G$, so does $e^{-tD_{\tilde{L}}^2}$, and this fact implies that the kernel $k_t^{\tilde{L}}$ satisfies $k_t^{\tilde{L}}(\tilde{x}g, \tilde{y}g) = k_t^{\tilde{L}}(\tilde{x}, \tilde{y})$. This G-invariance means that the kernel descends to $(\tilde{L} \times \tilde{L})/G$ which by the previous proposition is isomorphic to $(s \times r)^{-1}(L \times L)$. We shall also write $k_t^{\tilde{L}}$ for the kernel on either one of these spaces.

Over any $(x, y) \in L \times L$, the fiber of the projection $(s \times r)^{-1}(L \times L) \cong (\tilde{L} \times \tilde{L})/G \longrightarrow L \times L$ is $\{[\gamma] \in \mathcal{G} \mid s(\gamma) = x,\ r(\gamma) = y\} = G_y^x$. It is well-known that $k_t^L(x, y) = \sum_{[\gamma] \in G_y^x} k_t^{\tilde{L}}([\gamma])$. This follows from the uniqueness of the heat kernel together with the calculation:

$$\begin{aligned}
\int_L \sum_{G_y^x} k_t^{\tilde{L}}([x \xrightarrow{\gamma} y])\varphi(y)dy &= \int_{s^{-1}(x)} k_t^{\tilde{L}}([x \xrightarrow{\gamma} y])\varphi(y)dy \\
&= \int_{\tilde{L}} k_t^{\tilde{L}}(\tilde{x}, \tilde{y})\varphi(\pi(\tilde{y}))d\tilde{y} \\
&= (e^{-tD_{\tilde{L}}^2}\pi^*\varphi)(\tilde{x}) \\
&= \pi^* \circ e^{-tD_L^2}\varphi(\tilde{x}) \\
&= e^{-tD_L^2}\varphi(x) \\
&= \int_L k_t^L(x, y)\varphi(y)dy
\end{aligned}$$

where the fourth equality follows from the fact that $D_{\tilde{L}} \circ \pi^* = \pi^* \circ D_L$. Since the fibers of $s \times r$ are discrete, pushing down just means summing over the fibers, and thus we have shown that $(s \times r)_* k_t^{\tilde{L}} = k_t^L$.

For each leaf L, $k_t^{\tilde{L}}$ is a section of $\mathrm{Hom}(S,S) \otimes r^*|TL|$ over $(s \times r)^{-1}(L \times L) \subset \mathcal{G}$. Globally, this gives a leafwise smooth section of $\mathrm{Hom}(S,S) \otimes r^*|T\mathcal{F}|$ over $\mathcal{G}$ which will be denoted by $\tilde{k}_t$. This is almost a δ-section of $\mathrm{Hom}(S,S) \otimes \pi_2^*|TM|$ over M (recall that $\pi_2^*|TM| \otimes |N^*\mathcal{G}| \cong r^*|T\mathcal{F}|$) except that $\tilde{k}_t$ is, in general, not compactly supported and not smooth transversely in $\mathcal{G}$. Nevertheless, the previous paragraph shows that $\tilde{k}_t$ represents the Schwartz kernel of e^{-tD^2} on M. The verification is as follows. Given a smooth section φ of S on M, the value of $\pi_{1*}(\tilde{k}_t \cdot \pi_2^* \varphi)$ at x is given by $\int_{s^{-1}(x)} \tilde{k}_t([x \xrightarrow{\gamma} y])\varphi(y)dy$. The previous paragraph shows that with $L = L_x$, this $= \int_L k_t^L(x,y)\varphi(y)dy = (e^{-tD_L^2}\varphi_L)(x)$ which is the definition of $(e^{-tD^2}\varphi)(x)$.

In [21], Roe proved

Theorem 4.2.2 (Roe) *If D is a leafwise Dirac operator and f is a function on R whose Fourier transform $\hat{f}$ is smooth and compactly supported, then $f(D)$ has a kernel whose symbol is in $C_c^\infty(\mathrm{Hom}(S,S) \otimes r^*|T\mathcal{F}| \longrightarrow \mathcal{G})$.*

The proof uses the finite propagation speed property of e^{itD} and actually shows that if R is a number such that $\hat{f}$ is supported in the interval $[-R, R]$, then the kernel $k_{f(D)}$ has support in $\mathcal{G}^f{}_R$.

The function $g(x) = e^{-tx^2}$ is in the Schwartz space $\mathcal{S}$ and has Fourier transform $\frac{1}{\sqrt{2t}} e^{-\xi^2/4t}$. Although $\hat{g}$ does not have compact support, it is super-exponentially decreasing in the sense that it and all its derivatives are $o(e^{-b\xi})$ for any b as $\xi \to \pm\infty$. To approximate g, consider a sequence g_n of smooth even functions on R such that $g_n \to g$ in $\mathcal{S}$, $\widehat{g_n} \in C_c^\infty$, and $\widehat{g_n} \to \hat{g}$ super-exponentially in the sense that for every i and b, $\sup_\xi(|\widehat{g_n}^{(i)}(\xi) - \hat{g}^{(i)}(\xi)|e^{b\xi}) \to 0$ as $n \to \infty$. One way to do this is to use cutoff functions. Take $\phi : R \to [0,1]$ a smooth even function with support in $[-2,2]$ and $\phi \equiv 1$ on $[-1,1]$ and then let $\widehat{g_n}(\xi) = \phi(\xi/n)\frac{1}{\sqrt{2t}}e^{-\xi^2/4t}$.

For each of these g_n, $g_n(D)$ has a kernel that is compactly supported on $\mathcal{G}$ and thus $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ is well-defined and finite. We would like to show that $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ converges as $n \to \infty$. To do this we shall need a pointwise estimate on the kernels of $g_n(D)$. To get these pointwise estimates, let F be the disjoint union of the holonomy coverings of the leaves of $\mathcal{F}$. Then F is a manifold of bounded geometry, S is a Clifford bundle of bounded geometry, and we can apply the following theorem of Cheeger, Gromov, and Taylor [5].

Theorem 4.2.3 *If M is a manifold of bounded geometry and S is a Clifford bundle of bounded geometry over M, then for every $\delta > 0$, there is a constant C such that for any even function g in S and any points x_1, x_2 in M,*

$$|k_g(x_1, x_2)| \leq C \cdot \sum_{i=0}^{N} \int_{d-\delta}^{\infty} |\hat{g}^{(2i)}(\xi)| d\xi$$

where k_g is the kernel of $g(D)$, $N = [n/2] + 1$, and d is the distance between x_1 and x_2.

This theorem is proved using the finite propagation speed and a local Sobolev embedding theorem together with bounds on the elliptic constant for the Dirac operator D. Since the kernels on $\tilde{L} \times \tilde{L}$ descend to $\mathcal{G}$ as explained above, this theorem gives uniform pointwise bounds on the kernels of $g_n(D)$ on $\mathcal{G}$.

Recall that in the local expression for $\mathrm{Tr}_\nu(f^* \circ B)$ where B is a smooth leafwise operator on a bundle E, the contribution of U_α^γ is

$$\int_{U_\alpha^\gamma} \frac{\mathrm{tr}\ k_B^\gamma(f'(y), y', y'')}{|\mathrm{Id} - (h_\gamma \circ f)_*|(y'')} |dy'| \nu(y'')$$

where $k_B^\gamma(x', y', y'')|dy'|$ is the leafwise kernel of B on $U_\beta \overset{\gamma}{\times} U_\alpha$ with $U_\beta \supset f(U_\alpha)$. In estimating errors obtained by approximating an arbitrary kernel on $\mathcal{G}$ with compactly supported ones, we shall use the estimate that

$$\int_{U_\alpha^\gamma} \frac{\mathrm{tr}\ k_B^\gamma(f'(y), y', y'')}{|\mathrm{Id} - (h_\gamma \circ f)_*|(y'')} |dy'| \nu(y'') \leq$$
$$\sup_{U_\alpha^\gamma} |\mathrm{tr}\ k_B^\gamma(f'(y), y', y'')| \times \int_{y'' \in \Sigma_\alpha^\gamma} \left(\int_{P_{y''}} |dy'| \right) \frac{\nu(y'')}{|\mathrm{Id} - (h_\gamma \circ f)_*|(y'')}$$

The plaque volumes $\int_{P_{y''}} |dy'|$ are clearly globally bounded on M. Since ν is assumed to be an invariant transfixed k-density, the $\nu(y'')/|\mathrm{Id} - (h_\gamma \circ f)_*|(y'')$ factor is holonomy invariant. The fixed set M^f is compact and therefore this factor is bounded on M^f, and hence also on the flow-out of M^f. In proving the Lefschetz theorem, the important data is on the flow-out of M^f so there is no harm in assuming that $\nu/|\mathrm{Id}-(h_\gamma\circ f)_*|$ is globally bounded on $\mathcal{G}^f$. With this assumption, then, there is a number B such that

$$\int_{U_\alpha^\gamma}\frac{|dy'|\nu(y'')}{|\mathrm{Id}-(h_\gamma\circ f)_*|(y'')}<B$$

for all charts U_α^γ for $\mathcal{G}^f$.

Now we use Theorem 4.2.3 to obtain the following estimate.

Proposition 4.2.4 *Given $(M,\mathcal{F})$, (E^*,d_*), D, f, and ν as before, then for any $\delta>0$, there are constants b and C' such that for any even function $g\in\mathcal{S}$ with $\widehat{g}$ compactly supported and for any $R>\delta$,*

$$\int_{\mathcal{G}^f\backslash\mathcal{G}^f{}_R}\frac{tr(k_{g(D)}[f(x)\xrightarrow{\gamma}x])}{|\mathrm{Id}-(h_\gamma\circ f)_*|}|dx'|\nu\le C'\sum_{k=0}^{\infty}\sum_{i=0}^{N}e^{b(R+k)}\int_{R+k-\delta}^{\infty}|\widehat{g}^{(2i)}(\xi)|d\xi \qquad (4.1)$$

Proof. Take a finite covering $\{U_\beta\}$ of M by foliation charts, and recall that there are constants a and b so that $\mathcal{G}^f{}_R$ can be covered with no more than ae^{bR} charts of the form U_β^γ where γ is a leafwise path from $f(U_\beta)$ to U_β (Proposition 2.2.13). Decompose $\mathcal{G}^f\backslash\mathcal{G}^f{}_R$ as $\bigcup_{k=0}^{\infty}(\mathcal{G}^f{}_{R+k+2}\backslash\mathcal{G}^f{}_{R+k})$. For each chart U_β^γ covering $\mathcal{G}^f{}_{R+k+2}\backslash\mathcal{G}^f{}_{R+k}$, the most it can contribute to the integral is B times the supremum of the kernel of $g(D)$ on $\mathcal{G}\backslash\mathcal{G}_{R+k}$. By Theorem 4.2.3, there is a constant C such that this supremum is $<C\sum_{i=0}^{N}\int_{R+k-\delta}^{\infty}|\widehat{g}^{(2i)}(\xi)|d\xi$. Thus the contribution of $\mathcal{G}^f{}_{R+k+2}\backslash\mathcal{G}^f{}_{R+k}$ to the integral is $<ae^{b(R+k+2)}BC\sum_{i=0}^{N}\int_{R+k-\delta}^{\infty}|\widehat{g}^{(2i)}(\xi)|d\xi$. Taking $C'=ae^{2b}BC$ and summing over k then gives the result. ∎

Corollary 4.2.5 *There is a constant C'' such that for any even function g in $\mathcal{S}$ with $\widehat{g}$ compactly supported and any $R>\delta$,*

$$\int_{\mathcal{G}^f\backslash\mathcal{G}^f{}_R}\frac{tr(k_{g(D)}[f(x)\xrightarrow{\gamma}x])}{|\mathrm{Id}-(h_\gamma\circ f)_*|}|dx'|\nu<C''e^{-R}\sup_{|\xi|\ge R-\delta,\,0\le i\le N}|\widehat{g}^{(2i)}(\xi)|e^{(b+1)\xi}.$$

Proof. Suppose $|\widehat{g}^{(2i)}(\xi)|e^{(b+1)}\xi < C$ for all ξ with $|\xi| \geq R - \delta$ and for all i with $0 \leq i \leq N$. Then

$$\int_{R+k-\delta}^{\infty} |\widehat{g}^{(2i)}(\xi)|d\xi < C \int_{R+k-\delta}^{\infty} e^{-(b+1)\xi} d\xi \leq \frac{C}{b+1} e^{-(b+1)(R+k-\delta)}.$$

Hence the right-hand side of inequality 4.1 is

$$< C' \sum_{k=0}^{\infty} (N+1) e^{b(R+k)} \frac{C}{b+1} e^{-(b+1)(R+k-\delta)} = \frac{C'(N+1)}{b+1} e^{(b+1)\delta} \sum_{k=0}^{\infty} e^{-k} e^{-R} C.$$

Thus the result follows with $C'' = \frac{C'(N+1)}{b+1} e^{(b+1)\delta} \sum e^{-k}$. ∎

The point of this result is that it enables Tr_ν to be extended to operators $f^* \circ g(D)$ where $\widehat{g}$ is super-exponentially decaying.

Theorem 4.2.6 *If $\widehat{g}$ is super-exponentially decaying and $\widehat{g_n}$ are all compactly supported and converge to $\widehat{g}$ super-exponentially, then* $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ *converges as* $n \to \infty$.

Proof. First we claim that for any ϵ, there is an R such that, for all n sufficiently large, the contribution of the complement of $\mathcal{G}^f{}_R$ to $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ is less than ϵ. This follows from $|\widehat{g_n}^{(2i)}(\xi)| \leq |\widehat{g_n}^{(2i)}(\xi) - \widehat{g}^{(2i)}(\xi)| + |\widehat{g}^{(2i)}(\xi)|$ together with the previous corollary and the assumptions on $\widehat{g}$ and $\widehat{g_n} \to \widehat{g}$.

Thus to show that $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ converges, we only need to consider $\mathcal{G}^f{}_R$. Another application of Theorem 4.2.3 shows that the kernels of $g_n(D)$ converge uniformly on $\mathcal{G}^f{}_R$. Since the volume of $\mathcal{G}^f{}_R$ is finite, the $\mathcal{G}^f{}_R$ contribution to $\mathrm{Tr}_\nu(f^* \circ g_n(D))$ converges. ∎

We can now define $\mathrm{Tr}_\nu(f^* \circ e^{-tD^2})$ to be this limit.

4.3. Time Invariance

Let E^*, D, f, and ν be as before. Recall that $D = \bigoplus(d_i + d_i^*)$ so that $D^2 = \bigoplus \Delta_i$ and e^{-tD^2} is $e^{-t\Delta_i}$ on E^i. Thus $\mathrm{Tr}_\nu(f^* \circ e^{-tD^2}) = \sum_i \mathrm{Tr}_\nu(f^* \circ e^{-t\Delta_i})$. For the heat operator approach to the Lefschetz theorem, we need to consider instead $\sum_i(-1)^i \mathrm{Tr}_\nu(f^* \circ e^{-t\Delta_i})$. To do this, introduce a Z_2 grading by decomposing $S = \bigoplus_i E^i$ into $S^+ = \bigoplus_{i \text{ even}} E^i$ and $S^- = \bigoplus_{i \text{ odd}} E^i$. The Dirac operator then decomposes into $D^+ : C^\infty(S^+) \longrightarrow C^\infty(S^-)$ and $D^- : C^\infty(S^-) \longrightarrow C^\infty(S^+)$. Define the super-trace tr^s on the fibers of S to be the usual trace on S^+ and minus the usual trace on S^-. Then define Tr_ν^s exactly as for Tr_ν except that the fiber trace tr^s replaces tr. In other words, $\mathrm{Tr}_\nu^s(A) = \pi_*(\mathrm{tr}^s \Delta_\nu^*(k_A))$ and

$$\sum(-1)^i \mathrm{Tr}_\nu(f^* \circ e^{-t\Delta_i}) = \mathrm{Tr}_\nu^s(f^* \circ e^{-tD^2}) =$$

$$\lim_{n\to\infty} \mathrm{Tr}_\nu^s(f^* \circ g_n(D)) = \lim_{n\to\infty} \int_{\mathcal{G}^f} \frac{\mathrm{tr}^s(k_{g_n}([f(x) \xrightarrow{\gamma} x]))}{|\det(\mathrm{Id} - (h_\gamma \circ f)_*)|} |dx'|\nu.$$

The fiber trace tr^s has the property that $\mathrm{tr}^s(A \circ B) = \mathrm{tr}^s(B \circ A)$ if either A or B preserves the Z_2 grading but $\mathrm{tr}^s(A \circ B) = -\mathrm{tr}^s(B \circ A)$ if either one reverses the grading. Thus, using the super-trace instead of the trace, Theorem 3.4.1 becomes

$$\mathrm{Tr}_\nu^s(f^* \circ B_2 \circ B_1) = \mathrm{Tr}_\nu^s(B_1 \circ f^* \circ B_2)$$

if the operator B_1 preserves the grading and

$$\mathrm{Tr}_\nu^s(f^* \circ B_2 \circ B_1) = -\mathrm{Tr}_\nu^s(B_1 \circ f^* \circ B_2)$$

if it reverses the grading. Note that D, d, and d^* all reverse the grading while D^2, Δ_i, and f^* all preserve it.

We shall now show that $\mathrm{Tr}_\nu^s(f^* \circ e^{-tD^2})$ is constant as a function of t. To do this, we use a trick from Roe [21] that shows that one can find a sequence of approximations $g_{m,t}$ to e^{-tx^2} as in the previous section, such that $\mathrm{Tr}_\nu^s(f^* \circ g_{m,t}(D))$ is itself constant as a function of t for m fixed. Let φ_m be a sequence of smooth even functions converging to $e^{-x^2/2}$ in S and with $\widehat{\varphi_m}$ compactly supported and converging to $e^{-\xi^2/2}$ super-exponentially. Then φ_m^2 converges to e^{-x^2} in S, and the Fourier transforms $\widehat{\varphi_m} * \widehat{\varphi_m}$ are compactly supported and converge super-exponentially to $\frac{1}{\sqrt{2}}e^{-\xi^2/4}$. Since φ_m is even, we can write $\varphi_m(x) = \phi_m(x^2)$ where ϕ_m is smooth. Let $g_{m,t}(x) = \phi_m^2(tx^2) = \varphi_m^2(\sqrt{t}x)$. Then $g_{m,t}(x) \to e^{-tx^2}$ in $\mathcal{S}$ with Fourier transform $\widehat{g_{m,t}}(\xi) = \frac{1}{\sqrt{t}}\widehat{\varphi_m}*\widehat{\varphi_m}(\xi/\sqrt{t})$ compactly supported and converging super-exponentially to $\frac{1}{\sqrt{2t}}e^{-\xi^2/4t}$.

In this situation, the previous section shows that

$$\mathrm{Tr}_\nu^s(f^* \circ e^{-tD^2}) = \lim_{m\to\infty} \mathrm{Tr}_\nu^s(f^* \circ g_{m,t}(D)).$$

Before proving that $\mathrm{Tr}_\nu^s(f^* \circ g_{m,t}(D))$ is constant in t, we need the following lemma.

Lemma 4.3.1 *If $\phi \in S(R)$ is such that the function $\varphi(x) = \phi(x^2)$ has compactly supported Fourier transform, then so does the function $x \mapsto \phi'(x^2)$.*

Proof. Since $\varphi(x)$ has compactly supported Fourier transform, so does $\varphi'(x)$ since $\widehat{\varphi'}(\xi) = i\xi\widehat{\varphi}(\xi)$. Then $\phi'(x^2) = \varphi'(x)/2x$ has Fourier transform $\widehat{\varphi'} * \mathrm{sign}$ because the Fourier transform of $1/2x$ is the function

$$\mathrm{sign}(\xi) = \begin{cases} 1 & \text{if } \xi > 0 \\ -1 & \text{if } \xi < 0 \end{cases}$$

$$(\widehat{\varphi'} * \mathrm{sign})(\xi) = \int_{\eta\le\xi} \widehat{\varphi'}(\eta)d\eta - \int_{\eta\ge\xi} \widehat{\varphi'}(\eta)d\eta$$

If the support of $\widehat{\varphi'}$ is contained in $[a,b]$, then for any ξ outside of $[a,b]$,

$$(\widehat{\varphi'} * \mathrm{sign})(\xi) = \pm\int_{-\infty}^{\infty} \widehat{\varphi'}(\eta)d\eta$$

But φ is even, so φ' and thus also $\widehat{\varphi'}$ is odd. Thus the integral is 0, and we have shown that the Fourier transform of $\phi'(x^2)$ has support contained in $[a,b]$ if the support of $\widehat{\varphi'}$ is. ∎

Proposition 4.3.2 $\mathrm{Tr}_\nu^s(f^* \circ g_{m,t}(D))$ *is constant in t for $0 < t < \infty$.*

Proof. Differentiate $\mathrm{Tr}^s_\nu(f^* \circ \phi^2_m(tD^2))$ with respect to t to get

$$\begin{aligned}
&2\mathrm{Tr}^s_\nu(f^* D^2 \phi'_m(tD^2)\phi_m(tD^2)) \\
&= 2\mathrm{Tr}^s_\nu(f^*(dd^* + d^*d)\phi'_m(tD^2)\phi_m(tD^2)) \\
&= 2\mathrm{Tr}^s_\nu(f^*(dd^*)\phi'_m(tD^2)\phi_m(tD^2)) + 2\mathrm{Tr}^s_\nu(f^*(d^*d)\phi'_m(tD^2)\phi_m(tD^2)).
\end{aligned}$$

It is thus sufficient to show that

$$\mathrm{Tr}^s_\nu(f^*(dd^*)\phi'_m(tD^2)\phi_m(tD^2)) = -\mathrm{Tr}^s_\nu(f^*(d^*d)\phi'_m(tD^2)\phi_m(tD^2)).$$

We shall now establish this fact by a chain of equalities transforming the left-hand side into the right-hand side. First apply Theorem 3.4.1 with $B_2 = dd^*\phi'_m(tD^2)$ and $B_1 = \phi_m(tD^2)$. The lemma implies that $\phi'_m(tx^2)$ has compactly supported Fourier transform, and thus the kernel of $\phi'_m(tD^2)$ is in $C^\infty_c(\mathcal{G}^f)$. The same is then true of the kernel of $dd^*\phi'_m(tD^2)$ so that application of the theorem is justified. Also, since B_1 is a function of D^2, it preserves the grading, and we can conclude

$$\mathrm{Tr}^s_\nu(f^* \circ (dd^*)\phi'_m(tD^2) \circ \phi_m(tD^2)) = \mathrm{Tr}^s_\nu(\phi_m(tD^2) \circ f^* \circ dd^*\phi'_m(tD^2)).$$

Since f^* is a map of the Dirac complex (E^*, d), f^* commutes with d, and we get

$$\mathrm{Tr}^s_\nu(\phi_m(tD^2)df^*d^*\phi'_m(tD^2)).$$

Now apply Theorem 3.4.1 with $B_1 = \phi_m(tD^2)d$ and $B_2 = d^*\phi'_m(tD^2)$. This time B_1 reverses the grading so we get

$$\mathrm{Tr}^s_\nu(\phi_m(tD^2)d \circ f^* \circ d^*\phi'_m(tD^2)) = -\mathrm{Tr}^s_\nu(f^* \circ d^*\phi'_m(tD^2) \circ \phi_m(tD^2)d).$$

Finally, d commutes with $D^2 = dd^* + d^*d$ and hence with any function of D^2 so that

$$-\mathrm{Tr}^s_\nu(f^*d^*\phi'_m(tD^2)\phi_m(tD^2)d) = -\mathrm{Tr}^s_\nu(f^*(d^*d)\phi'_m(tD^2)\phi_m(tD^2))$$

which establishes the claim. ∎

Corollary 4.3.3 $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ *is constant in* t *for* $0 < t < \infty$.

Proof. This follows from $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2}) = \lim_{m\to\infty} \mathrm{Tr}^s_\nu(f^* \circ g_{m,t}(D))$. ■

4.4. The Time Zero Limit

Since $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ is independent of t for $0 < t < \infty$, the time zero limit and the time infinity limit must be equal. The time zero limit is determined by quantities defined over the fixed point set of f. First we shall show that as $t \to 0^+$, the integrand defining $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ concentrates on the fixed point set.

Proposition 4.4.1 *For any $R > 0$,*

$$\lim_{t\to 0+} (\textit{contribution of } \mathcal{G}^f \backslash \mathcal{G}^f{}_R \textit{ to } \mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})\ ) = 0.$$

Proof. Let $g_t(x) = e^{-tx^2}$ so that $\widehat{g}_t(\xi) = \frac{1}{\sqrt{2t}} e^{-\xi^2/4t}$. Then

$$\widehat{g_t}^{(k)}(\xi) = (1/2t)^{k+\frac{1}{2}} p_k(\xi, t) e^{-\xi^2/4t}$$

where p_k is a polynomial of degree k in ξ and t. Now pick a $\delta < R$ and apply Corollary 4.2.5. For any i,

$$\lim_{t\to 0^+} \sup_{|\xi|\geq R-\delta} |\widehat{g_t}^{(2i)}(\xi)| e^{(b+1)t} = 0.$$

■

This proposition shows that in determining the time zero limit, we only need consider $\mathcal{G}^f{}_R$ where R is arbitrarily small. Proposition 2.2.12 says that for R small enough, $\mathcal{G}^f{}_R$ is contained in the flow-out of the fixed set M^f. But the analysis of the time zero asymptotics of the leafwise heat operator near the fixed set reduces to the usual time zero asymptotics along the leaves integrated transversely.

Theorem 4.4.2 *Let f be a Lefschetz morphism of dimension k of $(M, \mathcal{F})$ and let ν be an invariant transfixed k-density. Let (E^*, d_*) be a leafwise Dirac complex and let (f, T_*) be a geometric endomorphism of it. For any x in the fixed set M^f, let*

$$a(x) = \frac{\sum_i (-1)^i \mathrm{tr}(T^i(x))}{|\det(\mathrm{Id} - T_x(f|_L))|},$$

the Atiyah-Bott local index for $f|_L$ at x. Then

$$\lim_{t\to 0^+}\sum_i(-1)^i\mathrm{Tr}_\nu((f,T^i)^*\circ e^{-t\Delta_i})=\int_{M^f}\frac{a(x)\nu(x)}{|\det(\mathrm{Id}-f_*)|(x)}.$$

Proof. The definitions of Dirac complex and super-trace imply that

$$\sum(-1)^i\mathrm{Tr}_\nu((f,T^i)^*\circ e^{-t\Delta_i})=\mathrm{Tr}^s_\nu((f,T)^*\circ e^{-tD^2}).$$

The previous proposition shows that to compute $\lim_{t\to 0^+}\mathrm{Tr}^s_\nu((f,T)^*\circ e^{-tD^2})$, we only need consider $\mathcal{G}^f{}_R$ for any positive R. Thus, using Proposition 3.3.3,

$$\lim_{t\to 0^+}\mathrm{Tr}^s_\nu((f,T)^*\circ e^{-tD^2})=\lim_{t\to 0^+}\int_{\mathcal{G}^f{}_R}\frac{\mathrm{tr}^s k_t([f(x)\to x])\otimes\nu}{|\det(\mathrm{Id}-(h_\gamma\circ f)_*)|}$$

where k_t is the (symbol of the) leafwise heat kernel on $\mathcal{G}$. Since M^f is a transverse embedded submanifold of M (Proposition 2.2.10), M^f can be covered by a finite number of foliation charts U_α so that for each U_α, $U_\alpha\cap M^f=\Sigma^f_\alpha$ is a k-dimensional submanifold transverse to the foliation. In $\mathcal{G}^f$, M^f is covered by the charts

$$\begin{aligned}U^0_\alpha &= \{[f(x)\xrightarrow{\gamma}x]\mid x\in U_\alpha,\ \text{the plaque } P_x \text{ intersects } \Sigma^f_\alpha,\ \text{and } \gamma \text{ is a plaque path}\}\\ &\cong \Sigma^f_\alpha\times R^p.\end{aligned}$$

For R small enough, Proposition 2.2.12 implies $\mathcal{G}^f{}_R\subset\bigcup U^0_\alpha$. Let $\{\varphi_\alpha\}$ be a partition of unity for M^f subordinate to Σ^f_α and extend φ_α to U^0_α by defining $\varphi_\alpha(x)=\varphi_\alpha(x'')$. Then the local expression for Tr^s_ν leads to

$$\lim_{t\to 0^+}\int_{\mathcal{G}^f{}_R}\frac{\mathrm{tr}^s k_t([f(x)\to x])\otimes\nu}{|\det(\mathrm{Id}-(h_\gamma\circ f)_*)|}=\lim_{t\to 0^+}\sum_\alpha\int\int_{U^0_\alpha}\frac{\varphi_\alpha(x)k_t(f'(x),x',x'')|dx'|\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')}$$

Note that on U^0_α, the holonomy h_γ is trivial. On each chart,

$$\int\int_{U^0_\alpha}\frac{\varphi_\alpha(x)k_t(f'(x),x',x'')|dx'|\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')}=$$
$$\int_{x''\in\Sigma^f_\alpha}\frac{\varphi_\alpha(x'')\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')}\int_{x'\in P_{x''}}k_t(f'(x),x',x'')|dx'|.$$

As $t\to 0^+$, for x'' fixed, the asymptotics of the heat kernel on the plaque $P_{x''}$ imply that $\int_{x'\in P_{x''}}k_t(f'(x),x',x'')|dx'|$ converges to the leafwise local index for f at the fixed point x'', i.e.,

$$\int_{x'\in P_{x''}}k_t(f'(x),x',x'')|dx'|\to a(x'')=\frac{\sum(-1)^i\mathrm{Tr}(T^i(x))}{|\det(\mathrm{Id}-T_{x''}f)|}$$

where $T_{x''}f$ is the derivative restricted to $T_x\mathcal{F}$ [12, Section 1.8] or [20, Chapter 8]. To establish the theorem, then, we only need the fact that this convergence is uniform in x'', for then we will have

$$\begin{aligned}
&\lim_{t\to 0+}\sum_\alpha \int\int_{U^0_\alpha} \frac{\varphi_\alpha(x)k_t(f'(x),x',x'')|dx'|\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')} \\
&= \sum_\alpha \int_{x''\in\Sigma'_\alpha} \frac{\varphi_\alpha(x'')\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')} \lim_{t\to 0+}\int_{x'\in P_{x''}} k_t(f'(x),x',x'')|dx'| \\
&= \sum_\alpha \int_{\Sigma'_\alpha} \frac{\varphi_\alpha(x'')\nu(x'')}{|\det(\mathrm{Id}-f_*)|(x'')} a(x'') \\
&= \int_{M^f} \frac{a(x)\nu(x)}{|\det(\mathrm{Id}-f_*)|(x)}.
\end{aligned}$$

It remains to show that the convergence is uniform in x'' on the chart U^0_α. But this follows by carrying out the Gilkey-Seeley expansion for the asymptotics of the leafwise heat kernel on U_α with the transverse variable x'' acting as a parameter and showing that the necessary estimates are all uniform in x''. Briefly, the argument is as follows. Let $k_t(y',x',x'')|dx'|$ be the leafwise heat kernel for e^{-tD^2} at the point (y',x'',x',x'') in $U_\alpha \overset{0}{\times} U_\alpha$ as before. Let $k'_t(y',x',x'')|dx'|$ be the kernel for the approximation constructed on the chart $U_\alpha \overset{0}{\times} U_\alpha$ using the resolvant. In Gilkey's notation, we let $r_n(x',x'',\xi',\lambda)$ be the homogeneous symbol of degree $-(n+2)$ in the local approximation for $(D^2-\lambda)^{-1}$. Here $((x',x''),\xi',\lambda)\in U_\alpha\times R^p\times\mathcal{R}$ where $\mathcal{R}$ is a region in the complex plane as in [12]. Then $e_n(t,x',x'',\xi')$ is defined to be $\frac{1}{2\pi i}\int_\gamma e^{-t\lambda}r_n(x',x'',\xi',\lambda)d\lambda$, and $K_n(t,y',x',x'')$ is defined to be $\int_{R^p} e^{i(y'-x')\cdot\xi'}e_n(t,y',x'',\xi')d\xi'$. For any k, if n_0 is large enough, there is an estimate [12, Lemma 1.7.3]

$$\sup_{x',y'\in P^\alpha_{x''}}\ \sup_{0<t<1} |k_t(y',x',x'') - \sum_{n=0}^{n_0} K_n(t,y',x',x'')| \le C_{k,x''}t^k.$$

This estimate is obtained on each plaque from a Sobolev embedding theorem together with an estimate on the local Sobolev norm of the operator

$$e^{-tD^2_{x''}} - \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}R_{x''}(\lambda)d\lambda$$

where $R_{x''}(\lambda)$ is the resolvant with symbol $\sum_{n=0}^{n_0} r_n(x',x'',\xi',\lambda)$. The constant involved in the Sobolev embedding theorem depends only on p, the leaf dimension, and is thus independent of the transverse parameter x''. The Sobolev norm estimates depend continuously on:

1. the local Sobolev norms of a finite number of powers of $D^2_{x''}$;
2. the elliptic constants of a finite number of powers of $D^2_{x''}$;
3. the distance from $\mathcal{R}$ to the spectrum of $D^2_{x''}$; and
4. the local symbol of $D^2_{x''}$.

Since all of these depend continuously on the transverse parameter x'', we see that $C_{k,x''}$ is a continuous function of x''. Thus on the compact set $\mathrm{supp}(\varphi_\alpha) \subset \Sigma^f_\alpha$, there is a C_k such that

$$\sup_{x'' \in \mathrm{supp}(\varphi_\alpha)} \sup_{x',y' \in P^\alpha_{x''}} \sup_{0<t<1} |k_t(y',x',x'') - \sum K_n(t,y',x',x'')| \leq C_k t^k.$$

Thus $\sum K_n(t,y',x',x'')$ gives an asymptotic expansion for the heat kernel uniformly in x''. Substituting $y' = f'(x)$ gives an asymptotic expansion for $k_t(f'(x),x',x'')$ that is still uniform in x'', and then taking integrals, we get that

$$\int_{P_{x''}} \left(k_t(f'(x),x',x'') - \sum K_n(t,y',x',x'')\right) |dx'| \leq C'_k t^k$$

where C'_k is independent of x''.

To calculate the limiting value as $t \to 0^+$, we can take k to be any positive number. Since we have assumed that on each leaf f has isolated nondegenerate fixed points, a calculation shows that only the K_0 term contributes terms of order 0 in t [12, 1.8.3] or [20, 1.8.11]. In fact,

$$\lim_{t \to 0^+} \int_{P_{x''}} K_0(t,u',x',x'')|dx'| = a(x''),$$

and a similar argument to that above, shows that the error is bounded by $Ct^{1/2}$ independently of x''. ∎

For the four classical complexes, Atiyah and Bott [3] described the local indices. These are as follows:

1. deRham: $a(x) = \text{sign } \det(\text{Id} - T_x(f|_L))$;

2. Dolbeault: $a(x) = 1/\det_{\mathbf{C}}(\text{Id} - T_x(f|_L))$;

3. Signature: $a(x) = \prod_j(-i\cot(\theta_j/2))$ where T_xL is decomposed into j 2-dimensional oriented orthogonal invariant subspaces so that $T_x(f|_L)$ is rotation through the angle θ_j on the jth subspace; and

4. Spin: $a(x) = \prod_j(\pm\frac{i}{2}\csc(\theta_j/2))$ where the angles θ_j are as before and the sign $\pm$ depends on the lifting of T^*f to the spin bundle.

CHAPTER 5

TIME INFINITY AND HOMOLOGY

We have established that if (f, T^*) is a Lefschetz morphism of a leafwise Dirac complex on a foliated manifold, then $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ is independent of t for $0 < t < \infty$ and the time zero limit can be evaluated by quantities defined on the fixed point set of f. We now consider the time infinity limit. The general philosophy here is that the time infinity limit is a global quantity involving the trace of (f, T^*) on the homology, in some sense, of the Dirac complex. This is modeled after the classical case of a (non-foliated) compact manifold where the heat kernel converges in the C^∞topology to the kernel for the projection on the (finite-dimensional) homology as $t \to +\infty$.

For the foliated Lefschetz theorem, the answer is not so straightforward. To identify some of the difficulties, we first recall that the definition of $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ is $\pi_* \mathrm{tr}^s \Delta^*((f \times \mathrm{id})^* k_t \otimes \nu)$ where k_t is the leafwise heat kernel on $M \times M$. The question of whether the time infinity limit commutes with the various operations involved in this definition thus arises naturally. A related question is what $\lim_{t\to\infty} k_t$ represents if it does exist as a distribution on M. Is it projection on the homology of the leafwise Dirac complex? In what sense should we consider the homology of the complex? We shall give partial answers to these questions in this chapter, and in the next chapter we shall study the variety of behavior that can result in specific examples.

For the first question, note that the push-forward map $\pi_* : C^{-\infty}(|TM|) \to C^{-\infty}(\{.\})$, which is just integration over M, is continuous (Section 3.1). Thus

$$\lim_{t\to\infty} \pi_* \mathrm{tr}^s \Delta^*((f \times \mathrm{id})^* k_t \otimes \nu) = \pi_* \lim_{t\to\infty} \mathrm{tr}^s \Delta^*((f \times \mathrm{id})^* k_t \otimes \nu)$$

provided that this second limit exists. Here we are considering $\mathrm{tr}^s \Delta^*((f \times \mathrm{id})^* k_t \otimes \nu)$ as a distributional section of $|TM|$ over M. The fiberwise supertrace, tr^s, is also continuous so that the time infinity limit can also be commuted past tr^s provided that the limit exists.

The next operation, $\Delta^*_\nu : k \mapsto \Delta^*(k \otimes \nu)$ from $C^\delta_{c(f\circ\mathcal{G})}(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S))$ to $C^\delta_{c(\mathcal{G}f)}(|TM| \otimes \mathrm{Hom}(S,S))$ is continuous if $C^\delta_{c(f\circ\mathcal{G})}$ and $C^\delta_{c(\mathcal{G}f)}$ are given the topologies induced from the inclusions into $C^{-\infty}$ (Section 3.1). However, as defined, the space $C^\delta_{c(f\circ\mathcal{G})}$ consists only of δ-sections. It does not contain smooth sections, and as mentioned in Section 3.1, the map Δ^*_ν is not a continuous extension of the pull-back map on smooth sections:

$$\Delta^* : C^\infty(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S) \to M \times M) \longrightarrow C^\infty(|TM| \otimes \mathrm{Hom}(S,S) \to M).$$

Although the kernel k_t is a δ-section on $\mathcal{G}$ for all finite $t > 0$, it may be that the time infinity limit is not. An example where the limiting distribution is not a δ-section is the irrational foliation of the torus discussed in the next chapter. In this case the time infinity limit is constant and thus an element of C^∞ rather than $C^\delta_{\mathcal{G}}$.

Nevertheless, in some cases the time infinity limit can be commuted. If the transfixed dimension $k = 0$ and ν is trivial, then Tr_ν is a continuous extension of the trace on operators with smooth kernels. Consider the immersion $M \xrightarrow{f\times\mathrm{id}} M \times M$. Recall from Section 3.1 that the conormal bundle of the map $f \times \mathrm{id}$ is defined by

$$N^*(f\times\mathrm{id}) = \{\eta \in T^*(M\times M) \mid \exists x \in M \text{ with } \eta \in T^*_{(f(x),x)}M\times M \text{ and } (f\times\mathrm{id})^*\eta = 0\}.$$

Recall also from Section 3.3 that if f is transfixed of dimension 0, then $M \xrightarrow{f\times\mathrm{id}} M \times M$ is transverse to the groupoid $\mathcal{G} \xrightarrow{s\times r} M \times M$ so that $N^*\mathcal{G}\backslash 0$ is disjoint from $N^*(f \times \mathrm{id})\backslash 0$. Suppose that the closure of $N^*\mathcal{G}\backslash 0$ in $T^*(M \times M)\backslash 0$ is also disjoint from $N^*(f \times \mathrm{id})\backslash 0$ and let Γ be a closed cone in the complement of $N^*(f \times \mathrm{id})\backslash 0$ that contains $N^*\mathcal{G}\backslash 0$.

Proposition 5.0.1 *If $k = 0$, ν is trivial, Γ is as above, and k_t converges to a limiting distribution k_∞ in $C^{-\infty}_\Gamma$, then*

$$\lim_{t\to\infty} \mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2}) = \mathrm{Tr}^s(f^* \circ \lim_{t\to\infty} e^{-tD^2}).$$

Proof. Since ν is trivial, $\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$ is defined by $\pi_* \mathrm{tr}^s(f \times \mathrm{id})^* k_t$. Note that $\pi_* \mathrm{tr}^s(f \times \mathrm{id})^* k$ is defined for all distributional sections k in $C^{-\infty}(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S))$ such that $\mathrm{WF}(k) \cap N^*(f \times \mathrm{id}) = \emptyset$. Also note that k_t is in $C_\Gamma^{-\infty}$ because $\mathrm{WF}(k_t) \subset \overline{N^*\mathcal{G}} \backslash 0 \subset \Gamma$. Theorem 8.2.4 of Hörmander [17] implies that $(f \times \mathrm{id})^*$ is continuous from $C_\Gamma^{-\infty}$ to $C^{-\infty}$. Since π_* and tr^s are continuous, the assumption that $k_t \to k_\infty$ in $C_\Gamma^{-\infty}$ implies that

$$
\begin{aligned}
\lim_{t\to\infty} \pi_* \mathrm{tr}^s(f \times \mathrm{id})^* k_t &= \pi_* \mathrm{tr}^s(f \times \mathrm{id})^* k_\infty \\
&= \mathrm{Tr}^s(f^* \circ \lim_{t\to\infty} e^{-tD^2}).
\end{aligned}
$$

■

Another situation where the limit commutes is when the limiting distribution k_∞ is a δ-section in the same space as k_t for t finite.

Proposition 5.0.2 *If k_t converges to a distribution k_∞ continuously in the space $C^\delta_\mathcal{G}(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S))$, then*

$$\lim_{t\to\infty} \mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2}) = \mathrm{Tr}^s_\nu(f^* \circ \lim_{t\to\infty} e^{-tD^2}).$$

Proof. The map $k_t \mapsto \pi_* \mathrm{tr}^s \Delta^*((f \times \mathrm{id})^* k_t \otimes \nu)$ is the composition of the continuous maps

$$
\begin{aligned}
C^\delta_\mathcal{G}(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S)) &\xrightarrow{(f\times\mathrm{id})^*} C^\delta_{f\circ\mathcal{G}}(\pi_2^*|TM| \otimes \mathrm{Hom}(S,S)) \\
&\xrightarrow{\Delta^*_\nu} C^\delta_{\mathcal{G}_f}(|TM| \otimes \mathrm{Hom}(S,S)) \\
&\xrightarrow{\mathrm{tr}^s} C^\delta_{\mathcal{G}_f}(|TM|) \\
&\xrightarrow{\pi_*} C^{-\infty}(\{.\}).
\end{aligned}
$$

■

Note that in these propositions, the limit of the operator e^{-tD^2} is being considered only in the sense that its kernel converges on $M \times M$. In other words, the topology of convergence for e^{-tD^2} is the topology of the Schwartz kernels theorem, that is as an operator from $C^\infty(M)$ to $C^{-\infty}(M)$.

If the time infinity limit does commute, as in the two cases just described, then the global part of the Lefschetz formula is the ν-trace of f^* on the image of P, where $P = \lim_{t\to\infty} e^{-tD^2}$. Thus we can state the following Lefschetz formula for foliated manifolds:

Theorem 5.0.3 *If f is a Lefschetz morphism of dimension k of a leafwise Dirac complex over $(M, \mathcal{F})$ and ν is an invariant transfixed k-density, and if* $\lim_{t\to\infty}$ *commutes with* $\mathrm{Tr}_\nu^s(f^*)$, *then*

$$\int_{M^f} \frac{a(x)}{|\mathrm{Id} - f_*|} \nu = \mathrm{Tr}_\nu^s(f^* \circ P)$$

where $a(x)$ is the leafwise index of f at the fixed point x and $P = \lim_{t\to\infty} e^{-tD^2}$.

The right-hand side of the equation is traditionally considered to be a trace on homology. In the case of compact leaves, discussed below, the time infinity limit P is projection onto the leafwise harmonic sections of the Dirac complex so that $\mathrm{Tr}_\nu^s(f^* \circ P)$ is the trace of f^* on this space. Even in this case, however, we shall see that the holonomy of the leaves cannot be ignored. The holonomy covers of leaves must be considered and the value of $\mathrm{Tr}_\nu^s(f^* \circ P)$ is a trace of f^* on the covers weighted by factors which take into account the holonomy transport along the various paths from $f(x)$ to x.

If the holonomy groupoid $\mathcal{G}$ is compact, the time infinity limit does commute. We now analyze this situation. First note that compactness of $\mathcal{G}$ is equivalent to several other conditions involving the orbit structure of $\mathcal{F}$.

Proposition 5.0.4 *For a foliated manifold $(M, \mathcal{F})$, compactness of $\mathcal{G}$ is equivalent to all leaves L being compact plus any one of the following conditions:*

1. *there is a bound on the volumes of leaves;*
2. *all leaves have finite holonomy;*
3. *each leaf has arbitrarily small saturated neighborhoods;*
4. *the quotient space of leaves of M is Hausdorff;*
5. *if $K \subseteq M$ is compact, then the saturation of K is also compact.*

Proof. Suppose $\mathcal{G}$ is compact. For any leaf L, pick an $x \in L$. The holonomy cover $\tilde{L}$ is isomorphic to the closed subset $r^{-1}(x) \subset \mathcal{G}$. Since $\mathcal{G}$ is compact, so is $\tilde{L}$. This implies that L is compact and has finite holonomy. Given that all leaves are compact, the equivalence of finite holonomy with any of the other four conditions is proved in Epstein [11].

The converse also follows from Epstein. If all leaves are compact and any one of the five equivalent conditions is satisfied, then there is a local model for the foliation in a saturated neighborhood U of any leaf L. This local model is as follows. There is a finite subgroup K of the orthogonal group $O(q)$ such that K is isomorphic to the holonomy covering group of $\tilde{L} \to L$ and the foliated neighborhood U is isomorphic to the foliated neighborhood $\tilde{L} \overset{K}{\times} D^q$ where D^q is the unit disk in R^q. The space $\tilde{L} \overset{K}{\times} D^q$ is the quotient of $\tilde{L} \times D^q$ obtained by identifying $(\tilde{x}, kd)$ with $(\tilde{x}k, d)$, and it is foliated by the quotients of the $\tilde{L} \times d$'s. With this local model, the holonomy groupoid over $U \times U \subset M \times M$ is isomorphic to $(\tilde{L} \times \tilde{L} \times D^q)/\sim$ where the equivalence relation is given by $(\tilde{x}, \tilde{y}, kd) \sim (\tilde{x}k, \tilde{y}k, d)$. With this isomorphism the source and range maps are given by $[(\tilde{x}, \tilde{y}, d)] \overset{s}{\mapsto} [(\tilde{x}, d)]$ and $[(\tilde{x}, \tilde{y}, d)] \overset{r}{\mapsto} [(\tilde{y}, d)]$. Since $\tilde{L}$ and D^q are both compact, this space $(s \times r)^{-1}(U \times U)$ is compact. Since M is compact, it can be covered by a finite number of U. But then $\mathcal{G}$ is covered by a finite number of the $(s \times r)^{-1}(U \times U)$ and hence is compact. ■

As explained in Section 4.2, the holonomy groupoid lying over $L \times L$ is isomorphic to $(\tilde{L} \times \tilde{L})/G$ where $G = G_x^x$ is the holonomy group of L. The symbol of the kernel k_t is equal to the equivariant kernel $k_t^{\tilde{L}}$ for $e^{-tD_{\tilde{L}}^2}$ on $(\tilde{L} \times \tilde{L})/G$. Since $\tilde{L}$ is compact, $k_t^{\tilde{L}} \to k_P^{\tilde{L}}$ in the C^∞topology as $t \to \infty$, where P is the L^2 projection onto the harmonic sections of $S|_{\tilde{L}}$. (See, for example, Lemma 8.4 of [20]). Thus the symbol of k_t converges pointwise on $\mathcal{G}$ to the kernel for leafwise projection onto the harmonic sections. Since $\mathcal{G}$ is compact and since there is a uniform pointwise bound on the kernels $k_t^{\tilde{L}}$ for all $\tilde{L}$ and t large (Theorem 4.2.3), the limiting symbol on $\mathcal{G}$ is a bounded Borel function. This limiting symbol h_∞ on $\mathcal{G}$ defined by $h_\infty = k_P^{\tilde{L}}$ on $(\tilde{L} \times \tilde{L})/G$ is the symbol of the limiting distribution because for $\phi_1 \in C^\infty(S)$ and

$\phi_2 \in C^\infty(S^* \otimes |TM|)$,

$$\begin{aligned} \langle \phi_2 \otimes \phi_1, k_\infty \rangle &= \lim_{t\to\infty} \langle \phi_2 \otimes \phi_1, k_t \rangle \\ &= \lim_{t\to\infty} \int_{\mathcal{G}} \phi_2(x) k_t(x \xrightarrow{\gamma} y) \phi_1(y) \\ &= \int_{\mathcal{G}} \phi_2(x) h_\infty(x \xrightarrow{\gamma} y) \phi_1(y) \end{aligned}$$

by the dominated convergence theorem. We have shown:

Proposition 5.0.5 *If $\mathcal{G}$ is compact, then the limit of the distributions k_t as $t \to \infty$ exists and is equal to the generalized section of $\pi_2^*|TM| \otimes \mathrm{Hom}(S, S)$ over $M \times M$ which is supported on $\mathcal{G}$ with symbol equal to leafwise projection onto harmonic sections of S.*

If we pass down from $\mathcal{G}$ to $M \times M$, what does k_∞ look like? First recall the local description of k_t on M from Proposition 3.2.4. Given sections $\phi_1 \in C^\infty(S)$ and $\phi_2 \in C^\infty(S^* \otimes |TM|)$,

$$\langle \phi_2 \otimes \phi_1, k_t \rangle = \int_M \phi_2(x) \left(\int_{x \xrightarrow{\gamma} y \in \tilde{L}_x} k_t^{\tilde{L}}(x \xrightarrow{\gamma} y) \phi_1(y) |dy'| \right) |dx|.$$

From Section 4.2, for fixed x

$$\begin{aligned} \int_{\gamma \in \tilde{L}_x} k_t^{\tilde{L}}(x \xrightarrow{\gamma} y) \phi_1(y) |dy'| &= \int_{y \in L_x} \sum_{g \in G_x^x} k_t^{\tilde{L}}(\tilde{x}, \tilde{y} g) \phi_1(y) |dy| \\ &= \int_{y \in L_x} k_t^L(x, y) \phi_1(y) |dy| \end{aligned}$$

so that

$$\langle \phi_2 \otimes \phi_1, k_t \rangle = \int_{x \in M} \phi_2(x) \left(\int_{y \in L_x} k_t^L(x, y) \phi_1(y) |dy| \right) |dx|.$$

Analyzing k_∞ is thus equivalent to analyzing the functional

$$\phi_2 \otimes \phi_1 \longmapsto \lim_{t\to\infty} \int_M \phi_2(x) \left(\int_{y \in L_x} k_t^L(x, y) \phi_1(y) |dy| \right) |dx|.$$

This description of k_∞ is valid regardless of any assumptions about compactness of leaves.

If the leaf L is compact, then

$$\lim_{t\to\infty} \int k_t^L(x, y) \phi(y) |dy| = \lim_{t\to\infty} e^{-tD_L^2} \phi(x) |dx| = P_L \phi(x)$$

where P_L is now the projection on the leaf L, rather than on the holonomy cover $\tilde{L}$. If all the leaves are compact with bounded volumes, then by the dominated convergence theorem applied to M, the time infinity limit commutes with the integral to give

$$\langle \phi_2 \otimes \phi_1, k_\infty \rangle = \int_M \phi_2(x) P_{L_x} \phi_1(x) |dx|.$$

This is just a different description of the distribution described in the proposition above. Note that if all leaves are compact, but their volumes are not bounded [23,10], then this description of k_∞ is still valid, provided only that there is a bound on $\int_{L_x} k_t^L(x,y)\phi_1(y)|dy|$ that is independent of x. For if there is, the dominated convergence theorem can be applied.

What is $\lim_{t\to\infty} \mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2})$? Expressed as an integral over $\mathcal{G}^f$,

$$\mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2}) = \int_{\mathcal{G}^f} \frac{\mathrm{tr}^s k_t([f(x) \xrightarrow{\gamma} x])}{|\mathrm{Id} - (h_\gamma \circ f)_*|} |dx| \nu$$

where $k_t|dx|$ is the symbol of e^{-tD^2} on $\mathcal{G}$.

Proposition 5.0.6 *If $\mathcal{G}$ is compact, then*

$$\lim_{t\to\infty} \mathrm{Tr}^s_\nu(f^* \circ e^{-tD^2}) = \int_{\mathcal{G}^f} \frac{\mathrm{tr}^s k_P^{\tilde{L}}([f(x) \xrightarrow{\gamma} x])}{|\mathrm{Id} - (h_\gamma \circ f)_*|} |dx| \nu = \mathrm{Tr}^s_\nu(f^* \circ P)$$

where P is the operator of leafwise projection onto harmonic forms on the holonomy covers of the leaves. Thus the Lefschetz theorem for compact leaves can be written as

$$\int_{M^f} \frac{a(x)}{|\mathrm{Id} - f_*|} \nu = \mathrm{Tr}^s_\nu(f^* \circ P).$$

Proof. From the assumption that $\mathcal{G}$ is compact, it follows that $\mathcal{G}^f$ is also, since $\mathcal{G}^f$ is a pull-back of $\mathcal{G}$ to the compact space M. Then the dominated convergence theorem together with the pointwise bounds on the k_t kernels implies that the time infinity limit commutes with the integral. For any $[\gamma]$ in $\mathcal{G}^f$, $\lim_{t\to\infty} k_t([f(x) \xrightarrow{\gamma} x]) = k_P^{\tilde{L}}([f(x) \xrightarrow{\gamma} x])$. This gives the first equality. The second equality is just the definition of Tr^s_ν. ∎

Note that the operator P in this proposition is the same as the operator in Proposition 5.0.5.

To understand this integral over $\mathcal{G}^f$, we first analyze $\mathcal{G}^f$. The following analysis is valid for any foliation, not just for those with compact leaves. Over a fixed leaf L, $\mathcal{G}^f$ may have several components. (See the discussion after Definition 2.2.7). To describe $\mathcal{G}^f|_L$, pick an $x \in L$. The points of $\mathcal{G}^f$ over x are the elements of $G_x^{f(x)} = \{[\gamma] \in \mathcal{G} \mid s(\gamma) = f(x),\ r(\gamma) = x\}$. For each $\gamma \in G_x^{f(x)}$, flowing out from γ defines a map C_γ from $s^{-1}(x)$ to $\mathcal{G}^f|_L$. In other words, for any path α starting at x, $C_\gamma(\alpha) = f(\alpha)^{-1} * \gamma * \alpha$. This gives a map of covers over L:

$$\tilde{L} \cong s^{-1}(x) \xrightarrow{C_\gamma} \mathrm{im}(C_\gamma) \subset \mathcal{G}^f|_L.$$

Note that $\mathrm{im}(C_\gamma) = \{$ elements of $\mathcal{G}^f|_L$ obtained by flowing out from γ $\}$. If γ_1 and γ_2 are two elements of $G_x^{f(x)}$ that can be obtained from one another by flowing out, then $\mathrm{im}(C_{\gamma_1}) = \mathrm{im}(C_{\gamma_2})$; otherwise $\mathrm{im}(C_{\gamma_1})$ and $\mathrm{im}(C_{\gamma_2})$ are disjoint. Since any element of $\mathcal{G}^f|_L$ can be obtained by flowing out from some element of $G_x^{f(x)}$, $\mathcal{G}^f|_L$ is partitioned by the $\mathrm{im}(C_\gamma)$'s. Since $\mathrm{im}(C_{\gamma_1}) = \mathrm{im}(C_{\gamma_2})$ if and only if γ_1 and γ_2 are in the same orbit of the flowing out action of G_x^x on $G_x^{f(x)}$, we have shown the following proposition:

Proposition 5.0.7 *Over L, $\mathcal{G}^f$ is a disjoint union of covers of L, one for each orbit of $G_x^{f(x)}$ under the action of G_x^x. The cover corresponding to $\gamma \in G_x^{f(x)}$ is a quotient of $\tilde{L}$ by the isotropy group of γ, that is by $\{\alpha \in G_x^x \mid f(\alpha)^{-1} * \gamma * \alpha = \gamma\}$.*

On each component of $\mathcal{G}^f|_L$, the factor $|\mathrm{Id} - (h_\gamma \circ f)_*|$ is constant, but these may vary from component to component.

Now return to the case of compact leaves with finite holonomy. Over a particular point x in a fixed leaf L, the integrand in the proposition is

$$\sum_{\gamma \in G_x^{f(x)}} \frac{\mathrm{tr}^s k_P^{\tilde{L}}([\gamma])}{|\mathrm{Id} - (h_\gamma \circ f)_*|}.$$

If all the denominator factors were equal to a constant A, this would equal $\mathrm{tr}^s k_P^L(f(x), x)/A$ because $\sum_{\gamma \in G_y^x} k^{\tilde{L}}([\gamma]) = k^L(x, y)$ (Section 4.2). If they are not all equal, then this is a weighted sum, the weights taking into account the transverse behavior of f near γ.

Consider the contribution of a single fixed leaf L to this integral. The previous proposition says that over L, $\mathcal{G}^f$ consists of a disjoint union of covers intermediate between L and $\tilde{L}$. With the finite holonomy assumption, there are only finitely many of these. Pick representatives $\gamma_1, \ldots, \gamma_m$ in $G_x^{f(x)}$ representing the orbits of the G_x^x action. Let G_i be the isotropy group of γ_i and let n_i be the cardinality of the i^{th} orbit. Let $L_i = \mathrm{im}(C_{\gamma_i}) \subset \mathcal{G}^f|_L$.

Proposition 5.0.8 *With the above assumptions:*

$$\begin{aligned}\int_{\mathcal{G}^f|_L} \frac{\mathrm{tr}^s k_P^{\tilde{L}}([f(x) \xrightarrow{\gamma} x])}{|\mathrm{Id} - (h_\gamma \circ f)_*|} |dx| &= \sum_{i=1}^m \frac{1}{|\mathrm{Id} - (h_{\gamma_i} \circ f)_*|} \cdot \mathrm{tr}_{G_i}^s(f^* \circ P_{\tilde{L}} \ on\ \tilde{L}) \\ &= \mathrm{tr}^s(f^*|_{H^\bullet(\tilde{L})}) \sum_{i=1}^m \frac{n_i}{|G_x^x|} \frac{1}{|\mathrm{Id} - (h_{\gamma_i} \circ f)_*|}\end{aligned}$$

where $\mathrm{tr}_{G_i}^s$ *is the supertrace analogue of the* Γ*-trace defined by Atiyah [1] for the covering* $\tilde{L} \longrightarrow L_i$.

Proof. Using the decomposition from the previous proposition, it is clear that the first integral is equal to

$$\sum_{i=1}^m \frac{1}{|\mathrm{Id} - (h_\gamma \circ f)_*|} \cdot \int_{L_i} \mathrm{tr}^s k_P^{\tilde{L}}([f(x) \xrightarrow{\gamma} x])|dx|.$$

The integral over L_i resembles the integrals used for the Γ-trace in Atiyah's index theory for coverings [1]. To prove the first equality, we apply that theory to the covering $\tilde{L} \xrightarrow{C_{\gamma_i}} L_i$ and the operator $f^* \circ P_{\tilde{L}}$ on $\tilde{L}$. To do this, we need to show that the kernel k for this operator is invariant by G_i and that for any $[f(y) \xrightarrow{\beta} y]$ in L_i and any α in $C_{\gamma_i}^{-1}(\beta)$, $k(\alpha, \alpha) = k_P^{\tilde{L}}([f(y) \xrightarrow{\beta} y])$. Throughout this discussion, x is fixed and $\tilde{L}$ is identified with $s^{-1}(x)$. Note that with this identification, an extension of $f : L \to L$ to $\tilde{f} : \tilde{L} \to \tilde{L}$ can be written as

$$\tilde{f}([x \xrightarrow{\alpha} y]) = \gamma_i^{-1} * f(\alpha) = [x \xrightarrow{\gamma_i^{-1}} f(x) \xrightarrow{f(\alpha)} f(y)].$$

The kernel for $P_{\tilde{L}}$ on $\tilde{L} \times \tilde{L} \cong s^{-1}(x) \times s^{-1}(x)$ is given by $(\alpha, \beta) \mapsto k_P^{\tilde{L}}(\alpha^{-1} * \beta)$. Thus the kernel for $\tilde{f}^* \circ P_{\tilde{L}}$ is

$$k(\alpha, \beta) = k_P^{\tilde{L}}(\tilde{f}(\alpha)^{-1} * \beta) = k_P^{\tilde{L}}(f(\alpha)^{-1} * \gamma_i * \beta).$$

The covering group G_i acts on $\tilde{L}$ by $\gamma(\alpha) = \gamma * \alpha$ so that

$$k(\gamma\alpha, \gamma\beta) = k_P^{\tilde{L}}(f(\alpha)^{-1} * f(\gamma)^{-1} * \gamma_i * \gamma * \beta).$$

But since γ is in G_i, $f(\gamma)^{-1} * \gamma_i * \gamma = \gamma_i$ which shows that k is G_i invariant. Also, if $C_{\gamma_i}(\alpha) = \beta$, then $f(\alpha)^{-1} * \gamma_i * \alpha = \beta$ so that $k(\alpha, \alpha) = k_P^{\tilde{L}}(f(\alpha)^{-1} * \gamma_i * \alpha) = k_P^{\tilde{L}}(\beta)$.

For the second equality, simply note that since $\tilde{L} \longrightarrow L_i$ is a finite covering,

$$\mathrm{tr}_{G_i}(f^* \circ P_{\tilde{L}}) = \frac{1}{|G_i|} \cdot \mathrm{tr}^s(f^* \circ P_{\tilde{L}} \text{ on } \tilde{L}).$$

But $1/|G_i| = n_i/|G_x^x|$ and $P_{\tilde{L}}$ is projection onto the finite dimensional homology of $\tilde{L}$. ■

The formula in the proposition for the integral over L can be viewed as a version of the Selberg trace formula applied to the covering $\mathcal{G}^f \to L$ as discussed by McKean in [18, Section 3]. In McKean's treatment, the trace of a kernel on the base space of a covering is decomposed as a sum of traces on various coverings, the coverings being determined by the conjugacy classes of the covering group. We have the same situation here with, in addition, a transverse weighting factor.

CHAPTER 6

EXAMPLES

6.1. Fibrations

The first example we consider is a fibration $M \xrightarrow{\pi} B$ where M is foliated by the fibers $F \cong \pi^{-1}(b)$. In contrast to a general foliation, the quotient space of leaves in this case is a manifold, namely B. At any point x in M, the projection π identifies a transversal Σ through x with a subspace of B through $\pi(x)$, and the tangent map π_* identifies the normal space $N\mathcal{F}_x$ with the tangent space $TB_{\pi(x)}$. If x and y are in the same fiber so that $\pi(x) = \pi(y) = b$, then the holonomy from x to y is simply the identification $N\mathcal{F}_x \overset{\pi_*}{\cong} TB_b \overset{\pi_*}{\cong} N\mathcal{F}_y$. Thus a transverse form or density on M that is holonomy invariant is always the pull-back of a unique form or density on B.

If $f : M \to M$ is a Lefschetz morphism, the requirement that it take leaves to leaves means that f induces a map f on the base space B. The condition that f has dimension k transverse fixed set means that f fixes a k-dimensional submanifold B^f of B. The morphism f is transfixed of dimension k if, in addition to this, for every $b \in B^f$, $\mathrm{Id} - f_*$ is invertible on TB/TB^f. Finally, the definition of a Lefschetz morphism includes the condition that f restricted to a fiber over B^f has only simple fixed points. Note that Proposition 2.2.10 implies that the fixed set M^f is a k-dimensional submanifold transverse to the fibers.

An invariant transfixed k-density ν on M descends to a density on B^f. This section of $|TB^f|$ will also be denoted by ν. For any x in M^f, let $a(x)$ be the Atiyah-Bott index at x for f restricted to the fiber through x. Then the local part of the Lefschetz theorem for a fibration is

$$\int_{M^f} \frac{a(x)}{|\mathrm{Id} - f_*|(x)} \nu = \int_{B^f} \left(\sum_{x \in \pi^{-1}(b) \cap M^f} a(x) \right) \frac{\nu(b)}{|\mathrm{Id} - f_*|(b)}.$$

In other words, the time zero index is the local index for f on each fiber integrated over the space of fixed fibers with respect to the measure $\nu/|\mathrm{Id} - f_*|$.

For the global part, note that since M is assumed to be compact, the fiber F must be compact. Since all leaves are compact and the holonomy is trivial, the groupoid $\mathcal{G}$ is compact and the analysis of Chapter 5 applies. From Proposition 5.0.7, the global part of the Lefschetz theorem is

$$\int_{\mathcal{G}^f} \frac{\mathrm{tr}^s k_P^{\tilde{L}}([f(x) \xrightarrow{\gamma} x])}{|\mathrm{Id} - (h_\gamma \circ f)_*|} |dx|\nu = \mathrm{Tr}_\nu^s(f^* \circ P).$$

Since the holonomy is trivial, $\tilde{L} = L = \pi^{-1}(b)$, $\mathcal{G}^f \cong \pi^{-1}(B^f)$, and the denominator factor is a function of b alone. Thus the integral over $\mathcal{G}^f$ decomposes to become

$$\int_{b\in B^f} \left(\int_{x\in\pi^{-1}(b)} k_P^{\pi^{-1}(b)}(f(x), x)|dx| \right) \cdot \frac{\nu(b)}{|\mathrm{Id} - f_*|(b)}.$$

The integral over the compact fiber $\pi^{-1}(b)$ is equal to the Lefschetz number for f restricted to that fiber:

$$\int_{x\in\pi^{-1}(b)} k_P^{\pi^{-1}(b)}(f(x), x)|dx| = \mathrm{Lef}\ (f^*|_{\pi^{-1}(b)}).$$

Thus the Lefschetz theorem

$$\int_{M^f} \frac{a(x)}{|\mathrm{Id} - f_*|(x)} \nu = \mathrm{Tr}_\nu^s(f^* \circ P)$$

reduces for a fibration to

$$\int_{B^f} \left(\sum_{x\in\pi^{-1}(b)\cap M^f} a(x) \right) \frac{\nu(b)}{|\mathrm{Id} - f_*|(b)} = \int_{B^f} \left(\mathrm{Lef}\ (f^*|_{\pi^{-1}(b)}) \right) \frac{\nu(b)}{|\mathrm{Id} - f_*|(b)}.$$

This is just an integrated version of the Lefschetz formula for the fibers that are fixed.

6.2. Irrational Flow on the Torus

The next example we consider is the torus foliated by lines with irrational slope. Here $M = T^2 = R^2/Z^2$. Coordinates (x, y) on T^2 shall mean the equivalence class $\{(x+m, y+n) \mid (m, n) \in Z^2\}$ of coordinates on R^2. The foliation is generated by the unit vector field $X = \cos(\theta)\partial/\partial x + \sin(\theta)\partial/\partial y$ where $\lambda = \tan(\theta)$ is irrational. Each leaf is a line $y = \lambda x + C$ in R^2 projected to T^2, so that the leaves are parametrized by $C \in R^1/Z^2$ where the action of $(a, b) \in Z^2$ on $C \in R^1$ is given by $(a, b)C = C - \lambda a + b$. Every leaf is dense and since all leaves are simply connected, the holonomy is trivial.

Linear morphisms $(x, y) \mapsto (mx, ny)$ that preserve the foliation must have $m = n$. Consider any such morphism $f(x, y) = (nx, ny)$ where $n \neq 1$.

Proposition 6.2.1 *The morphism f is transfixed of dimension 0 with $(n-1)^2$ fixed leaves. Each fixed leaf has a single non-degenerate fixed point so that f is a Lefschetz morphism.*

Proof. The morphism f takes the leaf $y = \lambda x + C$ to $y = \lambda x + nC$. Thus the leaf corresponding to C is mapped to itself if and only if there is $(a, b) \in Z^2$ such that $(n-1)C = -\lambda a + b$. Any such C is equivalent to one of the $(n-1)^2$ values $C_{k,l} = (-\lambda k + l)/|n-1|$ where k and l run from 0 to $|n-1|-1$. Furthermore, these $C_{k,l}$ are all inequivalent under the Z^2 action. For if $C_{k,l} \sim C_{0,0}$, then there is $(a, b) \in Z^2$ with $(-\lambda k + l)/|n-1| = -\lambda a + b$ which implies that $\lambda(a - k/|n-1|) = (b - l/|n-1|)$. Irrationality of λ then implies that $a - k/|n-1|$ and $b - l/|n-1|$ are both 0. Since k and l are in $\{0, \ldots, |n-1|-1\}$, $k = l = 0$. This shows that f fixes $(n-1)^2$ leaves.

It is easy to check that f has $(n-1)^2$ fixed points, namely the points $(k, l)/|n-1|$ where k and l are as before, and that these correspond to the $(n-1)^2$ fixed leaves. At a fixed point (x, y), the tangent map dilates the tangent space $T_{(x,y)}M$ by n. Since $n \neq 1$, the dilation in the transverse direction shows that f is transfixed of dimension 0 at (x, y), and since the holonomy is trivial, this is true on the entire leaf through (x, y). The dilation in the leafwise direction shows that f is also a Lefschetz morphism. ■

Note that in this example, the space $\mathcal{G}^f$ is just $(n-1)^2$ copies of the real line, each immersing as a dense leaf in T^2. Since $k = 0$, the transfixed density ν is just a holonomy invariant function on the $(n-1)^2$ fixed leaves. If ν is defined on M and not just on $\mathcal{G}^f$, the only possibility is that ν is constant. We shall take $\nu = 1$.

Since M^f is a discrete set of $(n-1)^2$ points and since $|\det(\mathrm{Id}-f_*)| = |1-n|$, the local part of the Lefschetz formula is

$$\frac{1}{|1-n|}\sum_{(x,y)\in Z^2/|n-1|} a(x,y)$$

where $a(x,y)$ is the leafwise local index at (x,y). For example, if the complex is the leafwise deRham complex, then the local index is $\mathrm{sign}\,(\det(\mathrm{Id} - T_x(f|_L))) = \mathrm{sign}(1-n)$ at each fixed point. In this case, then, the local index is

$$(n-1)^2\mathrm{sign}(1-n)/|1-n| = 1-n.$$

For the global part, since ν is trivial, we can use Proposition 5.0.1. To apply this proposition, we analyze the limiting heat kernel on $T^2 \times T^2$. First introduce some notation. For the leafwise deRham complex, let s_1 be the constant function 1 and s_2 be the constant leafwise 1-form of unit length. Thus $s_2 = \cos(\theta)dx + \sin(\theta)dy$ is the 1-form dual to the vector field X that generates the foliation. Using the basis $[s_1\ s_2]$, the Dirac operator

$$D = \begin{bmatrix} 0 & -X \\ X & 0 \end{bmatrix}$$

so that

$$D^2 = \begin{bmatrix} -X^2 & 0 \\ 0 & -X^2 \end{bmatrix}$$

and $e^{-tD^2} = e^{tX^2}\cdot\mathrm{Id}$. Each leaf L is a line; on this line the operator X^2 corresponds to d^2/dx^2. The kernel for the operator e^{td^2/dx^2} on the line is

$$k_t(x,y) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}|dy|.$$

This shows that the symbol of k_t on $L\times L$ is $1/\sqrt{4\pi t}\cdot e^{-|x-y|^2/4t}|dy|\cdot\mathrm{Id}$.

For the irrational foliation on the torus, the holonomy groupoid $\mathcal{G}$ is isomorphic to the space $T^2\times R^1$ with $T^2\times R^1 \xrightarrow{s\times r} T^2\times T^2$ given by $(x,y,r)\mapsto (x,y,x+r\cos\theta, y+r\sin\theta)$. The preceding discussion shows that the symbol of k_t on $T^2\times R^1$ is $1/\sqrt{4\pi t}\cdot e^{-r^2/4t}|dr|\cdot\mathrm{Id}$. As an operator on $C^\infty(S)$, then, e^{-tD^2} is

$$\left(e^{-tD^2}\phi\right)(x,y) = \int_r \frac{1}{\sqrt{4\pi t}}e^{-r^2/4t}\phi(x+r\cos\theta, y+r\sin\theta)dr$$

and its distributional kernel k_t is characterized by

$$\langle \phi_1 \otimes \phi_2, k_t \rangle = \int_{T^2} \phi_1(x,y) \left(\int_r \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t} \phi_2(x + r\cos\theta, y + r\sin\theta) dr \right) dx\, dy$$

where $\phi_2 \in C^\infty(S)$ and $\phi_1\, dx\, dy \in C^\infty(S^* \otimes |TM|)$. (We shall henceforth ignore the $|TM|$ factor.) Let P be the operator of projection onto constants so that $P\phi$ is the constant section of S with value $\int_{T^2} \phi(x,y)\, dx\, dy$. The kernel of P is $k_P(x_1, y_1, x_2, y_2) = 1\, dx_2\, dy_2$.

Proposition 6.2.2 *As a distribution on $T^2 \times T^2$, $k_t \to k_P$ as $t \to \infty$.*

Proof. The bundles S and S^* shall be dropped for notational clarity. The topology on the space $C^{-\infty}(M \times M)$ of distributions is that of uniform convergence on bounded sets. Thus we must show that

$$\langle \phi_1 \otimes \phi_2, k_t \rangle \to \int_{T^2} \int_{T^2} \phi_1(x_1, y_1) \phi_2(x_2, y_2)\, dx_1\, dy_1\, dx_2\, dy_2$$

uniformly for ϕ_1 and ϕ_2 lying in a bounded set of C^∞. A bounded set B in C^∞ is characterized by bounds on $\sup_{\phi \in B} \sup_{x \in T^2} |D^\alpha \phi(x)|$ for each multi-index α.

To prove the convergence, work on the Fourier transform side. Any $\phi \in C^\infty(T^2)$ can be expressed as $\phi = \sum_{Z^2} a_{m,n} e^{2\pi i(mx+ny)}$ with the coefficients $a_{m,n}$ rapidly decreasing. The uniform bounds on the sup norms for ϕ in a bounded set become uniform bounds on the decay constants of the $a_{m,n}$'s. On the transform side, the operator e^{-tD^2} becomes a multiplication operator. That is, if $\phi_{m,n} = e^{2\pi i(mx+ny)}$, then

$$\begin{aligned}
e^{-tD^2} \phi_{m,n}(x,y) &= \int_{R^1} \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t} e^{2\pi i(mx + mr\cos\theta + ny + nr\sin\theta)} dr \\
&= e^{2\pi i(mx+ny)} \int_{R^1} \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t} e^{2\pi i(m\cos\theta + n\sin\theta) r} dr \\
&= e^{2\pi i(mx+ny)} e^{-4\pi^2 t(m\cos\theta + n\sin\theta)^2} \\
&= e^{-Ct(m+n\lambda)^2} \phi_{m,n}(x,y)
\end{aligned}$$

where C is the constant $4\pi^2/\cos\theta$. Thus

$$e^{-tD^2} : \sum a_{m,n} \phi_{m,n} \mapsto \sum a_{m,n} e^{-Ct(m+n\lambda)^2} \phi_{m,n}.$$

Since λ is irrational, $m + n\lambda$ is zero only if $(m, n) = (0, 0)$. Hence each Fourier coefficient except for $a_{0,0}$ converges to zero as $t \to \infty$.

If $\phi_1 = \sum a_{m,n}\phi_{m,n}$ and $\phi_2 = \sum b_{m,n}\phi_{m,n}$, then

$$\langle \phi_1 \otimes \phi_2, k_t \rangle = \langle \phi_1, e^{-tD^2}\phi_2 \rangle = \sum_{Z^2} a_{m,n} b_{-m,-n} e^{-Ct(m+n\lambda)^2}.$$

The uniform bounds on the decay of $a_{m,n}$ and $b_{m,n}$ for ϕ_1 and ϕ_2 in a bounded subset together with the fact that each factor $e^{-Ct(m+n\lambda)^2}$ with $(m, n) \neq (0, 0)$ is individually converging to zero, then implies that

$$\langle \phi_1 \otimes \phi_2, k_t \rangle \to a_{0,0} b_{0,0}$$

uniformly on the bounded set. But $a_{0,0}b_{0,0} = \int \phi_1(x_1, y_1)\, dx_1\, dy_1 \cdot \int \phi_2(x_2, y_2)\, dx_2\, dy_2$.

■

Although the pointwise limit of the symbol of k_t is zero on the groupoid $\mathcal{G}$, the limiting distribution on $M \times M$ is not zero. The distribution k_∞ is not a δ-section supported on $\mathcal{G}$, but is rather a smooth section on the closure of $\mathcal{G}$ in $M \times M$, namely the entire space.

To apply Proposition 5.0.1, we need to know that $k_t \to k_P$ in $C_\Gamma^{-\infty}$ where Γ is a closed cone containing $N^*\mathcal{G}$ and disjoint from $N^*(f \times \mathrm{id})$. The proof of this fact shall be omitted. The image of P is the two-dimensional space of constant sections $a_1 s_1 + a_2 s_2$. Proposition 5.0.1 then implies that the global part of the Lefschetz formula is the supertrace of f^* on this space. With f as before, f^* maps s_1 to itself and s_2 to ns_2 so that the global index is $1 - n$ in agreement with our calculation of the local index.

6.3. Reeb Foliation of the Torus

This example concerns the suspension of a diffeomorphism of S^1 with fixed points. Let $g : S^1 \to S^1$ be a diffeomorphism and let $M = (S^1 \times R^1)/Z^1$ where the action of n is given by $(\theta, r) \mapsto (g^n(\theta), r - n)$ so that $(g^n(\theta), r) \sim (\theta, r + n)$. Note that $M \cong T^2$. Foliate $S^1 \times R^1$ by the R^1 factors. Since the Z action takes leaves to leaves, this foliation descends to the quotient. Parametrize S^1 by $s \in [0,1]$ with $s \mapsto e^{2\pi i s}$. We shall assume that

1. $g : S^1 \to S^1$ has a finite number of fixed points $0 \leq a_1 < a_2 < \ldots < a_n < 1$, and

2. at each fixed point $g'(a_i) \neq 1$.

This then implies that n is even and that on alternate intervals (a_i, a_{i+1}) (a_{i+1}, a_{i+2}) g is increasing and decreasing. With no loss of generality we may take $a_1 = 0$ and g decreasing on the interval $(0, a_2)$. For each fixed point a_i, the leaf through $(a_i, 0)$ is a circle, but for any other θ, the leaf through $(\theta, 0)$ is a line. The compact leaves have holonomy with the holonomy once around the circle $t \mapsto (a_i, t)$, $0 \leq t \leq 1$ given by the diffeomorphism g near a_i and the linear holonomy by $g'(a_i)$. If θ is in an interval (a_{2k}, a_{2k+1}) of increase of g, then the leaf (θ, r) through $(\theta, 0)$ approaches the leaf through a_{2k+1} as $r \to \infty$. This follows from $(\theta, n) = (g^n(\theta), 0) \to (a_{2k+1}, 0)$ as $n \to \infty$. It approaches the leaf through a_{2k} as $r \to -\infty$, and a similar result holds for the intervals of decrease of g.

Give M a metric so that the induced metrics on the leaves are dr^2 and consider the leafwise deRham complex over M. Just as in the example of irrational flow on the torus (which is the suspension of the diffeomorphism $g(\theta) = \theta + \lambda$), the symbol of k_t on the groupoid $\mathcal{G}$ is:

$$k_t(\theta, y, r) = \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t} \cdot \mathrm{Id}\, dr.$$

As an operator on $C^\infty(S)$,

$$(e^{-tD^2}\phi)(\theta, y) = \int_r \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t} \phi(\theta, y + r)\, dr.$$

Despite the similarity with the irrational flow example, the time infinity limit in this case is different. In fact $\lim_{t\to\infty} e^{-tD^2}\phi$ is not, in general, smooth or even continuous.

Proposition 6.3.1 *As an operator from $C^\infty(S)$ to $C^{-\infty}(S)$, e^{-tD^2} converges to the operator*

$$P\phi(\theta, y) = \frac{1}{2} \sum_{i=1}^{n} \chi_{[a_{i-1}, a_{i+1}]}(\theta) \int_0^1 \phi(a_i, r)\, dr$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$.

Proof. We first show that, in agreement with the above formula, for fixed i and δ, $e^{-tD^2}\phi(\theta, y)$ converges uniformly to the constant value

$$\frac{1}{2}\left(\int_0^1 \phi(a_i, r)dr + \int_0^1 \phi(a_{i+1}, r)dr\right)$$

for (θ, y) in the subset $(a_i+\delta, a_{i+1}-\delta)\times[0,1]$ of $M \times M$. Without loss of generality, we shall assume i is even so that g is increasing on the given interval. First write

$$\begin{aligned}(e^{-tD^2}\phi)(\theta, y) &= \int_r \frac{1}{\sqrt{4\pi t}}e^{-r^2/4t}\phi(\theta, y+r)\,dr \\ &= \sum_{k\in Z}\int_0^1 \frac{1}{\sqrt{4\pi t}}e^{-(r+k)^2/4t}\phi(g^k(\theta), y+r)\,dr.\end{aligned}$$

The idea is that for large t, the terms coming from k near zero can be neglected while those from large positive k give the $1/2\int \phi(a_{i+1}, r)dr$ term and those from large negative k give the $1/2\int \phi(a_i, r)dr$ term. Given an ϵ, there is a δ' depending only on $\sup|\partial\phi/\partial\theta|$, such that if $|\alpha - a_i| < \delta'$ and $|\beta - a_{i+1}| < \delta'$, then

$$\left|\int_0^1 \phi(\alpha, y+r)dr - \int_0^1 \phi(a_i, y+r)dr\right| < \epsilon$$

and

$$\left|\int_0^1 \phi(\beta, y+r)dr - \int_0^1 \phi(a_{i+1}, y+r)dr\right| < \epsilon.$$

Then there is an N, depending only on δ and δ', such that for all θ in the interval $(a_i+\delta, a_{i+1}-\delta)$, $|g^k(\theta) - a_i| < \delta'$ if $k < -N$ and $|g^k(\theta) - a_{i+1}| < \delta'$ if $k > N$. There is a T, depending only on N and $\sup|\phi|$, such that for $t \geq T$,

$$\begin{aligned}\left|\int_{-N}^N \frac{1}{\sqrt{4\pi t}}e^{-r^2/4t}\phi(\theta, y+r)\,dr\right| &< \epsilon \text{ and} \\ \left|\int_N^\infty \frac{1}{\sqrt{4\pi t}}e^{-r^2/4t}\,dr - \frac{1}{2}\right| &< \epsilon/\sup|\phi|\end{aligned}$$

Combining these estimates gives

$$\left|e^{-tD^2}\phi(\theta, y) - \frac{1}{2}\left(\int_0^1 \phi(a_i, y+r)\,dr + \int_0^1 \phi(a_{i+1}, y+r)\,dr\right)\right| < 5\epsilon$$

for all $t \geq T$. Since $\int_0^1 \phi(a_i, y+r)\,dr = \int_0^1 \phi(a_i, r)\,dr$, we get the desired result. The size of T depends only on ϵ, δ, $\sup|\phi|$, and $\sup|\partial\phi/\partial\theta|$.

Since we can do this for each i, we get uniform convergence of $e^{-tD^2}\phi$ to $P\phi$ for $(\theta, y) \in \cup_i(a_i + \delta, a_{i+1} - \delta) \times [0,1]$ and ϕ in a bounded set of $C^\infty(S)$. Furthermore, since on the deleted intervals $[a_i - \delta, a_i + \delta] \times [0,1]$, $|e^{-tD^2}\phi(\theta, y)|$ is clearly bounded by $\sup|\phi|$ for all t, and since these deleted sets have arbitrarily small measure, $e^{-tD^2}\phi$ converges in L^1 to $P\phi$ uniformly on bounded sets of $C^\infty(S)$. Since $L^1(T^2)$ includes continuously in $C^{-\infty}(T^2)$, we are done. ■

The image section $P\phi$ is constant on the regions $a_i < \theta < a_{i+1}$ but has jump discontinuities at the boundaries. The kernel for $\lim_{t\to\infty} e^{-tD^2}$ in this case can be written as:

$$k_P(\theta_1, r_1, \theta_2, r_2) = \frac{1}{2}\sum_i \chi_{[a_{i-1}, a_{i+1}]}(\theta_1)\delta(\theta_2 - a_i)\, d\theta_2\, dr_2.$$

We now consider morphisms of the Reeb torus that are transfixed of dimension 0. If f is a smooth map on S^1 with the property that $f \circ g = g^2 \circ f$, then we define a map on $S^1 \times R^1$ by $(\theta, r) \mapsto (f(\theta), 2r)$. This map descends to an endomorphism of the foliated torus because

$$(g^n(\theta), r - n) \mapsto (fg^n(\theta), 2r - 2n) \sim (g^{-2n} f g^n(\theta), 2r) = (f(\theta), 2r).$$

The induced map on T^2 will also be denoted by f.

From the property $f \circ g = g^2 \circ f$, it is clear that $f(a_i) = a_j$ for each fixed point a_i of g. One can actually show that each interval $[a_i, a_{i+1}]$ is either mapped to a point by f or is mapped onto another interval $[a_j, a_{j+1}]$; however we omit the proof. In order to get a nonzero Lefschetz index, we shall assume that $f(a_i) = a_i$ and each interval is mapped to itself.

Lemma 6.3.2 *If g is a diffeomorphism as before and $f : S^1 \to S^1$ is a smooth map such that $f \circ g = g^2 \circ f$ and $f(a_i) = a_i$, then $f'(a_i) = 0$ for every i and f has at least one fixed point on every open interval (a_i, a_{i+1}).*

Proof. Differentiating $f \circ g = g^2 \circ f$ at a_i gives $f'(a_i)g'(a_i) = g'(a_i)^2 f'(a_i)$. But $g'(a_i) \neq 0$ because g is a diffeomorphism and $g'(a_i) \neq 1$ by assumption. The only remaining possibility is $f'(a_i) = 0$. This shows that $f(x) - x$ is negative for x slightly larger than a_i. The fact that $f'(a_{i+1}) = 0$ implies that it is positive for x slightly smaller than a_{i+1}. Hence it has a root in the open interval as claimed. ∎

For simplicity we shall assume that in each interval (a_i, a_{i+1}), f has exactly one fixed point b_i.

Proposition 6.3.3 *The morphism f of M has $2n$ fixed leaves, n compact ones through the fixed points $(a_i, 0)$ and n noncompact ones through the fixed points $(b_i, 0)$. These are the only fixed points. If $f'(b_i) \neq 1$ for any i, then f is a Lefschetz morphism of dimension 0. If $\nu = 1$, then the local index is $\sum_i(-1 + -1/|1 - f'(b_i)|)$.*

Proof. Any point in M can be represented uniquely as (θ, r) with $\theta \in S^1$ and $0 \leq r < 1$. If (θ, r) is fixed by f, then $(\theta, r) \sim f(\theta, r) = (f(\theta), 2r)$. But if $0 \leq r < 1$, this implies that $r = 0$ and hence that $f(\theta) = \theta$. But by our assumptions, the only fixed points of f on S^1 are the a_i and b_i. Hence $M^f = \{(a_i, 0), (b_i, 0)\}$ as claimed.

Note that these $2n$ points are all on distinct leaves. Are there any other fixed leaves? If $f(\theta, r)$ is on the same leaf as (θ, r), then $g^n f(\theta) = \theta$ for some n. Let $\alpha = g^n(\theta)$. Then $f(\alpha) = fg^n(\theta) = g^{2n} f(\theta) = g^n(\theta) = \alpha$ so that α is one of the a_i or b_i. Hence (θ, r) is on one of the leaves through the fixed points.

Since the set of fixed leaves is discrete, f has dimension 0 transverse fixed set. The transfixed condition at the point $(b_i, 0)$ is that the transverse derivative should not be 1. But this is precisely the condition that $f'(b_i) \neq 1$. Since the leaf through $(b_i, 0)$ has trivial holonomy, the condition is satisfied everywhere on the leaf. For the fixed points $(a_i, 0)$, we need to consider not only f, but also the composition of f with the holonomy maps g^n. But since $f'(a_i) = 0$, we also have $(g^n \circ f)'(a_i) = 0$ for any n and the transfixed condition is satisfied.

Along any of the fixed leaves, f is a dilation by a factor of 2 so that f is a Lefschetz morphism. This also shows that the leafwise local index of f on the deRham complex at any of the fixed points is $1 - 2 = -1$. With $\nu = 1$, then, the foliation local index is the sum of the leafwise local indices divided by the factors $|1 - f'|$ at the $2n$ fixed points. ■

We now study the time infinity limit. Despite the fact that $k = 0$ and ν is trivial, the time infinity limit does not commute with Tr^s in this case. One can see this directly by computing $\pi_* \mathrm{tr}^s (f \times \mathrm{id})^* k_P$ where k_P is the kernel computed earlier.

$$(f \times \mathrm{id})^* k_P = \frac{1}{2} \sum_i \chi_{[a_{i-1}, a_{i+1}]}(f(\theta_2)) \delta(\theta_2 - a_i) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \, d\theta_2 \, dr_2$$

Taking the fiberwise supertrace replaces the matrix with a factor of -1 and integrating over M then gives:

$$\frac{1}{2} \sum_i (-1) \chi_{[a_{i-1}, a_{i+1}]}(f(a_i)) = -\frac{n}{2}$$

which does not agree with the local trace.

The reason that Proposition 5.0.1 does not apply to this example is that $N^*(f \times \mathrm{id})$ is not disjoint from the closure of $N^*\mathcal{G}$. Being transfixed of dimension zero guarantees that $N^*(f \times \mathrm{id})$ and $N^*\mathcal{G}$ are disjoint, but $N^*\mathcal{G}$ is not necessarily closed. In this example, $N^*\mathcal{G}$ includes the cotangent vectors $(\xi, 0, -(g^n)^*\xi, 0)$ in $T^*_{(a_i,0),(a_i,0)}(M \times M)$. As $n \to \infty$, $(g^n)^*(\xi) \to 0$ since $g'(a_i) < 1$ and thus $(\xi, 0, 0, 0)$ is in $\overline{N^*\mathcal{G}}$. But since $f'(a_i) = 0$, $(\xi, 0, 0, 0)$ is also in $N^*(f \times \mathrm{id})$.

To understand the time infinity limit in this case, then, we must study $\lim_{t \to \infty} (f \times \mathrm{id})^* k_t$ directly. The distribution $(f \times \mathrm{id})^* k_t$ is a δ-section supported on $\mathcal{G}^f$. In this example, $\mathcal{G}^f$ is a disjoint union of $2n$ lines immersing in M. If L is one of the non-compact leaves, the fact that $\mathcal{G}^f|_L \cong L$ follows immediately from the fact that L has no holonomy. For the circle leaves through the $(a_i, 0)$, use Proposition 5.0.7. Since f wraps the circle $t \mapsto (a_i, t)$, $0 \le t \le 1$ around itself twice, the flowing out action of $g \in G_x^x$ takes a $\gamma \in G_x^{f(x)}$ to $g^{-2} * \gamma * g = g^{-1} * \gamma$. Since this action is transitive on $G_x^{f(x)}$, $\mathcal{G}^f|_L$ is isomorphic to the holonomy cover $\tilde{L}$ which is a line.

Parametrizing $\mathcal{G}^f$ over L by (a_i, r), we find that the symbol of the pulled back distribution $(f \times \mathrm{id})^* k_t$ is:

$$(f \times \mathrm{id})^* k_t(a_i, r) = \frac{1}{\sqrt{4\pi t}} e^{-(2r-r)^2/4t} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{|1-0|}\, dr.$$

Since the transverse factor $1/|1-0|$ is the same for every point in $\mathcal{G}^f$ lying over a point x in L, this distribution is the same as the δ-section supported on the compact submanifold $L \to M$ with symbol $k_t^L(f(x), x)$. Thus:

$$\lim_{t\to\infty} \pi_* \mathrm{tr}^s (f \times \mathrm{id})^* k_t|_L = \lim_{t\to\infty} \int_L \mathrm{tr}^s k_t^L(f(x), x) = \mathrm{tr}^s(f^*|_{H^*(L)})$$

which in this example is -1.

For the non-compact leaves through $(b_i, 0)$, f maps (b_i, r) to $(b_i, 2r)$. The symbol of the pulled back distribution $(f \times \mathrm{id})^*$ at (b_i, r) is then:

$$\frac{1}{\sqrt{4\pi t}} e^{-(2r-r)^2/4t} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{|1 - f'(b_i)|}\, dr.$$

As above, this is the same as the δ-section supported on the (immersed) submanifold $L \to M$ with symbol $k_t^L(f(x), x) \cdot 1/|1 - f'(b_i)|$. But since L is now the line instead of the circle, the time infinity limit of this cannot be considered as the trace of f^* on the L^2-harmonic sections over L. As a distribution on M, the total mass $-1/|1 - f'(b_i)|$ of $\mathrm{tr}^s(f \times \mathrm{id})^* k_t$ concentrates half on the limiting circle through $(a_i, 0)$ and half on the limiting circle through $(a_{i+1}, 0)$ as $t \to \infty$. Thus $\lim_{t\to\infty} \mathrm{tr}^s(f \times \mathrm{id})^*$ is the distribution:

$$\sum_i \delta(\theta - a_i) \left(-1 + \frac{1}{2} \frac{-1}{|1 - f'(b_i)|} + \frac{1}{2} \frac{-1}{|1 - f'(b_{i-1})|} \right) d\theta\, dr.$$

In agreement with the irrational flow examples, then, the time infinity limit of Tr_ν for the noncompact fixed leaves is related to the behavior of the ends of the leaf. In the irrational flow case, an end of one of the lines "spreads out equally" over the torus in the sense that the percentage of time it is in a given neighborhood is proportional to the measure of that neighborhood. In the Reeb case, an end approaches one or the other of the bounding compact leaves.

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