Arithmetic

TIMS Tutor

SECTION 9

Outline

This tutor is organized as follows:

- Introduction
- Numeration and Place Value
- Basic Number Facts
- Concepts of Whole-Number Operations
- Whole-Number Computation in *MATH TRAILBLAZERS*
- Fractions and Decimals
- Conclusion
- References

Introduction

The last 40 years have seen many changes in elementary school mathematics. For many people, "mathematics" is synonymous with "arithmetic." But today, while reasonable people can still debate the proper content of an elementary mathematics curriculum, the pre-eminence of arithmetic in that curriculum has faded, both because of the availability of calculators and because of the realization that a curriculum focused on rote arithmetic will not meet the needs of students who will graduate from high school after the year 2010. Much more than just arithmetic is now expected—geometry, probability, statistics, measurement, graphing, even algebra. Technology is dramatically changing the world, making it hard to imagine what our children will need to know in 20 or 40 more years, but also ensuring that many of the skills of 40 years ago will be obsolete. Despite recent advances in psychology, educational research, and curriculum design, we are far from resolving all the uncertainties of the elementary school mathematics curriculum. A few things, however, do seem to be clear:

- The school mathematics curriculum can and must be more rigorous.
- Arithmetic is only one piece, albeit an important one, of a broad mathematics curriculum.
- We need to do better at helping children connect marks on paper and the real world. Too many children and adults fail to use common sense when they are dealing with mathematical symbolism. Discussing mathematics and integrating subject matter may help students make these connections.
- As we correct for the overemphasis on skills in the traditional mathematics curriculum, we should avoid over-correcting. Problem solving requires both procedural skill and conceptual understanding. As William Brownell (1956) noted during a previous period of reform, "In objecting to the emphasis on drill prevalent not so long ago, we may have failed to point out that practice for proficiency in skills has its place too." In *MATH TRAILBLAZERS,* we consistently seek a balance between conceptual understanding and procedural skill.

Two short sections of the *Curriculum and Evaluation Standards for School Mathematics* summarize recommendations for changes in the content and emphases of K–8 mathematics instruction (National Council of Teachers of Mathematics, 1989, pages 20–21 and 70–71). Most of the topics and methods that are to receive "decreased attention" are part of the traditional arithmetic curriculum: complex paper-and-pencil computation, long division, written practice, and so on. While it is important to keep in mind that "decreased attention" does not mean "eliminate," one might ask—and many parents do ask—how advisable *any* shift in emphasis is, especially since American students perform fairly well on arithmetic computation compared to students in other countries, in marked contrast with their performance on higher-level skills (McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, and Cooney, 1987).

Several arguments can be made in support of the National Council of Teachers of Mathematics (NCTM) position. One is that the computational proficiency of American students comes at too high a cost. The hundreds of hours devoted to arithmetic computation in elementary school leave too little time for other important topics. The traditional rote approach to computation undermines higher-level thinking: children learn that mathematics is blindly following rules, not thinking. Worst of all, the long hours of computational drudgery teach children that mathematics is a most unpleasant business.

Another argument for shifting away from the traditional arithmetic curriculum is that technology has made paper-and-pencil calculation, if not obsolete, then at least much less important. More practical topics—probability and statistics, geometry, measurement, mental computation and estimation—deserve more attention.

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A final and most important argument in favor of the changes recommended by the NCTM is that new approaches to instruction in arithmetic are more effective in helping students learn both appropriate calculation skills and how to apply those skills in solving problems. These new approaches build on students' own knowledge and intuitive methods and engage their common sense. This more meaningful approach helps students to be efficient and flexible in their computation and can reduce, though not eliminate, the amount of practice required. Thus, a greater emphasis on conceptual understanding can lead to better procedural skills and problem-solving abilities (Brown & Burton, 1978; Skemp, 1978; Hiebert, 1984; Van Lehn, 1986; Carpenter, 1986; Baroody and Ginsburg, 1986; Silver, 1986; Good, Mulryan, and McCaslin, 1992).

Numeration and Place Value

Even as the overriding emphasis in elementary mathematics shifts away from the traditional arithmetic topics, an understanding of our number system is as important as ever. This section sketches how *Math Trailblazers* helps children build this understanding.

In kindergarten and grade 1, students using *Math Trailblazers* practice their counting skills. They learn to count past 100 by 1s, 2s, 5s, and 10s. They count forward and backward from any given number. They group objects for counting. Students use counting to solve addition and subtraction problems. They learn to write numbers up to and beyond 100. The 100 chart is introduced and used for a variety of purposes, including solving problems and studying patterns. Students partition, or break apart, numbers in several ways (25 = 20 + 5, 25 = 10 + 10 + 5, and so on). These activities help children become familiar with the structure of the number system. Beginning in kindergarten, a ten frame is frequently used as a visual organizer. (See Figure 1.)

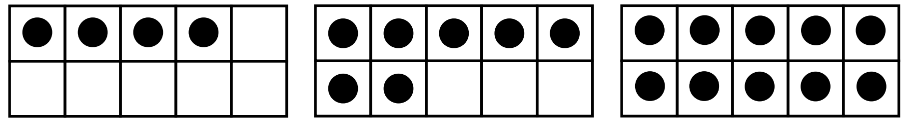

Figure 1: Sample ten frames

While formal study of place value is thus not a focus in kindergarten or grade 1, the multiple grouping and counting experiences described above provide children with a foundation for building an intuitive sense for the meaning of place value.

Work with counting continues in grade 2, especially skip counting and counting backwards. Place value is explored in the context of counting and grouping by tens. Children continue to count and group everyday objects such as buttons or peanuts. Connecting cubes are used to represent these objects and are often grouped by tens. Later, base-ten pieces are used to represent these same objects, thus linking the base-ten representations with quantities of actual objects.

Numbers well into the hundreds are explored. Counting is still used for problem solving, but more elaborate procedures may be employed. For example, a student may solve 885 – 255 by counting up, first by hundreds (355, 455, 555, 655, 755, 855) and then by tens (865, 875, 885), yielding 630 as the answer. Counting strategies for solving subtraction math facts are practiced.

More elaborate partitions of numbers are also investigated in grade 2. Particularly important are partitions in which every part is a single digit times 1, 10, 100, or 1000: 359 = 300 + 50 + 9 and so on. Attention is also given to partitioning numbers in more than one way:

$$
\begin{aligned}
359 &= 300 + 50 + 9 \\
&= 200 + 150 + 9 \\
&= 100 + 250 + 9
\end{aligned}
$$

This work with multiple partitioning is closely related to multidigit addition and subtraction. One way to think about addition and subtraction—indeed, one way to think about much of elementary mathematics—is as procedures for renaming numbers in more convenient forms. For example, 563 + 13 is a number that we usually rename as 576. 875 ÷ 25 is another number, renamable as 35 when it suits our purposes.

In grade 3, more formal study of place value takes place. Students work in varied contexts that involve numbers through the thousands. For example, students draw an outline of a coat to determine how many square centimeters of material would be required to make the coat. The use of contexts such as that example encourages students to associate quantities of actual objects with representations of the quantities using base-ten pieces and with the numerals that represent them. Most of this work is closely connected with investigations of addition and subtraction of multidigit numbers. Indeed, a good reason for studying computational algorithms is that they provide a context for learning about place value.

In grade 4, big numbers up to the millions are studied and used in various contexts. Continued efforts are made to connect the numbers with actual quantities. For example, the class creates its own base-ten pieces for large numbers—pieces we call super skinnies (10,000), super flats (100,000) and megabits (1,000,000). (See Figure 2.)

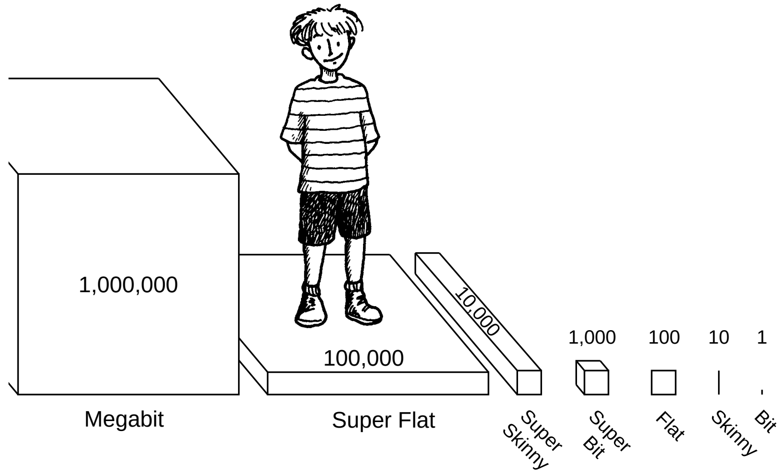

Figure 2: Base-ten pieces for small and large numbers

Once the basics of the base-ten place value system for whole numbers have been reviewed and established, the system is extended to include decimals. Decimals have been studied since grade 1, but as an alternative notation for writing certain fractions rather than as an extension of the place value system. In grade 4, the ten-for-one trading rules and other aspects of our number system are extended to the right of the decimal point for the first time.

In grade 5, the number system is extended to very large numbers. The story of Archimedes' attempt to calculate the number of grains of sand that would fill the observed universe introduces students to numbers far beyond what we encounter in everyday life. With the use of scientific calculators in grade 5, scientific notation for large numbers is needed and is introduced. Students also consider ideas of infinity, which open up new vistas of mathematical thought.

Basic Number Facts

The *MATH TRAILBLAZERS* approach to the basic facts differs from that in traditional textbooks. We seek a careful balance between strategies and drill, an approach based on research and advocated by the NCTM in the *Curriculum and Evaluation Standards for School Mathematics* (1989). Our approach is characterized by these elements:

- ***Early emphasis on problem solving.*** Students first approach the basic facts as problems to be solved rather than as facts to be memorized. They invent their own strategies to solve these problems or learn appropriate strategies from others through class discussion. Children's natural strategies, especially counting strategies, are explicitly encouraged.
- ***De-emphasis of rote work.*** We believe that children must indeed learn their math facts, but we de-emphasize rote memorization and the frequent administration of timed tests. Both of these can produce undesirable results. Instead, our primary goal is that students learn that they can find answers using strategies they understand.
- ***Ongoing practice.*** Work on the math facts is distributed throughout the curriculum, especially in the *Daily Practice and Problems* and in the games. This practice for facility, however, takes place only after students have a conceptual understanding of the operations and have achieved proficiency with strategies for solving basic fact problems. Delaying

practice in this way means that less practice is required for facility with the number facts.

- ***Gradual and systematic introduction of facts.*** Students study the facts in small groups that can be solved by a single strategy. Early on, for example, they study facts that can be solved by counting on 1, 2, or 3. Students first work on simple strategies for easy facts, and then progress to more sophisticated strategies and harder facts.
- ***Appropriate assessment.*** Students are assessed on the facts through teacher observation as well as through the appropriate use of written tests and quizzes.
- ***Facts are not gatekeepers.*** Students are not prevented from learning more complex mathematics because they do not perform well on fact tests.

The *MATH TRAILBLAZERS* approach to the math facts is discussed more fully in the TIMS Tutor: *Math Facts.*

Concepts of Whole-Number Operations

Concepts and Skills

Over the past 150 years, numerous attempts have been made to teach mathematics meaningfully rather than by rote. Unfortunately, as Lauren Resnick points out in *Syntax and Semantics in Learning to Subtract,* "… the conceptual teaching methods of the past were inadequate to the extent that they taught concepts *instead of* procedures and left it entirely to students to discover how computational procedures could be derived from the basic structure of the number and numeration system." (1987, p. 136).

Many educators have long recognized, however, that there is no real conflict between skills and concepts. (Whitehead, 1929; Dewey, 1938; Brownell, 1956; May, 1995) New conceptual understandings are built on existing skills and concepts; these new understandings in turn support the further development of skills and concepts. Thomas Carpenter describes the relationship in this way: "… It is an iterative process. Procedures are taught that can be supported by existing conceptual knowledge, and the conceptual knowledge base is extended to provide a basis for developing more advanced concepts. At every point during instruction, procedures are taught that can be connected to existing conceptual knowledge." (1986, p. 130) This integration of concepts and skills underlies our work with arithmetic in *MATH TRAILBLAZERS.*

Subtraction in Grades K to 4

To illustrate how concepts and skills are balanced in *MATH TRAILBLAZERS,* we outline in this section how one operation—subtraction—is developed with whole numbers in grades K–4. Though the details may differ for the other operations, the same general approach described here with subtraction applies to all four arithmetic operations.

In kindergarten and grade 1 of *MATH TRAILBLAZERS,* students solve a variety of problems involving subtraction of numbers up to and even beyond 100. These problems are based on hands-on classroom activities and realistic situations from children's experiences. A wide variety of subtraction problem types is represented, including take-away, comparison, part-whole, and missing addend problems. (See the TIMS Tutor: *Word Problems* for a discussion of these and other problem types.)

Students in even the earliest grades are thus faced with problems for which they have no ready solution methods, problems that in the traditional view are beyond their ability. They do, however, have much prior knowledge that is relevant. They have their common sense—their conceptual knowledge of the problem situations. They know, for example, that if Grace begins with 50 troll dolls and loses 17, then she must end with fewer than 50 troll dolls. Young students also have considerable procedural knowledge of our number system, including especially their skills at various kinds of counting: ordinary counting, counting on, counting back, and skip counting by 2s, 5s, and 10s. In addition, the students have tools they can use in solving the problems. These might include connecting cubes and links in kindergarten; cubes, links, 100 charts, ten frames, other manipulatives, paper and pencil, calculators, number lines, and so on in first grade.

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As the students apply their resources to solve the problems, they build their conceptual and procedural understanding of subtraction. They devise methods for solving the problems; they make records of their work; they discuss their methods with their teacher and classmates.

On one side, then, are hard problems involving many varieties of subtraction. On the other side are the kindergartners and first-graders, with their common sense, their counting skills, and various tools. As the students apply their resources to solve the problems, they build their conceptual and procedural understanding of subtraction. They devise methods for solving the problems; they make records of their work; they discuss their methods with their teacher and classmates. Their new knowledge about subtraction is closely linked to their prior knowledge, especially their out-of-school knowledge and their counting skills. (Baroody and Ginsburg, 1986)

The teacher comments on students' methods and may show students how to use conventional symbols to describe their work, but he or she makes no attempt to standardize students' methods. Any method that yields a correct result is acceptable—as long as it makes sense. The goal is to encourage students to apply their prior knowledge to problems they encounter and to let students know that their intuitive methods are valid.

In addition, first grade focuses on various strategies that can be used to solve single-digit subtraction problems. For example, a problem like 9 – 3 can be solved by counting back 3 from 9: 8, 7, 6. This work aims not at achieving quick facility with the subtraction facts, but rather at building conceptual understanding of subtraction and procedural skill with various strategies. (See the TIMS Tutor: *Math Facts.)*

In the beginning of grade 2, the problem-solving approach to subtraction continues. Problems with numbers up to 1000 are introduced, but again no standard solution method is taught. Students devise their own ways to solve the problems, drawing on their prior knowledge of the problem situations and the number system, and share their thinking with the class.

The strategies approach to the subtraction facts also continues in grade 2. As students' facility with the addition facts and simple subtraction facts increases, more sophisticated strategies become feasible. For example, a child may solve 14 – 6 by reasoning that "to take away 6 from 14, I first can take away 4, which leaves 10. Then I take away 2 more, which equals 8." These new strategies, sometimes called derived fact strategies, illustrate how new knowledge builds on prior skills.

Later in grade 2, systematic work begins on paper-and-pencil methods for subtracting two digit numbers. Students are asked to solve two-digit subtraction problems using their own methods and to record their solutions on paper. The class examines and discusses the various procedures that students devise. At this time, if no student introduces a standard subtraction algorithm, then the teacher does so, explaining that it is a subtraction method that many people use. The standard method is examined and discussed, just as the invented methods were. Students who do not have an effective method of their own are urged to adopt the standard method.

Problems that require borrowing are included from the beginning. Though this differs markedly from traditional approaches, we view it as important in developing a sound conception of subtraction algorithms. Giving children only multidigit problems that do not involve borrowing encourages the development of a rote and faulty algorithm that may not carry over into problems that require borrowing.

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By the beginning of grade 3, students have a strong conceptual understanding of subtraction and significant experience devising procedures to solve subtraction problems with numbers up to 1000. They also have some experience with standard and invented paper-and-pencil algorithms for solving two-digit subtraction problems. In grade 3, this prior knowledge is extended in a systematic examination of paper-and-pencil methods for multidigit subtraction.

This work begins with a series of multidigit subtraction problems that students solve in various ways. Many of these problems are set in a whimsical context, the TIMS Candy Company, a business that uses base-ten pieces to keep track of its production and sales. Other problems are based on student-collected data, such as a reading survey.

As in grade 2, the class discusses and compares the several methods students use to solve these problems. Again, any method that yields correct results is acceptable, but now a greater emphasis is given to methods that are efficient and compact. This work leads to a close examination of one particular subtraction algorithm. (See Figure 3.) Students solve several problems with base-ten pieces and with this standard algorithm, making connections between actions with the manipulatives and steps in the algorithm. After a thorough analysis of the algorithm, including a comparison of the standard algorithm and other methods, students are given opportunities to practice the algorithm. Practice in paper-and-pencil methods for multidigit subtraction is distributed throughout grades 3 and 4.

$$\begin{array}{rccc} & & 3 & 16 \\ & 7 & \not{4} & \not{6} \\ - & 4 & 3 & 9 \\ \hline & 3 & 0 & 7 \end{array}$$

Figure 3: A standard subtraction algorithm

Students are also expected to achieve facility with basic subtraction facts in grade 3. Since by the end of grade 2, students have worked for two years on strategies for simple subtraction problems, and since by that time they should also have facility with the addition facts, the expectation of facility with subtraction facts by the end of grade 3 is reasonable. To make this expectation a reality for all students, a systematic program of assessment, review, and practice of basic subtraction facts is built into the *Daily Practice and Problems* for grade 3.

By the beginning of grade 4, basic work with whole number subtraction is complete. Students have a firm conceptual understanding of subtraction and they have a diverse repertoire of methods that they can use to solve subtraction problems. They also know their subtraction facts. Work in grade 4 is designed to maintain and extend these skills and understandings. Facility with

Grade 5

In grade 5 students continue to use subtraction in activities and labs. As in earlier grades, they are encouraged to decide when it is appropriate to use paper and pencil, calculators, or estimation. A review of subtraction modeled with base-ten pieces is included for students who have not used MATH TRAILBLAZERS *in previous grades. Distributed practice is provided in the* Daily Practice and Problems *and the* Home Practice. *Facility with the subtraction facts is assessed in the first unit, and remediation is provided in the* Addition and Subtraction Math Fact Review *in the Unit Resource Guide File.*

the subtraction facts is verified and remediation is provided for students who need it. In the laboratory experiments and other work, students solve a wide variety of subtraction problems using methods of their own choosing. The *Daily Practice and Problems* and *Daily Home Practice* include distributed practice in paper-and-pencil subtraction so that those skills do not deteriorate.

Whole-Number Computation in *MATH TRAILBLAZERS*

We seek a balance between conceptual understanding and procedural skill. For all operations, standard methods for solving problems are not introduced until students have developed good conceptual and procedural understandings—the too-early introduction of such procedures may short-circuit students' common sense, encouraging mechanical and uncritical behavior.

Many other topics in *MATH TRAILBLAZERS*—addition, multiplication, division, fractions, and decimals—are treated in ways similar to that sketched for subtraction above. *In all these areas, we seek a balance between conceptual understanding and procedural skill.* For all operations, standard methods for solving problems are not introduced until students have developed good conceptual and procedural understandings—the too-early introduction of such procedures may short-circuit students' common sense, encouraging mechanical and uncritical behavior. (Brownell & Chazall, 1935; Resnick & Omanson, 1987; Rathmell & Huinker, 1989; Perry, 1991)

Grade	Addition	Subtraction	Multiplication	Division
K	• concepts of the operation	• concepts of the operation	• concepts of the operation	• concepts of the operation
1	• concepts of the operation • informal methods	• concepts of the operation • informal methods	• concepts of the operation	• concepts of the operation
2	• concepts of the operation • invented algorithms • standard methods for small numbers	• concepts of the operation • invented algorithms • standard methods for small numbers	• concepts of the operation • informal methods	• concepts of the operation • informal methods • strategies
3	• invented algorithms • standard methods for larger numbers	• invented algorithms • standard methods for larger numbers	• concepts of the operation • invented algorithms • standard methods for larger numbers	• concepts of the operation
4	• review, practice, apply, and extend	• review, practice, apply, and extend	• invented algorithms • standard methods for larger numbers	• invented algorithms • standard methods for larger numbers
5	• review, practice, apply, and extend	• review, practice, apply, and extend	• review, practice, apply, and extend	• standard methods for larger numbers

Table 1: Whole-number operations scope and sequence

In *Math Trailblazers*, instruction in standard procedures is delayed slightly beyond the traditional time, but problems that would normally be solved by standard procedures are often introduced sooner than is customary. This forces students to use their prior knowledge to devise ways to solve the problems "from first principles," thus promoting students' construction of their own understandings.

Even after standard methods have been analyzed and practiced, students are still encouraged to solve problems in more than one way.

Even after standard methods have been analyzed and practiced, students are still encouraged to solve problems in more than one way. Flexible thinking and mathematical power are our goals, not rote facility with a handful of standard algorithms.

Varieties of Computation

There is much more to computation than the standard paper-and-pencil algorithms for adding, subtracting, multiplying, and dividing. These algorithms are good for obtaining exact answers with simple technology, but, depending on the resources available and the result desired, there are many other kinds of computation.

For example, if you are in a supermarket check-out line with several items and you find only $10 in your wallet, then a quick judgment whether you have sufficient funds is desirable. In this case, a rough mental estimate of the total cost of your purchases is what you want. If you are planning an addition to your house, however, different computational demands must be met. The situation is more complex than the supermarket check-out, and the penalty for making a mistake is more severe, so greater care must be taken. You will want more resources—paper and pencil, a calculator, time to work, perhaps a computer spreadsheet—and you will probably want rather precise estimates for the cost of various alternative designs for the addition.

A well-rounded mathematics program should prepare students to compute accurately, flexibly, and appropriately in all situations. Figure 4 shows a classification of computational situations using two criteria, the result desired and the resources available. Although you may want to move some of the questions to other cells or insert your own examples, these six categories of computation indicate the scope required of a modern mathematics curriculum (Coburn, 1989).

	Resources Available		
	Paper & Pencil	Machine	Mental
Exact	How many students are in the three third grades at my school?	How much will my monthly payment be on my car loan?	How much baking soda do I need if I am tripling a recipe that calls for 2 teaspoons?
Approximate	What is my share of the national debt?	House remodeling: Which design(s) can I afford?	Supermarket checkout: Do I have enough money for these items?

Figure 4: Six varieties of computation

The TIMS Philosophy: Meaning, Invention, Efficiency, Power

The treatment of computation in *Math Trailblazers* proceeds in several stages. The grade levels for the stages vary with the operation—ideas of division, for example, develop long after addition—but the general pattern is similar for all the operations. Roughly speaking, the stages are:

- developing meaning for the operation,
- inventing procedures for solving problems, and
- becoming more efficient at carrying out procedures, all leading to
- developing mathematical power.

The goal of the first stage is to help students understand the meaning of the operation. Most of the work involves solving problems, writing or telling "stories" that involve operations, and sharing solution strategies. These methods typically involve a great deal of mental arithmetic and creative thinking. The use of manipulatives, pictures, and counting is encouraged at this stage. Discussing these informal methods helps develop students' understanding of the operation.

In the next stage, the focus shifts from developing the concept of the operation to devising and analyzing procedures to carry out the operation. At this stage, students "invent" methods for carrying out the operation, explaining, discussing, and comparing their procedures. Multiple solution strategies—mental, paper and pencil, manipulative, calculator—are encouraged, and parallels between various methods are explored. Evidence is accumulating that this "invented algorithms" approach enhances students' number and operation sense and problem-solving abilities (Madell, 1985; Sawada, 1985; Kamii, Lewis, & Jones, 1991; Burns, 1992; Kamii, Lewis, & Livingston, 1993; Porter & Carroll, 1995; Carroll & Porter, in press). Inventing their own methods helps make mathematics meaningful for children by connecting school mathematics to their own ways of thinking. The expectation that mathematics should make sense is reinforced.

In the third stage, a standard algorithm for the operation is introduced. This algorithm is not presented as the one, true, and official way to solve problems, but rather as yet another procedure to be examined. The algorithms used in *Math Trailblazers* are not all identical to the traditional ones taught in school. The addition and subtraction algorithms are only a little different, but the procedures for multiplication and division are considerably different. (See Figures 5 and 6.)

$$
\begin{array}{r}
58 \\
\times\ 36 \\
\hline
48 \\
300 \\
240 \\
1500 \\
\hline
2088
\end{array}
$$

Figure 5: All-partials multiplication

```
           1 9 R 31
  32 ) 6 3 9 |
     - 3 2 0 | 10
       3 1 9 |
     - 1 6 0 |  5
       1 5 9 |
     -   9 6 |  3
         6 3 |
     -   3 2 |  1
         3 1 | 19
```

Figure 6: A division algorithm

These alternative algorithms for multiplication and division have been chosen for several reasons. First, they are easier to learn than the traditional methods. Second, they are more transparent, revealing better what is actually happening. Third, they provide practice in multiplying by numbers ending in zero, an important skill for estimation. Finally, even though they are less efficient than the traditional algorithms, they are good enough for most purposes—any problem that is awkward to solve by these methods should probably be done by machine anyway.

Students who have no reliable method of their own are urged to adopt the standard algorithm. However, even after a standard algorithm for an operation has been introduced and analyzed, alternative methods are still accepted, even encouraged, for students who are comfortable with them. In particular, the standard algorithm is very inefficient with some problems. For example, students using *Math Trailblazers* should be able to compute 40 × 30 mentally to get 1200. Using the standard algorithm here would be inefficient. Or consider 16,000 – 5. Using a standard algorithm to solve this problem is tedious and often results in errors.

In the last stage, students achieve mathematical power through the mastery of procedures that solve entire classes of problems: efficient and reliable computational algorithms. This procedural facility, moreover, is based on solid conceptual understandings so that it can be applied flexibly to solve problems. These procedures become part of the students' base of prior knowledge– on which they can build more advanced conceptual and procedural understandings.

Fractions and Decimals

The approach to fractions and decimals in *Math Trailblazers* parallels that for whole numbers. At first, the focus is on developing concepts and meanings for fractions and decimals. Next comes a period in which students invent procedures for solving problems, connecting school mathematics to their own informal methods and common sense. Finally, formal procedures are investigated, not as substitutes for common sense, but as more efficient methods for achieving desired results.

Fraction Meanings

One of the problems with fractions is that they are so useful. Consider some of the meanings for ½:

- half of a cookie
  (a part-whole fraction)
- 1 ÷ 2
  (division)
- one cup water to two cups flour
  (a ratio, sometimes written 1:2)
- ½ mile
  (a measurement)
- the point midway between 0 and 1
  (the name of a point on a number line)
- the square root of ¼
  (a pure number)
- the chance a fair coin will land heads up
  (a probability)

Because the same notation can mean so many different things, children and even adults sometimes become confused and may manipulate fraction symbols haphazardly, often with unfortunate results. A better approach develops sound meanings for the symbols before focusing on how to manipulate them.

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Part-Whole Fractions

Beginning fraction work focuses on part-whole fractions. Many fractions in daily life are part-whole fractions, so even young children are familiar with terms like *one-half* and *three-fourths* in part-whole contexts. Also, many key ideas about fractions are well illustrated in part-whole situations.

There are two concepts that are fundamental in understanding part-whole fractions: knowing what the "whole" is and understanding what a part is in relation to the whole. For example, to understand the statement, "Last night I ate three-fourths of a carton of ice cream," requires knowing what the whole is: Just how big a carton of ice cream was it? One must also understand that the parts into which the whole is divided must be equal—they should have the same area or mass or number, etc. A way to make this clear to children is to talk about "fair shares."

The whole in a part-whole fraction can be either a single thing (e.g., a pizza) or a collection (e.g., a class of students). When the whole is a collection, then counting is generally used to make fair shares; when the unit is a single thing, the fairness of the shares depends on some measurable quantity. Half of one pizza is different from half of three pizzas. Often, area is the variable that must be equally allocated among the parts; such a situation may be called an area model for fractions.

Symbols and Referents

A key idea in the *Math Trailblazers* approach to fractions is that fractions should be represented in several ways and that students should be able to make connections between those representations. The fraction two-thirds, for example, can be expressed in words, symbols, pictures, or real objects (Figure 7).

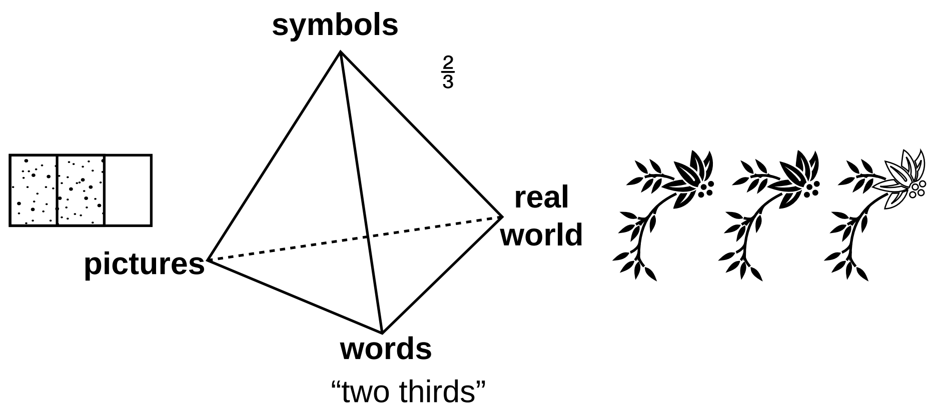

Figure 7: Representations of ⅔

The ability to move freely between these several representations is an essential component of the mathematical understanding of fractions. Especially important are connections between fraction symbols and the real-world situations to which those fractions refer. Given symbols for a fraction, can the student draw an illustrative picture or tell a story? Can students explain the relationship between a group of five girls and two boys and the fraction 5/7?

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Decimals

Decimals are treated in two ways in *MATH TRAILBLAZERS:* first, as another way to write certain common fractions—those with denominators that are powers of ten—and second, as an extension of the whole number place value system. Connections between fractions and decimals are stressed throughout.

A Developmental Approach

Fraction work in grades 1–3 of *MATH TRAILBLAZERS* focuses primarily on establishing links between symbols and referents for part-whole fractions. Children learn to make connections between marks on paper and the real world. Concepts of the unit—identifying the unit, knowing how the size of the unit affects the value of the fraction, appreciating the importance of fair shares—are also explored (Figure 8). Decimals are treated primarily as an alternative notation for certain fractions, and some attention is given to the interpretation of decimals that appear on calculators. Real-world situations, especially fair sharing, and area models predominate.

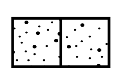
If  is $\frac{1}{3}$,
then what is one whole?

Figure 8: A concept-of-unit exercise

In grade 3, work continues with concepts of the unit, but more varied models begin to be used. Children explore the relative size of fractions, especially with respect to the landmark numbers 0, ½, and 1. Equivalent fractions are also studied in grade 3. Fractions are represented with a variety of manipulatives—paper folding, pattern blocks, geoboards. Collections of objects are divided to represent fractions.

Decimals are investigated more extensively in grade 3, again being treated as a kind of fraction. Decimals in metric length measurement and on number lines are investigated. Base-ten pieces are used to create concrete and visual representations of decimal fractions. Translating between common fractions and decimals is again stressed.

In grade 4, work on operations with fractions and decimals begins, but standard procedures are not taught. As discussed above, modeling concepts and procedures with manipulatives is intended to establish a basis for more algorithmic work in grade 5.

Operations with fractions and decimals include more than simply the four arithmetic operations. Putting fractions and decimals in order by size, renaming the same number in several equivalent forms, and estimating sums, differences, and so on, are all operations that can be carried out with fractions and decimals. Again, manipulatives, such as pattern blocks and base-ten pieces, are used to develop conceptual understanding and provide a concrete representation of the symbols.

In grade 5, paper-and-pencil procedures for addition, subtraction, and multiplication of fractions and decimals are explored, including use of common denominators and reducing. Repeating decimals are introduced. Students also use calculators to rename fractions as decimals for comparing and ordering common fractions.

Conclusion

Our goal in developing *MATH TRAILBLAZERS* has been to create a balanced program that will promote the coordinated development of both procedural skill and conceptual understanding. Moreover, by connecting school mathematics with intuitive knowledge and informal procedures and by using a variety of techniques and manipulatives for modeling the mathematical ideas, students will not only develop skills and concepts, but will be able to use those skills and concepts to solve problems.

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