

ON THE THEORIES OF MCDUFF'S II_1 FACTORS

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ABSTRACT. Recently, Boutonnet, Chifan, and Ioana proved that McDuff's family of continuum many pairwise nonisomorphic separable II_1 factors are in fact pairwise non-elementarily equivalent by proving that any ultrapowers of two distinct members of the family are nonsiomorphic. We use Ehrenfeucht-Fraisse games to provide an upper bound on the quantifier-depth of sentences which distinguish these theories.

1. INTRODUCTION

Constructing non-isomorphic separable II_1 factors has an interesting history. Murray and von Neumann [7] gave the first example of two non-isomorphic separable II_1 factors by proving that the hyperfinite II_1 factor \mathcal{R} was not isomorphic to $L(\mathbb{F}_2)$, the group von Neumann algebra associated to the free group on two generators. The way they proved this was by considering an isomorphism invariant, namely property Gamma, and proving that \mathcal{R} has property Gamma whilst $L(\mathbb{F}_2)$ does not. Dixmier and Lance [3] produced a new isomorphism class by constructing a separable II_1 factor that does have property Gamma but does not have another property, nowadays called being McDuff, that \mathcal{R} does have. Work of Zeller-Meier [9] and Sakai [8] led to several more isomorphism classes. The lingering question remained: are there infinitely many isomorphism classes of separable II_1 factors? In [5], McDuff constructed a countably infinite set of isomorphism classes of separable II_1 factors; in the sequel [6], she extends her technique to construct a family $(\mathcal{M}_\alpha)_{\alpha \in 2^\omega}$ of pairwise non-isomorphic separable II_1 factors. Throughout this paper, we will refer to this family as the family of *McDuff examples*. We will describe in detail the construction of the McDuff examples later in this paper.

The model-theoretic study of tracial von Neumann algebras began in earnest in [4], where it was shown that both property Gamma and being McDuff are axiomatizable properties (in the appropriate continuous first-order language for studying tracial von Neumann algebras). It follows that \mathcal{R} , $L(\mathbb{F}_2)$, and the Dixmier-Lance example are pairwise non-elementarily equivalent. However, it proved difficult to find new elementary equivalence classes of II_1 factors, although it was generally agreed upon by researchers in the model theory of operator algebras that there should be continuum many pairwise non-elementarily equivalent II_1 factors. The current authors recognized that one of the properties considered by Zeller-Meier in [9] was axiomatizable, thus providing a fourth elementary equivalence class; we include a proof of this observation in the last section.

In the recent paper [2], Boutonnet, Chifan, and Ioana prove that the McDuff examples are pairwise non-elementarily equivalent. They do not, however, exhibit sentences that distinguish these examples. Indeed, their main result is the following: if $\alpha, \beta \in 2^\omega$ are distinct, then for any nonprincipal ultrafilters \mathcal{U}, \mathcal{V} on arbitrary index sets, one has that $\mathcal{M}_\alpha^\mathcal{U} \not\cong \mathcal{M}_\beta^\mathcal{V}$. It is now routine to see that \mathcal{M}_α

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and \mathcal{M}_β are not elementarily equivalent. Indeed, since the question of whether or not \mathcal{M}_α and \mathcal{M}_β are elementarily equivalent is absolute, one can safely assume CH, whence \mathcal{M}_α elementarily equivalent to \mathcal{M}_β would imply that, for any nonprincipal ultrafilter \mathcal{U} on \mathbb{N} , one has that $\mathcal{M}_\alpha^\mathcal{U}$ and $\mathcal{M}_\beta^\mathcal{U}$ are saturated models of the same theory and a familiar back-and-forth argument shows that they are isomorphic.¹

To a model-theorist, it is interesting to know what sentences separate these examples. Indeed, to show that \mathcal{M}_α and \mathcal{M}_β are not elementarily equivalent, it would be interesting to write down an explicit set of sentences T such that, for some $\sigma \in T$, we have $\sigma^{\mathcal{M}_\alpha} \neq \sigma^{\mathcal{M}_\beta}$. At the end of this paper, we show how to do this when $\alpha(0) \neq \beta(0)$; for the general case, we do not know how to do this.

The main result of this paper is instead quantitative in nature. For II_1 factors \mathcal{M} and \mathcal{N} and $k \geq 1$, we say that $\mathcal{M} \equiv_k \mathcal{N}$ if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for any sentence σ of “complexity” at most k . (The precise notion of complexity will be defined in the next section.) Here is our main result:

Theorem. Suppose that $\alpha, \beta \in 2^\omega$ are distinct and $k \in \omega$ is least such that $\alpha(k) \neq \beta(k)$. Then $M_\alpha \not\equiv_{5k+3} M_\beta$.

In the next section, we describe the needed facts from logic as well as the parts of the paper [2] that we will use in our argument. In Section 3, we prove the main result; the proof uses Ehrenfeucht-Fr aisse games. In Section 4, we take care of some miscellaneous facts. First, we write down an explicit list of sentences that distinguish M_α from M_β when $\alpha(0) \neq \beta(0)$. Next we discuss how the model-theoretic behavior of “good unitaries” underlies much of the argument in [2]. We then go on to show how Zeller-Meier’s notion of *inner asymptotic commutativity* is axiomatizable and discuss another of Zeller-Meier’s notions (which he does not name but we call “super McDuff”), giving some evidence as to why it might be axiomatizable. Finally, we bring up the notion of the *first-order fundamental group* of a II_1 factor and show how finding a II_1 factor with proper first-order fundamental group would give a different proof of the existence of continuum many theories of II_1 factors.

We list here some conventions used throughout the paper. First, we follow set theoretic notation and view $k \in \omega$ as the set of natural numbers less than k : $k = \{0, 1, \dots, k-1\}$. In particular, 2^k denotes the set of functions $\{0, 1, \dots, k-1\} \rightarrow \{0, 1\}$. If $\alpha \in 2^k$, then we set $\alpha_i := \alpha(i)$ for $i = 0, 1, \dots, k-1$ and we let $\alpha^\# \in 2^{k-1}$ be such that α is the concatenation of (α_0) and $\alpha^\#$. If $\alpha \in 2^\omega$, then $\alpha|k$ denotes the restriction of α to $\{0, 1, \dots, k-1\}$.

Whenever we write a tuple \vec{x} , it will be understood that the length of the tuple is countable (that is, finite or countably infinite).

We use \subset (as opposed to \subseteq) to denote proper inclusion of sets.

If \mathcal{M} is a von Neumann algebra and A is a subalgebra of \mathcal{M} , then

$$A' \cap \mathcal{M} := \{x \in \mathcal{M} \mid [x, a] = 0 \text{ for all } a \in A\}.$$

In particular, the center of \mathcal{M} is $Z(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}$. For a tuple \vec{a} from \mathcal{M} , we write $C(\vec{a})$ to denote $A' \cap \mathcal{M}$, where A is the subalgebra of \mathcal{M} generated by the coordinates of \vec{a} . (Technically, this notation should also mention \mathcal{M} , but the ambient algebra will always be clear from context, whence we omit any mention of it in the notation.)

¹For those uncomfortable with the use of CH here, one can alternatively quote the Keisler-Shelah theorem as done in [2].

2. PRELIMINARIES

2.1. Logic.

Definition 2.1. We define the *quantifier-depth* $\text{depth}(\varphi)$ of a formula φ by induction on the complexity of φ .

- If φ is atomic, then $\text{depth}(\varphi) = 0$.
- If $\varphi_1, \dots, \varphi_n$ are formulae, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function and $\varphi = f(\varphi_1, \dots, \varphi_n)$, then $\text{depth}(\varphi) = \max_{1 \leq i \leq n} \text{depth}(\varphi_i)$.
- If $\varphi = \sup_{\vec{x}} \psi$ or $\varphi = \inf_{\vec{x}} \psi$, then $\text{depth}(\varphi) = \text{depth}(\psi) + 1$.

The main tool in this paper is the following variant of the usual Ehrenfeucht-Fraïssé game.

Definition 2.2. Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures and let $k \in \mathbb{N}$. $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ denotes the following game played by two players. First, player I plays either a tuple $\vec{x}_1 \in \mathcal{M}$ or a tuple $\vec{y}_1 \in \mathcal{N}$. Player II then responds with a tuple $\vec{y}_1 \in \mathcal{N}$ or $\vec{x}_1 \in \mathcal{M}$. The play continues in this way for k rounds. We say that *Player II wins* $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ if there is an isomorphism between the substructures generated by $\{\vec{x}_1, \dots, \vec{x}_k\}$ and $\{\vec{y}_1, \dots, \vec{y}_k\}$ that maps \vec{x}_i to \vec{y}_i .

Definition 2.3. Suppose that \mathcal{M} and \mathcal{N} are \mathcal{L} -structures.

- (1) We write $\mathcal{M} \equiv_k \mathcal{N}$ if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ whenever $\text{depth}(\sigma) \leq k$.
- (2) We write $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ if II has a winning strategy for $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$.

It is a routine induction to show that $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ implies $\mathcal{M} \equiv_k \mathcal{N}$.

Lemma 2.4. *Suppose that \mathcal{M} and \mathcal{N} are \aleph_1 -saturated. Then $\mathcal{M} \equiv_k \mathcal{N}$ if and only if $\mathcal{M} \equiv_k^{EF} \mathcal{N}$.*

Proof. We prove the lemma by induction on k . Suppose first that $k = 0$ and that $\mathcal{M} \equiv_0 \mathcal{N}$. Let \mathcal{M}_0 and \mathcal{N}_0 be the substructures of \mathcal{M} and \mathcal{N} respectively generated by the emptyset. It follows immediately that there is an isomorphism between \mathcal{M}_0 and \mathcal{N}_0 that sends $c^{\mathcal{M}}$ to $c^{\mathcal{N}}$ for each constant symbol c , whence II always wins $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 0)$.

Now suppose that $k > 0$ and inductively assume that the lemma holds for all integers smaller than k . We now describe a winning strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$. Suppose that I first plays $\vec{a}_1 \in \mathcal{M}$ (the case that I's first move is in \mathcal{N} is analogous). Consider the set $\Gamma(\vec{x})$ given by

$$\Gamma(\vec{x}) := \{|\varphi(\vec{x}) - r| = 0 : \text{depth}(\varphi) < k, \varphi^{\mathcal{M}}(\vec{a}_1) = 0\}.$$

We claim that $\Gamma(\vec{x})$ is finitely satisfiable in \mathcal{N} . Towards this end, consider conditions “ $|\varphi_i(\vec{x}) - r_i| = 0$ ” in Γ , $i = 1, \dots, p$. Let $\sigma := \inf_{\vec{x}} \max_{1 \leq i \leq p} |\varphi_i(\vec{x}) - r_i|$. Note that $\sigma^{\mathcal{M}} = 0$ and that $\text{depth}(\sigma) \leq k$. Since $\mathcal{M} \equiv_k \mathcal{N}$, we have that $\sigma^{\mathcal{N}} = 0$, witnessing that Γ is finitely satisfiable. Since \mathcal{N} is \aleph_1 -saturated, it follows that there is $\vec{b}_1 \in \mathcal{N}$ satisfying Γ . The strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ starts by demanding that II play \vec{b}_1 . Note now that $(\mathcal{M}, \vec{a}_1) \equiv_{k-1} (\mathcal{N}, \vec{b}_1)$, so by induction we have that $(\mathcal{M}, \vec{a}_1) \equiv_{k-1}^{EF} (\mathcal{N}, \vec{b}_1)$. The rest of the strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ is to have II play according to the winning strategy for II in $\mathfrak{G}((\mathcal{M}, \vec{a}_1), (\mathcal{N}, \vec{b}_1), k-1)$, where, for $p \geq 2$, round p in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ is viewed as round $p-1$ in $\mathfrak{G}((\mathcal{M}, \vec{a}_1), (\mathcal{N}, \vec{b}_1), k-1)$. This strategy is clearly a winning strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$, whence $\mathcal{M} \equiv_k^{EF} \mathcal{N}$. \square

In the sequel, we will often assume that $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ and that \mathcal{M} is nonseparable. For reasons that will become clear in the next section, we actually want to know that \mathcal{N} is also nonseparable.

Lemma 2.5. *Suppose that $\mathcal{M} \equiv_2^{EF} \mathcal{N}$ and \mathcal{M} is nonseparable. Then \mathcal{N} is nonseparable.*

Proof. Let \vec{b} be any tuple from \mathcal{N} . Let I play \vec{b} and have II respond with \vec{a} from \mathcal{M} . Since \mathcal{M} is nonseparable, there is $\epsilon > 0$ and $c \in \mathcal{M}$ such that $d(c, a_i) \geq \epsilon$ for all i . Have I play c and II responds with $d \in \mathcal{N}$. Since II wins $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 2)$, we have that $d(d, b_i) \geq \epsilon$ for all i , whence \vec{b} is not dense in \mathcal{N} . \square

2.2. McDuff's examples and property \tilde{V} . First, we recall McDuff's examples. Let Γ be a countable group. For $i \geq 1$, let Γ_i denote an isomorphic copy of Γ and let Λ_i denote an isomorphic copy of \mathbb{Z} . Let $\tilde{\Gamma} := \bigoplus_{i \geq 1} \Gamma_i$. If S_∞ denotes the group of permutations of \mathbb{N} with finite support, then there is a natural action of S_∞ on $\bigoplus_{i \geq 1} \Gamma$ (given by permutation of indices), whence we may consider the semidirect product $\tilde{\Gamma} \rtimes S_\infty$. Given these conventions, we can now define two new groups:

$$T_0(\Gamma) := \langle \tilde{\Gamma}, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = \text{ for } i \geq j \rangle$$

and

$$T_1(\Gamma) := \langle \tilde{\Gamma} \rtimes S_\infty, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = \text{ for } i \geq j \rangle.$$

Note that if Δ is a subgroup of Γ and $\alpha \in \{0, 1\}$, then $T_\alpha(\Delta)$ is a subgroup of $T_\alpha(\Gamma)$. Given a sequence $\alpha \in 2^{\leq \omega}$, we define a group $K_\alpha(\Gamma)$ as follows:

- (1) $K_\alpha(\Gamma) := \Gamma$ if $\alpha = \emptyset$;
- (2) $K_\alpha(\Gamma) := (T_{\alpha_0} \circ T_{\alpha_1} \circ \cdots \circ T_{\alpha_{n-1}})(\Gamma)$ if $\alpha \in 2^n$;
- (3) K_α is the inductive limit of $(K_{\alpha|n})_n$ if $\alpha \in 2^\omega$.

We then set $\mathcal{M}_\alpha(\Gamma) := L(T_\alpha(\Gamma))$. When $\Gamma = \mathbb{F}_2$, we simply write \mathcal{M}_α instead of $\mathcal{M}_\alpha(\mathbb{F}_2)$; these are the McDuff examples referred to the introduction.

Given $n \geq 1$, we let $\tilde{\Gamma}_{\alpha, n}$ denote the subgroup of $T_{\alpha_0}(K_{\alpha^\#}(\Gamma))$ given by the direct sum of the copies of $K_{\alpha^\#}(\Gamma)$ indexed by those $i \geq n$ and we let $P_{\alpha, n} := L(\tilde{\Gamma}_{\alpha, n})$. We define a *generalized McDuff ultraproduct corresponding to α* and Γ to be an ultraproduct of the form $\prod_{\mathcal{U}} \mathcal{M}_\alpha(\Gamma)^{\otimes t_s}$ and we refer to subalgebras of the form $\prod_{\mathcal{U}} P_{\alpha, n_s}^{\otimes t_s}$ as *special*.

We will need the following key facts:

Facts 2.6. *Suppose that $\alpha \in 2^{< \omega}$ is nonempty, Γ is a countable group, and (t_s) is a sequence of natural numbers.*

- (1) *Suppose that (m_s) and (n_s) are two sequences of natural numbers such that $n_s < m_s$ for all s . Then $(\prod_{\mathcal{U}} P_{\alpha, m_s}^{\otimes t_s})' \cap (\prod_{\mathcal{U}} P_{\alpha, n_s}^{\otimes t_s})$ is a generalized McDuff ultraproduct corresponding to $\alpha^\#$ and Γ .*
- (2) *For any sequence (n_s) , there is a pair of unitaries \vec{a} from $\prod_{\mathcal{U}} \mathcal{M}_\alpha^{\otimes t_s}$ such that $\prod_{\mathcal{U}} P_{\alpha, n_s}^{\otimes t_s} = C(\vec{a})$.*
- (3) *Given any separable subalgebra A of $\prod_{\mathcal{U}} \mathcal{M}_\alpha^{\otimes t_s}$, there is a sequence (n_s) such that $\prod_{\mathcal{U}} P_{\alpha, n_s}^{\otimes t_s} \subset A' \cap \prod_{\mathcal{U}} \mathcal{M}_\alpha^{\otimes t_s}$.*

The proofs of the above facts are contained in [2, Sections 2 and 3]. In particular, the proof of (1) is embedded in the proof of [2, Lemma 3.11].

We recall the definition of property \tilde{V} .

Definition 2.7. Let \mathcal{M} be a nonseparable von Neumann algebra. We say that \mathcal{M} has property \tilde{V} if there is a separable subalgebra $A \subseteq \mathcal{M}$ such that, for all separable $B \subseteq A' \cap \mathcal{M}$ and all separable $C \subseteq \mathcal{M}$, there is $u \in U(\mathcal{M})$ such that $uBu^* \subseteq C' \cap \mathcal{M}$.

The following is [2, Lemma 4.4].

Fact 2.8. *If Γ is any countable group, then $\prod_{\mathcal{U}} L(T_1(\Gamma))^{\otimes t_s}$ has \tilde{V} .*

Notation. If \vec{a} and \vec{b} are tuples from \mathcal{M} , we set $\vec{a} \leq \vec{b}$ if and only if $C(\vec{b}) \subseteq C(\vec{a})$. As with any preorder, we write $\vec{a} < \vec{b}$ to indicate that $\vec{a} \leq \vec{b}$ but $\vec{b} \not\leq \vec{a}$.

Definition 2.9. Let k be a natural number. We define what it means for a nonseparable von Neumann algebra \mathcal{M} to have \tilde{V} at depth k :

- \mathcal{M} has \tilde{V} at depth 0 if it has \tilde{V} ;
- If $k > 0$, then \mathcal{M} has \tilde{V} at depth k if for any \vec{a} , there is $\vec{b} > \vec{a}$ such that, for all $\vec{c} > \vec{b}$, there is $\vec{d} > \vec{c}$ for which there is a von Neumann algebra \mathcal{N} with $C(\vec{c})' \cap C(\vec{b}) \subseteq \mathcal{N} \subseteq C(\vec{d})' \cap C(\vec{a})$ and such that \mathcal{N} has \tilde{V} at depth $k - 1$.

In connection with this definition, let us set up some further notation.

Notation. Let \mathcal{M} be a nonseparable von Neumann algebra and let \vec{a} , \vec{b} , \vec{c} , and \vec{d} range over tuples from \mathcal{M} . Furthermore, let $k \geq 1$.

- (1) For $\vec{a} < \vec{b} < \vec{c} < \vec{d}$, $\Phi(\vec{a}, \vec{b}, \vec{c}, \vec{d}; k)$ denotes the statement “there is a von Neumann algebra \mathcal{N} with $C(\vec{c})' \cap C(\vec{b}) \subseteq \mathcal{N} \subseteq C(\vec{d})' \cap C(\vec{a})$ and such that \mathcal{N} has \tilde{V} at depth $k - 1$.”
- (2) For $\vec{a} < \vec{b} < \vec{c}$, $\Phi(\vec{a}, \vec{b}, \vec{c}; k)$ denotes the statement “there is $\vec{d} > \vec{c}$ such that $\Phi(\vec{a}, \vec{b}, \vec{c}, \vec{d}; k)$ holds.”
- (3) For $\vec{a} < \vec{b}$, $\Phi(\vec{a}, \vec{b}; k)$ denotes the statement “for all $\vec{c} > \vec{b}$, $\Phi(\vec{a}, \vec{b}, \vec{c}; k)$ holds.”
- (4) $\Phi(\vec{a}; k)$ denotes the statement “there is $\vec{b} > \vec{a}$ such that $\Phi(\vec{a}, \vec{b}; k)$ holds.”

The definition of \mathcal{M} having \tilde{V} at depth k can thus be recast as: for every \vec{a} , $\Phi(\vec{a}; k)$ holds. The following is the main result of [2] and appears there as Theorem 4.2 (really, Remark 4.3).

Fact 2.10. *Suppose that $\alpha \in 2^\omega$. Then $\mathcal{M}_\alpha^{\mathcal{U}}$ has \tilde{V} at depth k if and only if $\alpha_k = 1$.*

3. THE MAIN RESULT

Proposition 3.1. *Suppose that \mathcal{M} and \mathcal{N} are nonseparable with $\mathcal{M} \equiv_3^{EF} \mathcal{N}$ and \mathcal{M} has \tilde{V} . Then \mathcal{N} has \tilde{V} .*

Proof. Let $A \subseteq \mathcal{M}$ witness that \mathcal{M} has \tilde{V} . Let \vec{a} enumerate a countable dense subset of A and let I play \vec{a} . II then plays $\vec{a}_1 \in \mathcal{N}$. Let A_1 denote the subalgebra of \mathcal{N} generated by \vec{a}_1 . We claim that A_1 witnesses that \mathcal{N} has \tilde{V} . Towards this end, take separable $B_1 \subseteq A'_1 \cap \mathcal{N}$ and $C_1 \subseteq \mathcal{N}$. Let \vec{b}_1 and \vec{c}_1 enumerate countable dense subsets of B_1 and C_1 respectively. I then plays \vec{b}_1 and \vec{c}_1 . II

then responds with $\vec{b}, \vec{c} \in \mathcal{M}$. Since II wins, it follows that $B \subseteq A' \cap \mathcal{M}$, so there is $u \in U(\mathcal{M})$ such that $uBu^* \subseteq C' \cap \mathcal{M}$. I finally plays u and II responds with $u_1 \in \mathcal{N}$. It remains to observe that $u_1 \in U(\mathcal{N})$ and $u_1 B_1 u_1^* \subseteq C'_1 \cap \mathcal{N}$. \square

Theorem 3.2. *Suppose that $\alpha \in 2^{k+1}$ with $\alpha(k) = 1$. Further suppose that Γ is any countable group and that \mathcal{M} is a generalized McDuff ultraproduct corresponding to α and Γ . Finally suppose that $\mathcal{M} \equiv_{5k+3}^{EF} \mathcal{N}$. Then \mathcal{N} has property \tilde{V} at depth k .*

Proof. We proceed by induction on k . Fact 2.8 and the previous proposition establishes the case $k = 0$. So suppose that $k > 0$ and the result holds for all smaller k . Choose a \vec{b}_0 from \mathcal{N} and we would like to show that $\Phi(\vec{b}_0; k)$ holds. We obtain this by having player I play cooperatively in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 5k + 3)$. View \vec{b}_0 as the first play for player I; II responds with \vec{a}_0 from \mathcal{M} according to her winning strategy. Let P be a special subalgebra of \mathcal{M} such that $P \subset C(\vec{a}_0)$. At the next round, I plays \vec{a}_1 from \mathcal{M} such that $P = C(\vec{a}_1)$ (so $\vec{a}_0 < \vec{a}_1$) and II responds with \vec{b}_1 from \mathcal{N} .

Claim 1 $\vec{b}_0 < \vec{b}_1$.

Proof of Claim 1: We show that otherwise, player I could win the game. First suppose that there is $y \in C(\vec{b}_1) \setminus C(\vec{b}_0)$; since $5k + 3 \geq 3$, we can have I play y and II responds with $x \in \mathcal{M}$ according to her winning strategy. We have that $x \in C(\vec{a}_1) \setminus C(\vec{a}_0)$, a contradiction. This shows that $\vec{b}_0 \leq \vec{b}_1$. Now suppose that $z \in C(\vec{a}_0) \setminus C(\vec{a}_1)$ and have I play z , II responding with $w \in \mathcal{N}$; since II wins, it follows that $w \in C(\vec{b}_0) \setminus C(\vec{b}_1)$, whence $\vec{b}_0 < \vec{b}_1$.

Now we would like to show that $\Phi(\vec{b}_0, \vec{b}_1; k)$ holds. Choose any $\vec{b}_2 > \vec{b}_1$ and we will show $\Phi(\vec{b}_0, \vec{b}_1, \vec{b}_2; k)$ holds. Agreeably I plays \vec{b}_2 , II responding with \vec{a}_2 from \mathcal{M} . Since $5k + 3 \geq 4$, the proof of Claim 1 shows that $\vec{a}_2 > \vec{a}_1$. Now choose a special subalgebra Q such that $Q \subset C(\vec{a}_2)$ and $Q = C(\vec{a}_3)$. Player I now plays \vec{a}_3 and II responds with $\vec{b}_3 \in \mathcal{N}$. Since $5k + 3 \geq 5$, repeating Claim 1 shows that $\vec{b}_3 > \vec{b}_2$. To finish, we show that $\Phi(\vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3; k)$ holds.

Set $\mathcal{M}_1 := C(\vec{a}_3)' \cap C(\vec{a}_1)$ and $\mathcal{N}_1 := C(\vec{b}_3)' \cap C(\vec{b}_1)$. Note that \mathcal{M}_1 is a generalized McDuff ultraproduct corresponding to $\alpha^\# \in 2^k$ and Γ and that $\alpha^\#(k - 1) = 1$.

Claim 2: $\mathcal{M}_1 \equiv_{5k-2}^{EF} \mathcal{N}_1$.

Proof of Claim 2: We view any round p in $\mathfrak{G}(\mathcal{M}_1, \mathcal{N}_1, 5k - 2)$ as round $p + 4$ in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 5k + 3)$ where the first four rounds are played out as above II plays according to the winning strategy for that game. A priori II's moves come from \mathcal{M} or \mathcal{N} but if they do not land in \mathcal{M}_1 or \mathcal{N}_1 then I can win the game in 1 more step since $p + 4 + 1 \leq 5k - 2 + 5 = 5k + 3$ and this would be a contradiction.

Since $5k - 2 = 5(k - 1) + 3$, by induction we see that \mathcal{N}_1 has \tilde{V} at depth $k - 1$. Since we have $C(\vec{b}_2)' \cap C(\vec{b}_1) \subset \mathcal{N}_1 \subset C(\vec{b}_3)' \cap C(\vec{b}_0)$, it follows that $\Phi(\vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3; k)$. \square

Corollary 3.3. *Suppose that $\alpha, \beta \in 2^\omega$ and k are such that $\alpha|k = \beta|k$, $\alpha(k) = 1$, $\beta(k) = 0$. Then $\mathcal{M}_\alpha \not\equiv_{5k+3} \mathcal{M}_\beta$.*

Proof. Fix $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. If $\mathcal{M}_\alpha \equiv_{5k+3} \mathcal{M}_\beta$, then $\mathcal{M}_\alpha^\mathcal{U} \equiv_{5k+3}^{EF} \mathcal{M}_\beta^\mathcal{U}$. Since $\mathcal{M}_\alpha = L(K_{\alpha|(k+1)}(\Gamma))$ for some group Γ , the previous theorem implies that $\mathcal{M}_\beta^\mathcal{U}$ has \tilde{V} at depth k , contradicting Fact 2.10. \square

4. MISCELLANEA

4.1. Distinguishing \tilde{V} with a sentence. As mentioned in the introduction, it would be interesting to find concrete sentences that are actually distinguishing the McDuff examples. In this subsection, we show how we can find a set of sentences to distinguish \mathcal{M}_α from \mathcal{M}_β when $\alpha(0) = 1$ and $\beta(0) = 0$.

Suppose that \mathcal{M} is a separable McDuff II₁ factor for which $\mathcal{M}^\mathcal{U}$ has \tilde{V} as witnessed by separable $A \subseteq \mathcal{M}^\mathcal{U}$. Since any separable subalgebra of $\mathcal{M}^\mathcal{U}$ containing A also witnesses that $\mathcal{M}^\mathcal{U}$ has \tilde{V} , by considering a separable elementary substructure of $\mathcal{M}^\mathcal{U}$ containing A , we may assume that A is a separable McDuff II₁ factor, whence singly generated, say by $a \in A$. Fix $n \in \mathbb{N}$ and let $\theta_n(w)$ be the meta-statement

$$\forall \vec{x}, \vec{y} \left(\max_{1 \leq i \leq n} \|[w, x_i]\|_2 = 0 \rightarrow \inf_{u \in U} \max_{1 \leq i, j \leq n} \|[ux_iu^*, y_j]\|_2 = 0 \right).$$

Now $\theta_n(w)$ is not an official statement of continuous logic, but [1, Proposition 7.14] together with the fact that $(\mathcal{M}^\mathcal{U}, a)$ is \aleph_1 -saturated and $\theta_n(a)$ holds in $(\mathcal{M}^\mathcal{U}, a)$ implies that there are continuous functions $\gamma_n : [0, 1] \rightarrow [0, 1]$ with $\gamma_n(0) = 0$ such that $\psi_n(a)^{(\mathcal{M}^\mathcal{U}, a)} = 0$ for each n , where $\psi_n(w)$ is the formula

$$\sup_{\vec{x}, \vec{y}} \left(\left(\inf_u \max_{1 \leq i, j \leq n} \|[ux_iu^*, y_j]\|_2 \right) \div \gamma_n \left(\max_{1 \leq i \leq n} \|[w, x_i]\|_2 \right) \right).$$

Proposition 4.1. *Suppose that $\alpha, \beta \in 2^\omega$ are such that $\alpha(0) = 1$ and $\beta(0) = 0$. Then there are γ_n, ψ_n as above such that:*

- (1) For each $n \geq 1$, $(\inf_w \psi_n(w))^{\mathcal{M}_\alpha} = 0$.
- (2) There is $n \geq 1$ such that $(\inf_w \psi_n(w))^{\mathcal{M}_\beta} \neq 0$.

Proof. Let $\mathcal{M} := \mathcal{M}_\alpha^\mathcal{U}$, $\mathcal{N} := \mathcal{M}_\beta^\mathcal{U}$. Then \mathcal{M} has \tilde{V} , whence the discussion preceding the current proposition holds and we have γ_n, ψ_n satisfying (1). Suppose, towards a contradiction, that (2) fails, namely that $(\inf_w \psi_n(w))^{\mathcal{M}_\beta} = 0$ for all n . We claim that \mathcal{N} has \tilde{V} , a contradiction. By saturation (together with the fact that the ψ_n 's get successively stronger), there is $a_1 \in \mathcal{N}$ such that $\psi_n(a_1) = 0$ for all n . Let A_1 be the subalgebra of \mathcal{N} generated by a_1 . We claim that A_1 witnesses that \mathcal{N} has \tilde{V} . Towards this end, fix separable $B \subseteq A_1' \cap \mathcal{N}$ and separable $C \subseteq \mathcal{N}$. Let \vec{b} and \vec{c} enumerate countable dense subsets of B and C respectively. Set

$$\Omega(u) := \{u \in U\} \cup \{\|[ub_iu^*, c_j]\|_2 = 0 : i, j \in \mathcal{N}\}.$$

By choice of a_1 , $\Omega(u)$ is finitely satisfiable in \mathcal{N} , whence satisfiable in \mathcal{N} ; if u satisfies Ω , then $uBu^* \subseteq C' \cap \mathcal{N}$, yielding the desired contradiction. \square

Notice that each $\inf_w \psi_n(w)$ has depth 3 which agrees with the 3 appearing in Proposition 3.1. Also note that the above discussion goes through with $\mathcal{M}_\alpha^\mathcal{U}$ replaced with any generalized McDuff ultraproduct corresponding to α and any countable group Γ and likewise for $\mathcal{M}_\beta^\mathcal{U}$.

4.2. Good unitaries and definable sets. We would like to draw the reader's attention to some of the underlying model theory in [2] and recast Theorem 3.2. We highlight and give a name to the following concept that played a critical role in [2].

Definition 4.2. We say that a pair of unitaries u, v in a II_1 factor \mathcal{M} are *good unitaries* if $C(u, v)$ is a (2,100)-residual subalgebra of \mathcal{M} (in the terminology of [2]) with respect to the unitaries u and v , that is, for all $\zeta \in \mathcal{M}$,

$$\inf_{\eta \in C(u, v)} \|\zeta - \eta\|_2 \leq 100(\|\zeta, u\|_2^2 + \|\zeta, v\|_2^2).$$

We will call $C(u, v)$ a *good subalgebra* with respect to u and v .

If u and v are good unitaries, then $C(u, v)$ is a $\{u, v\}$ -definable set, which follows immediately from [1, Proposition 9.19]. Moreover, we claim that if u_1, v_1 are another pair of good unitaries for which $C(u_1, v_1) \subseteq C(u, v)$, then $C(u_1, v_1)' \cap C(u, v)$ is $\{u, v, u_1, v_1\}$ -definable. To see this, we first recall the following fact, due to Sorin Popa and communicated to us by David Sherman.

Fact 4.3. *Suppose that \mathcal{M} is a tracial von Neumann algebra with subalgebra \mathcal{N} . Let $E : \mathcal{M} \rightarrow \mathcal{N}' \cap \mathcal{M}$ denote the conditional expectation map. Then for any $x \in \mathcal{M}$, we have*

$$\|E(x) - x\|_2 \leq \sup_{y \in \mathcal{N}_{\leq 1}} \|[x, y]\|_2.$$

Note already that this fact shows $C(u, v)' \cap \mathcal{M}$ is $\{u, v\}$ -definable for any pair of good unitaries u, v . In general, intersections of definable subsets of metric structures need not be definable, so to show that $C(u_1, v_1)' \cap C(u, v)$ is definable, we need to do a bit more.

Lemma 4.4. *Suppose that \mathcal{M} is a tracial von Neumann algebra with subalgebra \mathcal{N} . Let $E : \mathcal{M} \rightarrow \mathcal{N}' \cap \mathcal{M}$ denote the conditional expectation and let $P(x) := d(x, \mathcal{N})$ for all $x \in \mathcal{M}$. Then E is an A -definable function if and only if P is an A -definable predicate.*

Proof. If E is an A -definable function, then $P(x) := \|E(x) - x\|_2$ is an A -definable predicate. Conversely, if P is an A -definable predicate, then for any $x, y \in \mathcal{M}$, we have $\|E(x) - y\|_2^2 = \|x - y\|_2^2 - P(x - y)^2 + P(y)^2$, whence E is an A -definable function. \square

Lemma 4.5. *Suppose that \mathcal{M} is a tracial von Neumann algebra with subalgebras $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \mathcal{M}$. Furthermore suppose that \mathcal{N}_1 and \mathcal{N}_2 are A -definable subsets of \mathcal{M} . Then $\mathcal{N}_2' \cap \mathcal{N}_1$ is an A -definable subset of \mathcal{M} .*

Proof. Since $\mathcal{N}_2' \cap \mathcal{N}_1 = (\mathcal{N}_2' \cap \mathcal{M}) \cap \mathcal{N}_1$ and the intersection of two zerosets is again a zeroset, it suffices to show that the distance to $\mathcal{N}_2' \cap \mathcal{N}_1$ is a definable predicate. To keep things straight, let $E_1 : \mathcal{M} \rightarrow \mathcal{N}_1$ and $E_2 : \mathcal{M} \rightarrow \mathcal{N}_2' \cap \mathcal{N}_1$ denote the respective conditional expectations. By assumption, E_1 is A -definable. If $x \in \mathcal{M}$, we have

$$\begin{aligned} \|E_2(x) - x\|_2 &\leq \|E_2(x) - E_2(E_1(x))\|_2 + \|E_2(E_1(x)) - E_1(x)\|_2 + \|E_1(x) - x\|_2 \\ &\leq 2\|x - E_1(x)\|_2 + \sqrt{\sup_{y \in (\mathcal{N}_2')_{\leq 1}} \|[E_1(x), y]\|_2}. \end{aligned}$$

Since \mathcal{N}_2' is an A -definable set and E_1 is an A -definable function, we see that $\mathcal{N}_2' \cap \mathcal{N}_1$ is an A -definable set. \square

In particular, if u, v, u_1, v_1 are as above, then $C(u_1, v_1)' \cap C(u, v)$ is an $\{u, v, u_1, v_1\}$ -definable subset of \mathcal{M} .

We note that Fact 2.6 (and the proof of Lemma 2.9 of [2]) shows that a special subalgebra of a generalized McDuff ultraproduct is a good subalgebra with respect to some pair of good unitaries.

In the definition of \tilde{V} at depth k , one could modify the definition to only work with pairs of good unitaries instead of arbitrary countable tuples. It follows from the work in [2] that if \mathcal{M} is a generalized McDuff ultraproduct, then \mathcal{M} has \tilde{V} at depth k if and only if \mathcal{M} has \tilde{V} at depth k in this augmented sense.

Returning now to the proof of Theorem 3.2, by the previous paragraph we see that at each play of the game, I could have chosen a pair of good unitaries instead of a countable sequence. Moreover, player I would also choose good unitaries corresponding to special subalgebras whenever they played a special subalgebra. Since II has a winning strategy by assumption, it follows that II always responds with pairs of good unitaries. Indeed, suppose that I plays good unitaries u, v (say in \mathcal{M}) and then II responds with $u_1, v_1 \in \mathcal{N}$. Since II wins, we have that u_1, v_1 are unitaries. To see that they are good, we need to play two more rounds of a side-game. Fix $\zeta_1 \in \mathcal{N}$ and $\epsilon > 0$. Have I play ζ_1 and have II reply with $\zeta \in \mathcal{M}$. Since u, v are good, there is $\eta \in C(u, v)$ such that

$$\|\zeta - \eta\|_2 < 100(\|\zeta, u\|_2^2 + \|\zeta, v\|_2^2) + \epsilon.$$

Have I play η and II responds with $\eta_1 \in \mathcal{N}$. It follows that $\eta_1 \in C(u_1, v_1)$ and

$$\|\zeta_1 - \eta_1\|_2 < 100(\|\zeta_1, u_1\|_2^2 + \|\zeta_1, v_1\|_2^2) + \epsilon.$$

Since $\zeta_1 \in \mathcal{N}$ and $\epsilon > 0$ were arbitrary, it follows that u_1, v_1 are good.

We see then that the subalgebras called \mathcal{M}_1 and \mathcal{N}_1 in the proof were in fact definable subalgebras defined over the parameters picked during the game.

We conclude this subsection with a discussion of what goes wrong when trying to distinguish \mathcal{M}_α and \mathcal{M}_β with a sentence when $\alpha(0) = \beta(0)$ but $\alpha(1) = 1$ and $\beta(1) = 0$. Motivated by the game played in the previous section, it seems natural to try to use sentences of the form

$$\sup_{u_1, v_1} \inf_{u_2, v_2} \sup_{u_3, v_3} \inf_{u_4, v_4} \chi,$$

where at every stage we quantify only over good unitaries above the previous unitaries in the partial order on tuples and χ expresses that $C(u_4, v_4)' \cap C(u_2, v_2)$ has \tilde{V} . There is no issue in saying that the unitaries involved are good and get progressively stronger; moreover, if the unitaries “played” at the inf stages yield a special subalgebra, then $C(u_4, v_4)' \cap C(u_2, v_2)$ is definable and so one can relativize the sentences from the previous subsection to this definable set and indeed express that this commutant has \tilde{V} . The issue arises in that there were “mystery” connectives γ_n used in the sentences from the previous subsection and for different choices of good unitaries u_3, v_3 , the generalized McDuff ultraproducts corresponding to $\alpha^\#$, $C(u_4, v_4)' \cap C(u_2, v_2)$, may require different connectives to express that they have \tilde{V} . Of course, a positive answer to the following question alleviates this concern and shows how one can find sentences distinguishing \mathcal{M}_α from \mathcal{M}_β when α and β differ for the first time at the second digit (and by induction one could in theory find sentences distinguishing all McDuff examples):

Question 4.6. *Given $\alpha \in 2^\omega$ and a countable group Γ , are all generalized McDuff ultraproducts corresponding to α and Γ elementarily equivalent?*

4.3. Inner asymptotic commutativity and super McDuffness. Motivated by Sakai’s definition of asymptotically commutative II_1 factors from [8], Zeller-Meier introduced the following notion in [9]:

Definition 4.7. Suppose that \mathcal{M} is a separable II_1 factor. We say that \mathcal{M} is *inner asymptotically commutative* (IAC) if and only if there is a sequence of unitaries (u_n) such that, for all $x, y \in \mathcal{M}$, we have $\lim_n \|[u_n x u_n^*, y]\|_2 = 0$.

Proposition 4.8. *Inner asymptotic commutativity is an axiomatizable property.*

Proof. For $n \geq 1$, consider the sentence

$$\sigma_n := \sup_{\vec{x}, \vec{y}} \inf_u \max_{1 \leq i, j \leq n} \|[u x_i u^*, y_j]\|_2.$$

We claim that a separable II_1 factor \mathcal{M} is IAC if and only if $\sigma_n^{\mathcal{M}} = 0$ for all n . The forward implication is clear. For the converse, suppose that $\sigma_n^{\mathcal{M}} = 0$ for all n . Let $\{a_i : i \in \mathcal{N}\}$ be a dense subset of \mathcal{M} . For each n , let $u_n \in U(\mathcal{M})$ be such that $\|[u_n a_i u_n^*, a_j]\|_2 < 1/n$ for all $i, j \leq n$. It then follows that (u_n) witnesses that \mathcal{M} is IAC. \square

Zeller-Meier also considers another property that may or may not hold for separable II_1 factors. Before we can define this property, we need some preparation:

Proposition 4.9. *Suppose that \mathcal{M} is a separable McDuff II_1 factor and $\mathcal{M} \preceq \mathcal{C} \preceq \tilde{\mathcal{C}}$ with \mathcal{C} and $\tilde{\mathcal{C}}$ both \aleph_1 -saturated. Then the following are equivalent:*

- (1) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$
- (2) $Z(\mathcal{M}' \cap \tilde{\mathcal{C}}) = \mathbb{C}$.

Proof. First suppose that (2) fails, so there is $a \in Z(\mathcal{M}' \cap \tilde{\mathcal{C}})$ such that $d(a, \text{tr}(a) \cdot 1) \geq \epsilon$. Since \mathcal{M} is McDuff, it is singly generated, say by $m \in \mathcal{M}$. Since $(\tilde{\mathcal{C}}, a, m)$ is \aleph_1 -saturated, there is a continuous function $\gamma : [0, 1] \rightarrow [0, 1]$ with $\gamma(0) = 0$ such that

$$(\tilde{\mathcal{C}}, a, m) \models \sup_y (\|[a, y]\|_2 \div \gamma(\|[y, m]\|_2)) = 0.$$

It follows that

$$(\tilde{\mathcal{C}}, m) \models \inf_x \max(\|[x, m]\|_2, \sup_y (\|[x, y]\|_2 \div \gamma(\|[y, m]\|_2)), \epsilon \div d(x, \text{tr}(x) \cdot 1)) = 0.$$

By elementarity, the same statement holds in (\mathcal{C}, m) ; by saturation, the infimum is realized by $b \in \mathcal{C}$. It follows that $b \in Z(\mathcal{M}' \cap \mathcal{C}) \setminus \mathbb{C}$, so (1) fails.

Now suppose that (2) holds and consider $a \in Z(\mathcal{M}' \cap \mathcal{C})$. Then there is a continuous function $\eta : [0, 1] \rightarrow [0, 1]$ with $\eta(0) = 0$ such that

$$\mathcal{C} \models \sup_y (\|[y, a]\|_2 \div \eta(\|[y, m]\|_2)) = 0.$$

By elementarity, the same statement holds in $\tilde{\mathcal{C}}$, that is, $a \in Z(\mathcal{M}' \cap \tilde{\mathcal{C}}) = \mathbb{C}$, whence (1) holds. \square

Observe that the end of the above proof actually shows that, under the same hypotheses as in the proposition, we have $Z(\mathcal{M}' \cap \mathcal{C}) \subseteq Z(\mathcal{M}' \cap \tilde{\mathcal{C}})$.

Corollary 4.10. *Suppose that \mathcal{M} is a separable McDuff II_1 factor. Then the following are equivalent:*

- (1) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$ for every \aleph_1 -saturated elementary extension \mathcal{C} of \mathcal{M} .
- (2) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$ for some \aleph_1 -saturated elementary extension \mathcal{C} of \mathcal{M} .

Definition 4.11. If \mathcal{M} is a separable McDuff II_1 factor, we say that \mathcal{M} is *super McDuff* if either of the equivalent conditions of the previous corollary hold.

It would be nice to know if being super McDuff is axiomatizable, for then [9] gives another example of a theory of II_1 factors. At the moment, the following proposition is the best that we can do.

Proposition 4.12. *Suppose that \mathcal{M}, \mathcal{N} are separable McDuff II_1 factors with $\mathcal{M} \preceq \mathcal{N}$. If \mathcal{N} is super McDuff, then so is \mathcal{M} .*

First, we need a little bit of preparation. Given $p \in S(\mathcal{M})$, we define $p^\mathcal{U} \in S(\mathcal{M}^\mathcal{U})$ by declaring, for every formula $\varphi(x, y)$ and every element $a := (a_i)^\bullet \in \mathcal{M}^\mathcal{U}$, $\varphi(x, a)^{p^\mathcal{U}} := \lim_{\mathcal{U}} \varphi(x, a_i)^p$.

Lemma 4.13. *If $p \in S(\mathcal{M})$ is not algebraic, then neither is $p^\mathcal{U} \in S(\mathcal{M}^\mathcal{U})$.*

Proof. Suppose that $p^\mathcal{U}$ is algebraic. Let \mathcal{N} be an elementary extension of \mathcal{M} containing a realization a of p . Then $a^\bullet \in \mathcal{N}^\mathcal{U}$ is a realization of $p^\mathcal{U}$, whence it belongs to $\mathcal{M}^\mathcal{U}$ by algebraicity of $p^\mathcal{U}$. It follows that a is the limit of a sequence from \mathcal{M} , whence it belongs to \mathcal{M} as well. Since a was an arbitrary realization of p , we conclude that p is algebraic. \square

Proof of Proposition 4.12. Fix a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} . Without loss of generality, we may assume that $\mathcal{M} \preceq \mathcal{N} \preceq \mathcal{M}^\mathcal{U}$. Suppose that \mathcal{M} is not super McDuff as witnessed by $a \in Z(\mathcal{M}' \cap \mathcal{M}^\mathcal{U}) \setminus \mathbb{C}$. Let $p := \text{tp}(a/\mathcal{M})$; since \mathcal{M} is a II_1 factor, p is not algebraic, whence neither is $p^\mathcal{U}$. Let \mathcal{C} be a $(2^{\aleph_0})^+$ -saturated elementary extension of $\mathcal{M}^\mathcal{U}$.

Claim 1: $p^\mathcal{U}(\mathcal{C}) \subseteq (\mathcal{M}^\mathcal{U})' \cap \mathcal{C}$.

Proof of Claim 1: Let $\varphi(x, y)$ denote the formula $\|[x, y]\|_2$. Then for any $b \in \mathcal{M}$, we have that $\varphi(x, b)^p = 0$, whence it follows that for any element $b \in \mathcal{M}^\mathcal{U}$ we have $\varphi(x, b)^{p^\mathcal{U}} = 0$, verifying the claim.

Claim 2: $p(\mathcal{M}^\mathcal{U}) \subseteq Z(\mathcal{M}' \cap \mathcal{M}^\mathcal{U})$.

Proof of Claim 2: Fix $\epsilon > 0$. Then the following set of conditions is unsatisfiable in $\mathcal{M}^\mathcal{U}$:

$$\{\|[x, b]\|_2 = 0 : b \in \mathcal{M}\} \cup \{\|[x, a]\|_2 \geq \epsilon\}.$$

By saturation, there are $b_1, \dots, b_n \in \mathcal{M}$ such that the following meta-statement is true in $\mathcal{M}^\mathcal{U}$:

$$\mathcal{M}^\mathcal{U} \models \forall x \left(\max_{1 \leq i \leq n} \|[x, b_i]\|_2 = 0 \rightarrow (\|[x, a]\|_2 \div \epsilon) = 0 \right).$$

As above, by saturation this meta-statement can be made into an actual first-order formula with parameters from \mathcal{M} that holds of a , whence it holds of any other realization of p in $\mathcal{M}^\mathcal{U}$. This shows that if $a' \in p(\mathcal{M}^\mathcal{U})$ and $c \in \mathcal{M}' \cap \mathcal{M}^\mathcal{U}$, then $\|[a', c]\|_2 \leq \epsilon$; since $\epsilon > 0$ is arbitrary, this proves the claim.

Claim 3: $p^\mathcal{U}(\mathcal{C}) \subseteq Z((\mathcal{M}^\mathcal{U})' \cap \mathcal{C})$.

Proof of Claim 3: Suppose that $a' \in \mathcal{C}$ realizes $p^\mathcal{U}$. Fix $b' \in (\mathcal{M}^\mathcal{U})' \cap \mathcal{C}$. Take $a'', b'' \in \mathcal{M}^\mathcal{U}$ such that $\text{tp}(a', b'/\mathcal{M}) = \text{tp}(a'', b''/\mathcal{M})$. By Claim 2, $a'' \in Z(\mathcal{M}' \cap \mathcal{M}^\mathcal{U})$. Note also that $b'' \in \mathcal{M}' \cap \mathcal{M}^\mathcal{U}$. It follows that $\|[a', b']\|_2 = \|[a'', b'']\|_2 = 0$, yielding the desired conclusion.

In order to establish that \mathcal{N} is not super McDuff, by Lemma 4.13, it suffices to establish the following claim:

Claim 4: $p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}}) \subseteq Z(\mathcal{N}' \cap \mathcal{M}^{\mathcal{U}})$.

Proof of Claim 4: Arguing as in the proof of Claim 1, we see that $p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}}) \subseteq \mathcal{N}' \cap \mathcal{M}^{\mathcal{U}}$. Now suppose that $a' \in p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}})$ and $b' \in \mathcal{N}' \cap \mathcal{M}^{\mathcal{U}}$. Then $a' \in Z(\mathcal{M}' \cap \mathcal{M}^{\mathcal{U}})$ by Claim 2 and $b' \in \mathcal{M}' \cap \mathcal{M}^{\mathcal{U}}$, so $[a, b] = 0$ as desired. \square

4.4. The first-order fundamental group. For a II_1 factor \mathcal{M} and $t \in \mathbb{R}_+$, we let \mathcal{M}_t denote the amplification of \mathcal{M} by t . Note that if \mathcal{U} is an ultrafilter, then $(\mathcal{M}^{\mathcal{U}})_t$ is canonically isomorphic to $(\mathcal{M}_t)^{\mathcal{U}}$, whence we can unambiguously write $\mathcal{M}_t^{\mathcal{U}}$.

Recall that the fundamental group of \mathcal{M} is the set $\mathcal{F}(\mathcal{M}) := \{t \in \mathbb{R}_+ : \mathcal{M}_t \cong \mathcal{M}\}$. $\mathcal{F}(\mathcal{M})$ is a (not necessarily closed) subgroup of \mathbb{R}_+ . We now consider the *first-order fundamental group* of \mathcal{M} , $\mathcal{F}_{\text{fo}}(\mathcal{M}) := \{t \in \mathbb{R}_+ : \mathcal{M}_t \equiv \mathcal{M}\}$. Clearly $\mathcal{F}(\mathcal{M}) \subseteq \mathcal{F}_{\text{fo}}(\mathcal{M})$. As the name indicates, $\mathcal{F}_{\text{fo}}(\mathcal{M})$ is actually a group. The easiest way to see this is to recognize that $\mathcal{F}_{\text{fo}}(\mathcal{M})$ is absolute, whence, assuming CH, we have $\mathcal{F}_{\text{fo}}(\mathcal{M}) = \mathcal{F}(\mathcal{M}^{\mathcal{U}})$ for a fixed ultrafilter \mathcal{U} on \mathbb{N} . Alternatively, one can use Keisler-Shelah as follows. Suppose that $s, t \in \mathcal{F}_{\text{fo}}(\mathcal{M})$. By Keisler-Shelah, there is \mathcal{U} such that $\mathcal{M}^{\mathcal{U}} \cong \mathcal{M}_s^{\mathcal{U}}$. Note now that $\mathcal{M}^{\mathcal{U}} \equiv \mathcal{M}_t^{\mathcal{U}}$, whence there is \mathcal{V} such that $\mathcal{M}^{\mathcal{U}} \cong (\mathcal{M}_t^{\mathcal{U}})^{\mathcal{V}}$. We then have

$$(\mathcal{M}^{\mathcal{U}})^{\mathcal{V}} \cong (\mathcal{M}_t^{\mathcal{U}})^{\mathcal{V}} \cong ((\mathcal{M}_s^{\mathcal{U}})_t)^{\mathcal{V}} = (\mathcal{M}_{st}^{\mathcal{U}})^{\mathcal{V}} \cong ((\mathcal{M}_{st})^{\mathcal{U}})^{\mathcal{V}},$$

whence it follows that $\mathcal{M} \equiv \mathcal{M}_{st}$.

Unlike the ordinary fundamental group, the first-order fundamental group is a closed subgroup of \mathbb{R}_+ . Indeed, if (r_k) is a sequence from \mathbb{R}_+ with limit $r \in \mathbb{R}_+$, it is easy to verify that $\prod_{\mathcal{U}} \mathcal{M}_{r_k} \cong \mathcal{M}_r^{\mathcal{U}}$ for any nonprincipal ultrafilter \mathcal{U} on \mathbb{N} ; if each $r_k \in \mathcal{F}_{\text{fo}}(\mathcal{M})$, then $\prod_{\mathcal{U}} \mathcal{M}_{r_k} \equiv \mathcal{M}$, whence it follows that $r \in \mathcal{F}_{\text{fo}}(\mathcal{M})$.

In summary:

Proposition 4.14. $\mathcal{F}_{\text{fo}}(\mathcal{M})$ is a closed subgroup of \mathbb{R}_+ containing $\mathcal{F}(\mathcal{M})$.

Question 4.15. Does there exist a separable II_1 factor \mathcal{M} for which $\mathcal{F}_{\text{fo}}(\mathcal{M}) \neq \mathbb{R}_+$?

Recall that II_1 factors \mathcal{M} and \mathcal{N} are said to be *stably isomorphic* if $\mathcal{M} \cong \mathcal{N}_t$ for some $t \in \mathbb{R}_+$. So the above question is equivalent to the question: does stable isomorphism imply elementary equivalence? Since all of the free group factors are stably isomorphic, a special case of the above question is whether or not all of the free group factors are elementarily equivalent (a question Thomas Sinclair has called the *noncommutative Tarski problem*).

In connection with the number of theories of II_1 factors, we have:

Proposition 4.16. Suppose that \mathcal{M} is a separable II_1 factor with $\mathcal{F}_{\text{fo}}(\mathcal{M}) \neq \mathbb{R}_+$. Then

$$|\{\text{Th}(\mathcal{M}_t) : t \in \mathbb{R}_+\}| = 2^{\aleph_0}.$$

Proof. Since the map $t\mathcal{F}_{\text{fo}}(\mathcal{M}) \mapsto \text{Th}(\mathcal{M}_t)$ is injective, the result follows from the fact that closed subgroups of \mathbb{R}_+ are countable. \square

It seems very unlikely that $\mathcal{F}_{\text{fo}}(\mathcal{M}) = \mathbb{R}_+$ for all separable II_1 factors \mathcal{M} . In fact, it seems very unlikely that $\mathcal{M} \equiv \mathcal{M}_2(\mathcal{M})$ for all separable II_1 factors \mathcal{M} . Let $\mathcal{F}_{\forall}(\mathcal{M}) := \{t \in \mathbb{R}_+ : \mathcal{M} \equiv_{\forall} \mathcal{M}_t\}$. Of course, if CEP holds, then $\mathcal{F}_{\forall}(\mathcal{M}) = \mathbb{R}_+$ for any II_1 factor \mathcal{M} , so what follows is only interesting if CEP fails.

Proposition 4.17. *The following statements are equivalent:*

- (1) *If \mathcal{M} is existentially closed, then \mathcal{M} is McDuff.*
- (2) *If \mathcal{M} is existentially closed, then $2 \in \mathcal{F}_\forall(\mathcal{M})$.*
- (3) *For any II_1 factor \mathcal{M} , $2 \in \mathcal{F}_\forall(\mathcal{M})$.*
- (4) *For any II_1 factor \mathcal{M} , $\mathcal{M} \equiv_\forall \mathcal{M} \otimes \mathcal{R}$.*
- (5) *For any II_1 factor \mathcal{M} , $\mathcal{F}_\forall(\mathcal{M}) = \mathbb{R}_+$.*

In the statement of the proposition, when we say that \mathcal{M} is existentially closed, we mean that \mathcal{M} is an existentially closed model of its theory.

Proof of Proposition 4.17. Since McDuff II_1 factors have full fundamental group, (1) implies (2) is trivial. (2) implies (3) follows from the fact that $\mathcal{M} \equiv_\forall \mathcal{N}$ implies $\mathcal{F}_\forall(\mathcal{M}) = \mathcal{F}_\forall(\mathcal{N})$. (3) implies (4) follows from the fact that $\mathcal{M} \otimes \mathcal{R}$ embeds into $\prod_{\mathcal{U}} M_{2^n}(\mathcal{M})$. Now suppose that (4) holds and fix an arbitrary II_1 factor \mathcal{M} . Since $\mathcal{M} \otimes \mathcal{R}$ is McDuff, for any $t \in \mathbb{R}_+$ we have that

$$\mathcal{M} \equiv_\forall \mathcal{M} \otimes \mathcal{R} \cong (\mathcal{M} \otimes \mathcal{R})_t \equiv_\forall \mathcal{M}_t,$$

whence (5) holds. Finally assume that (5) holds and assume that \mathcal{M} is existentially closed. By considering the chain

$$\mathcal{M} \subseteq M_2(\mathcal{M}) \subseteq M_4(\mathcal{M}) \subseteq M_8(\mathcal{M}) \subseteq \dots$$

and noting that each element of the chain has the same universal theory as \mathcal{M} by (5), we see that \mathcal{M} is existentially closed in the union $\mathcal{M} \otimes \mathcal{R}$. Since $\mathcal{M} \otimes \mathcal{R}$ is McDuff and being McDuff is $\forall\exists$ -axiomatizable, we have that \mathcal{M} is McDuff as well. \square

Note that it is not always true that $\mathcal{M} \equiv_{\forall\exists} \mathcal{M} \otimes \mathcal{R}$ (e.g. when \mathcal{M} is not McDuff).

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