COMPUTABILITY AND THE CONNES EMBEDDING PROBLEM

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Abstract. The Connes Embedding Problem (CEP) asks whether every separable II_1 factor embeds into an ultrapower of the hyperfinite II_1 factor. We show that the CEP is equivalent to the statement that every type II_1 tracial von Neumann algebra has a computable universal theory.

1. Introduction

With the advent of continuous model theory, model theoretic studies of operator algebras began in earnest (see [11, 12]). The class of C*-algebras and tracial von Neumann algebras were seen to be elementary classes as were many interesting subclasses - II_1 factors, particular von Neumann algebras, being one such; see [10] for other examples. Naturally one would look at existentially closed models in these classes ([8]) and attempt to identify model complete theories and theories with significant quantifier simplification. The results in this direction have been decidably negative. For tracial von Neumann algebras, it is known that there is no model companion ([7]); in fact, no known theory of II_1 factors is model complete ([8]). For C*-algebras, the only theory of an infinite-dimensional algebra which has quantifier elimination is the entirely atypical theory of $C(X)$, continuous functions from X into $\mathbb{C}$ where X is Cantor space ([5]). It follows that the class of C*-algebras does not have a model companion.

Most of the work to date in this area has focused on identifying useful elementary properties, classes and their axioms. The axioms for all known elementary classes have been recursive and indeed of low quantifier complexity ($\forall\exists$-axiomatizable or better). In this paper, we would like to address the other end of the spectrum: we want to consider the possibility that the theories of even iconic operator algebras such as $\mathcal{R}$ are not computable. By computable, in the continuous setting, we mean: is there an algorithm such that given a sentence φ and $\epsilon > 0$, the algorithm successfully computes $\varphi^{\mathcal{R}}$ to within ϵ?

To apparently make it easier, we ask if the universal theory of $\mathcal{R}$ is computable. We find ourselves at the doorstep of one of the most celebrated problems in the theory of operator algebras, the Connes embedding problem

1991 *Mathematics Subject Classification.* 46L10, 03C65, 03D45.

Goldbring's work was partially supported by NSF grant DMS-1262210.

(CEP): does every separable II_1 factor embed into an ultrapower of $\mathcal{R}$? In [12], it is shown that CEP is equivalent to the logical statement that every II_1 factor has the same *universal theory* as $\mathcal{R}$. In this paper, we show that CEP is equivalent to the statement that every type II_1 tracial von Neumann algebra has a computable universal theory. Although this result does not resolve the question of the computability of the theory of $\mathcal{R}$, we view this result as cautionary. CEP is a well-studied problem (see the survey [3] for many equivalences) and its resolution would only deal with the issue of the universal theory if one believes that $\mathrm{Th}(\mathcal{R})$ is computable.

We would like to thank David Sherman for a helpful conversation regarding this project.

2. Prerequisites from von Neumann Algebras

The study of what are now called von Neumann algebras began in the 1930's with the work of Murray and von Neumann [9] and was motivated by von Neumann's work in the foundations of quantum mechanics. In this section, we recall the basic definitions needed from the theory of von Neumann algebras. If H is a complex Hilbert space, $\mathcal{B}(H)$ denotes the set of bounded operators on H. $\mathcal{B}(H)$ has the structure of a unital $*$-algebra and a unital $*$-subalgebra A of $\mathcal{B}(H)$ is said to be a *von Neumann algebra* if A is closed in the strong operator topology, which is the weakest topology on $\mathcal{B}(H)$ that makes, for each $x \in H$, the map $T \mapsto \|T(x)\| : \mathcal{B}(H) \to \mathbb{C}$ continuous.

Example 2.1. For any Hilbert space H, $\mathcal{B}(H)$ is a von Neumann algebra. In particular, if H is n-dimensional, then $\mathcal{B}(H)$ in this case is isomorphic to $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over $\mathbb{C}$.

Of particular importance in von Neumann algebras are the *projections*. If A is a von Neumann algebra then $p \in A$ is a projection if $p^2 = p = p^*$. Any von Neumann algebra is generated by its projections. The role of von Neumann algebras and projections in the study of representation theory and invariant subspaces is highlighted by the following example taken from [6].

Example 2.2. Suppose that Γ is a group and H is a Hilbert space. Consider $U(H)$, the unitary group of $\mathcal{B}(H)$; that is, the set of $u \in B(H)$ such that $u^* = u^{-1}$. Fix a group homomorphism $f : \Gamma \to U(H)$; f is called a *unitary representation* of Γ. An important object of study in representation theory are subspaces of H which are invariant under the representation f. We say that a closed subspace $K \subseteq H$ is f-invariant if for all $\gamma \in \Gamma$, $f(\gamma)(K) \subseteq K$. Now if K is any f-invariant closed subspace and p is the orthogonal projection of H onto K then a small calculation (see section 2.2 of [6]) shows that the projection p commutes with $f(\gamma)$ for all $\gamma \in \Gamma$. Conversely, if p is any projection which commutes with all elements of $f(\Gamma)$, then the associated closed subspace of H is f-invariant. In this way, the study of f-invariant closed subspaces of H becomes the study of projections in the von Neumann algebra

$$f(\Gamma)' = \{a \in B(H) : a \text{ commutes with } f(\gamma) \text{ for all } \gamma \in \Gamma\},$$

known as the commutant of $f(\Gamma)$. More about the relationship between representation theory and von Neumann algebras can be found in [13].

If A is a von Neumann algebra, then a *trace* on A is a linear functional $\mathrm{tr} : A \to \mathbb{C}$ such that

(1) $\mathrm{tr}(1) = 1$;
(2) tr is positive, that is, for all $a \in A, \mathrm{tr}(a^*a) \geq 0$;
(3) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$;
(4) tr is faithful, that is, $\mathrm{tr}(x) = 0$ if and only if $x = 0$; and
(5) tr is normal, that is, whenever (p_α) is a collection of mutually orthogonal projections from A with join $\bigvee p_\alpha$, then $\mathrm{tr}(\bigvee p_\alpha) = \sum \mathrm{tr}(p_\alpha)$.

If tr is a trace on A, then tr induces a norm on A, called the *2-norm*, defined by $\|x\|_2 := \sqrt{\mathrm{tr}(x^*x)}$; A is then called *separable* if it is separable in the topology induced by the 2-norm.

Example 2.3. In the case of $M_n(\mathbb{C})$, there is a trace which is the normalized version of the usual trace on matrices, that is, if $A = (a_{ij})$ then $\mathrm{tr}(A) = \frac{1}{n}\sum_i a_{ii}$.

Example 2.4. Here is a more general example related to our comments about representation theory above. Suppose that Γ is a group. Let H be the Hilbert space formally generated by an orthogonal basis ζ_h for all $h \in \Gamma$. For any $g \in \Gamma$, define $u_g := f(g)$ to be the linear operator on H determined by $u_g(\zeta_h) = \zeta_{gh}$ for all $h \in \Gamma$. Notice that $f(g)$ is unitary for all $g \in \Gamma$ (since $u_g^* = u_g^{-1} = u_{g^{-1}}$) and so f is a unitary representation of Γ; it is called the *left regular representation*. The von Neumann algebra generated by $f(\Gamma)$ is called the group von Neumann algebra $L(\Gamma)$. Any element of $L(\Gamma)$ has the form $\sum_{g\in\Gamma} c_g u_g$ for complex numbers c_g (NB: not all such expressions define elements of $L(\Gamma)$). One can define a trace on $L(\Gamma)$ as follows:

$$\mathrm{tr}(\sum_{g\in\Gamma} c_g u_g) = c_e$$

where e is the identity in Γ.

If A is a von Neumann algebra, then the *center of* A is the set $Z(A) := \{x \in A \ : \ xy = yx \text{ for all } y \in A\}$. The center of A is a $*$-subalgebra of A and thus contains (a copy of) $\mathbb{C}$. A is said to be a *factor* if the center of A is as small as possible, that is, when $Z(A) = \mathbb{C}$. $\mathcal{B}(H)$ is an example of a factor for any Hilbert space H. One can check that $L(\Gamma)$ is a factor if and only if every nontrivial conjugacy class in Γ is infinite.

A *tracial von Neumann algebra* is a pair (A, tr), where A is a von Neumann algebra and tr is a trace on A. We will often suppress mention of the trace and simply say "Let A be a tracial von Neumann algebra..." It is a fact that a factor admits at most one trace, so this abuse in notation should cause no confusion in the case of factors.

An infinite-dimensional factor that admits a trace is called a *II_1 factor*. Of particular importance in this paper is the *hyperfinite II_1 factor $\mathcal{R}$*, which can be described as follows. The map

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

which sends $M_{2^n}(\mathbb{C})$ to $M_{2^{n+1}}(\mathbb{C})$ is an embedding of tracial von Neumann algebras; by definition, $\mathcal{R}$ is the inductive limit of these embeddings. The trace on $\mathcal{R}$ is induced by the normalized traces on the matrix algebras. It is a fact that $\mathcal{R}$ embeds into any II_1 factor. In general, a tracial von Neumann algebra is said to be of type II_1 if it contains a copy of $\mathcal{R}$.

There is an ultraproduct construction for tracial von Neumann algebras; for the details on this tracial ultraproduct, see [13]. If A is a tracial von Neumann algebra and ω is a nonprincipal ultrafilter on $\mathbb{N}$, we let A^ω denote the tracial ultrapower of A. We say that a separable II_1 factor is *embeddable* if it embeds into $\mathcal{R}^\omega$ for some (equivalently, any) nonprincipal ultrapower on $\mathbb{N}$. As alluded to in the introduction, the Connes Embedding Problem (CEP) asks whether or not every separable II_1 factor is embeddable.

3. Prerequisites from Logic

In this section, we describe an appropriate language in continuous logic for studying tracial von Neumann algebras.

For a von Neumann algebra A, we let A_1 denote the operator norm unit ball. Let $\mathcal{F}$ denote the set of all *-polynomials $p(x_1, \ldots, x_n)$ ($n \geq 0$) such that, for *any* von Neumann algebra A, we have $p(A_1^n) \subseteq A_1$. For example, the following functions belong to $\mathcal{F}$:

- the "constant symbols" 0 and 1 (thought of as 0-ary functions);
- $x \mapsto x^*$;
- $x \mapsto \lambda x$ ($|\lambda| \leq 1$);
- $(x, y) \mapsto xy$;
- $(x, y) \mapsto \frac{x+y}{2}$.

We then work in the *language* $\mathcal{L} := \mathcal{F} \cup \{\mathrm{tr}_{\mathcal{R}}, \mathrm{tr}_{\Im}, d\}$, where $\mathrm{tr}_{\Re}$ (resp. $\mathrm{tr}_{\Im}$) denote the real (resp. imaginary) parts of the trace and d denotes the metric on A_1 given by $d(x, y) := \|x - y\|_2$. We can then formulate certain properties of tracial von Neumann algebras using the language $\mathcal{L}$ as follows.

Basic $\mathcal{L}$-formulae will be formulae of the form $\mathrm{tr}_{\Re}(p(\vec{x}))$ or $\mathrm{tr}_{\Im}(p(\vec{x}))$ for $p \in \mathcal{F}$. *Quantifier-free $\mathcal{L}$-formulae* are formulae of the form $f(\varphi_1, \ldots, \varphi_m)$, where $f : \mathbb{R}^m \to \mathbb{R}$ is a continuous function and $\varphi_1, \ldots, \varphi_m$ are basic $\mathcal{L}$-formulae. Finally, an arbitrary $\mathcal{L}$-formula is of the form

$$Q^1_{x_1 \in A_1} \cdots Q^k_{x_k \in A_1} \varphi(x_1, \ldots, x_n),$$

where $k \leq n$, $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula, and each Q^i is either sup or inf; we think of these Q_i's as *quantifiers over the unit ball of the algebra.*

Remarks 3.1.

(1) Our setup here is a bit more specialized than the general treatment of continuous logic in [1], but a dense set of the formulae in [1] are logically equivalent to formulae in the above form, so there is no loss of generality in our treatment here.

(2) In order to keep the set of formulae "separable", when forming the set of quantifier-free formulae, we restrict ourselves to a countable dense subset of the set of all continuous functions $\mathbb{R}^m \to \mathbb{R}$ as m ranges over $\mathbb{N}$. In fact, one can take this countable dense set to be "finitely generated" which is important for our computability-theoretic considerations. (See [2].)

Suppose that $\varphi(\vec{x})$ is a formula, A is a tracial von Neumann algebra, and $\vec{a} \in A_1^n$, where n is the length of the tuple $\vec{x}$. We let $\varphi(\vec{a})^A$ denote the real number obtained by replacing the variables $\vec{x}$ with the tuple $\vec{a}$; we may think of $\varphi(\vec{a})^A$ as the truth value of $\varphi(\vec{x})$ in A when $\vec{x}$ is replaced by $\vec{a}$. For example, if $\varphi(x_1)$ is the formula $\sup_{x_2} d(x_1x_2, x_2x_1)$, then $\varphi(a)^A = 0$ if and only if a is in the center of A.

If φ has no *free variables* (that is, all variables occurring in φ are bounded by some quantifier), then we say that φ is a *sentence* and we observe that φ^A is a real number. Given a tracial von Neumann algebra, the *theory of* A is the function $\mathrm{Th}(A)$ which maps the sentence φ to the real number φ^A. Sometimes authors define $\mathrm{Th}(A)$ to consists of the set of sentences φ for which $\varphi^A = 0$; since $\mathrm{Th}(A)$, as we have defined it, is determined by its zeroset, these two formulations are equivalent.

If $\varphi(\vec{x})$ is a formula, then there is a bounded interval $[m_\varphi, M_\varphi] \subseteq \mathbb{R}$ called the range of φ such that, for any tracial von Neumann algebra A and any $\vec{a} \in A$, we have $\varphi(\vec{a})^A \in [m_\varphi, M_\varphi]$. A sentence of the form $\sup_{x_1 \in A_1} \cdot \sup_{x_n \in A_1} \varphi$ is called universal if the range of φ is non-negative and similarly existential if all the quantifiers are inf. This terminology is justified if one thinks of the value 0 as "true". If we restrict the function $\mathrm{Th}(A)$ to the set of all universal (resp. existential) sentences, the resulting function is defined to be the *universal* (resp. *existential*) theory of A, denoted $\mathrm{Th}_\forall(A)$ (resp. $\mathrm{Th}_\exists(A)$). We should also mention that, as a consequence of *Łos' theorem* (and the fact that the tracial ultraproduct construction is the continuous logic ultraproduct construction), we have $\mathrm{Th}(A) = \mathrm{Th}(A^\omega)$ for any ultrafilter ω.

Remark 3.2. In what follows, we will restrict ourselves to $\mathcal{L}$-structures that are tracial von Neumann algebras. We can do this because it is shown in [11] that the class of (unit balls of) tracial von Neumann algebras forms a universally axiomatizable class of $\mathcal{L}$-structures.

Let T be a set of $\mathcal{L}$-sentences. We say that a tracial von Neumann algebra A *models* T, written $A \models T$, if $\varphi^A = 0$ for each $\varphi \in T$. It is shown in [11] that there is a set T_{II_1} of $\mathcal{L}$-sentences such that $A \models T_{II_1}$ if and only if A is a II_1 factor. In fact, by examining T_{II_1}, one can show that there is a *recursively enumerable* such set T_{II_1}, meaning that there is an algorithm that runs forever and continually returns the axioms of T_{II_1}. The aforementioned observation will be crucial for what is to follow and so we isolate it:

Fact 3.3. The class of II_1 factors has a recursively enumerable axiomatization.

Up until now, we have been treating tracial von Neumann algebras *semantically*. It will be crucial to also treat them *syntactically*. In [2], a *proof system* for continuous logic is established. In our context, this gives meaning to the phrase "the axioms T_{II_1} can prove the sentence σ," which we denote $T_{II_1} \vdash \sigma$.

Fact 3.4. The set $\{\sigma \ : \ T_{II_1} \vdash \sigma\}$ is *recursively enumerable.*

Proof. This follows immediately from the existence of the proof system developed in [2] together with Fact 3.3. □

There is a connection between the semantic and syntactic treatments developed above (which [2] refers to as "Pavelka-style completeness"). Let $\dot{-} : \mathbb{R}^2 \to \mathbb{R}$ be the function $x \dot{-} y := \max(x - y, 0)$ and let $\mathbb{D}$ denote the set of dyadic rational numbers.

Fact 3.5. ([2, Corollary 9.8]) For a sentence φ, we have

$$\sup\{\varphi^A \ : \ A \models T_{II_1}\} = \inf\{r \in \mathbb{D}^{>0} \ : \ T_{II_1} \vdash \varphi \dot{-} r\}.$$

We denote this common value by $\varphi_{T_{II_1}}$.

Remark 3.6. By Downward Löwenheim-Skolem, every tracial von Neumann algebra has a separable subalgebra with the same theory. Consequently, we have that

$$\varphi_{T_{II_1}} = \sup\{\varphi^A \ : \ A \models T_{II_1} \text{ and } A \text{ is separable}\}.$$

CEP and Model Theory: At this point, it is convenient to recall the connection between CEP and model theory. If A, B are tracial von Neumann algebras and A is a subalgebra of B, then $\mathrm{Th}_\forall(A) \leq \mathrm{Th}_\forall(B)$ (as functions). Since $\mathcal{R}$ embeds into any II_1 factor, we have that $\mathrm{Th}_\forall(\mathcal{R}) \leq \mathrm{Th}_\forall(A)$ for every II_1 factor A. If A is an embeddable II_1 factor, then certainly $\mathrm{Th}_\forall(A) \leq \mathrm{Th}_\forall(\mathcal{R})$ (as $\mathrm{Th}(\mathcal{R}) = \mathrm{Th}(\mathcal{R}^\omega)$). Conversely, suppose that A is a separable tracial von Neumann algebra such that $\mathrm{Th}_\forall(A) \leq \mathrm{Th}_\forall(\mathcal{R})$. It is then a standard fact of model theory that A is embeddable. We thus see that CEP is equivalent to the statement that, for every II_1 factor A, we have that $\mathrm{Th}_\forall(A) = \mathrm{Th}_\forall(\mathcal{R})$. (Actually, we just saw that CEP is equivalent to the statement that $\mathrm{Th}_\forall(A) = \mathrm{Th}_\forall(\mathcal{R})$ for every separable type II_1 tracial von Neumann algebra.) As a side remark, note that, for tracial von Neumann algebras A and B, we have $\mathrm{Th}_\forall(A) = \mathrm{Th}_\forall(B)$ if and only if $\mathrm{Th}_\exists(A) = \mathrm{Th}_\exists(B)$, which is easily seen to be equivalent to the operator algebraic conjecture known as the Microstate Conjecture.

4. CEP implies Computability

In this section, we assume that CEP holds. For ease of notation, we set $T := T_{II_1}$.

Lemma 4.1. *Suppose that σ is universal. Then $\sigma_T = \sigma^{\mathcal{R}}$.*

Proof. By definition, $\sigma^{\mathcal{R}} \leq \sigma_T$. Now fix a separable II_1 factor M; we must show $\sigma^M \leq \sigma^{\mathcal{R}}$. This follows immediately from the fact that M is embeddable. □

Lemma 4.2. *Suppose that σ is existential. Then $\sigma_T = \sigma^{\mathcal{R}}$.*

Proof. Again, it suffices to show that $\sigma^M \leq \sigma^{\mathcal{R}}$ for arbitrary $M \models T$. But this follows from the fact that M contains a copy of $\mathcal{R}$. □

Corollary 4.3. *If σ is a universal sentence, then $(M_\sigma \dot{-} \sigma)_T = M_\sigma \dot{-} \sigma_T$.*

Proof. Observe that $M_\sigma \dot{-} \sigma$ is logically equivalent to an existential sentence. Using the previous two lemmas, we have

$$(M_\sigma \dot{-} \sigma)_T = (M_\sigma \dot{-} \sigma)^{\mathcal{R}} = M_\sigma \dot{-} \sigma^{\mathcal{R}} = M_\sigma \dot{-} \sigma_T. \quad \square$$

If A is a tracial von Neumann algebra, we say that $\mathrm{Th}_\forall(A)$ is *computable* if there is an algorithm such that, upon inputs universal sentence σ and positive dyadic rational number ϵ, returns an interval $I \subseteq \mathbb{R}$ of length at most ϵ with dyadic rational endpoints such that $\sigma^A \in I$. One defines $\mathrm{Th}_\exists(A)$ being computable in an analogous way.

Remark 4.4. This is **not** the same notion of computable theory as defined in [2] but is more appropriate for our needs.

Corollary 4.5. $\mathrm{Th}_\forall(\mathcal{R})$ *and* $\mathrm{Th}_\exists(\mathcal{R})$ *are computable.*

Proof. Here is the algorithm: given universal σ and positive dyadic rational ϵ, run all proofs from T and wait until you see that $T \vdash \sigma \dot{-} r$ and $T \vdash (M_\sigma \dot{-} \sigma) \dot{-} s$ where $r - (M_\sigma - s) \leq \epsilon$. By the previous corollary, this algorithm will eventually halt and the interval $[M_\sigma - s, r]$ will be the desired interval. □

5. Computability implies CEP

Recall that if CEP is false, then there are at least two distinct universal (equivalently existential) theories of type II_1 algebras. In fact:

Proposition 5.1. *Suppose that CEP fails. Then there are continuum many different universal (equivalently existential) theories of type II_1 algebras. In fact, there is a single existential sentence σ such that σ^M takes on continuum many values as M ranges over all type II_1 algebras.*

Proof. For $N \in \mathbb{N}$, A a type II_1 algebra, a a tuple from A, and $\epsilon > 0$, let $\sigma_{N,A,a,\epsilon}$ be the existential sentence

$$\inf_x \left(\max_{\deg p \leq N} \max(|\operatorname{tr}_{\Re}(p(x)) - \operatorname{tr}_{\Re}(p(a))|, |\operatorname{tr}_{\Im}(p(x)) - \operatorname{tr}_{\Im}(p(a))|) \right) \dot{-} \epsilon.$$

Here the max is over all *-monomials p of degree at most N with complex coefficient 1. Since CEP fails, there are N, A, a, and $\epsilon > 0$ such that $\sigma^{\mathcal{R}}_{N,A,a,\epsilon} > 0$. (Of course $\sigma^{A}_{N,A,a,\epsilon} = 0$.) For simplicity, set $\sigma := \sigma_{N,A,a,\epsilon}$ and $r := \sigma^{\mathcal{R}}$. For each $t \in [0,1]$, set $A_t := t\mathcal{R} \oplus (1-t)A$, which denotes the direct sum of $\mathcal{R}$ and A with trace $\operatorname{tr}_t := t \operatorname{tr}_{\mathcal{R}} + (1-t) \operatorname{tr}_A$. Note that each A_t is a type II_1 algebra and the map $t \mapsto \sigma^{A_t} : [0,1] \to \mathbb{R}$ is continuous. Since $\sigma^{A_0} = 0$ and $\sigma^{A_1} = r$, the proof of the proposition is complete. □

Corollary 5.2. *Suppose that the universal theory of every type II_1 algebra is computable. Then CEP holds.*

Proof. Suppose that CEP fails. By the previous lemma, there are uncountably many universal theories of type II_1 algebras. But there are only countably many programs that could be computing universal theories of type II_1 algebras, whence not every type II_1 algebra has a computable universal theory. □

6. Further computability-theoretic consequences of the CEP

In this section, we assume that CEP holds and we derive some further computability-theoretic results. Unlike Section 4, in this section, we let T denote the set of sentences whose models are the tracial von Neumann algebras (see Remark 3.2).

Fix a separable II_1 factor A with enumerated subset $X = (a_0, a_1, a_2, \ldots)$ that generates A (as a von Neumann algebra). We now pass to a language $\mathcal{L}_X$ containing $\mathcal{L}$ obtained by adding to $\mathcal{L}$ new constant symbols for each a_i. We now add to T sentences of the form $\max(r_n \dot{-} f(\vec{a}), f(\vec{a}) \dot{-} s_n)$, where $f \in \mathcal{F}$ and (r_n, s_n) is a sequence of intervals of dyadic rationals containing $f(\vec{a})$ with $s_n - r_n \to 0$; we call the resulting theory $T_{(A,X)}$. (In model-theoretic lingo: we are just adding the *atomic diagram* of A to T.) Note that a model of $T_{(A,X)}$ is a tracial von Neumann algebra B whose interpretations of the new constants generate a von Neumann subalgebra of B isomorphic to A.

We say that (A, X) as above is *recursively presented* if there is an algorithm that enumerates each sequence of intervals (r_n, s_n) for each $f \in \mathcal{F}$. It is a standard construction in recursion theory to code a recursively presented tracial von Neumann algebra (A, X) by a single natural number, which we refer to as the *Gödel code* of (A, X).

Fix a recursively presented II_1 factor (A, X). Suppose that $\sigma = \sup_x \varphi(x)$ is a universal sentence and ϵ is a positive dyadic rational. Then clearly there is $n \in \mathbb{N}$ such that $\sigma^A \leq \max_{i \leq n} \varphi(a_i)^A + \epsilon$; we will say that such an n is *good for* (A, X, σ, ϵ). Consider the following algorithmic question: is there a way of computably determining some n that is good for (A, X, σ, ϵ)? The next result tells us that CEP implies that there is a *single* algorithm that works for all recursively presented (A, X) and all σ and ϵ.

Theorem 6.1. *There is a computable partial function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{D}^{>0} \rightharpoonup \mathbb{N}$ such that, if e is the Gödel code of a recursively presented separable II_1 factor (A, X) and n is the Gödel code of a universal sentence $\sigma = \sup_x \varphi(x)$, then $f(m, n, \epsilon)$ is good for (A, X, σ, ϵ).*

Proof. Here is the algorithm for determining $f(m, n, \epsilon)$. First, use the computability of $\mathrm{Th}_\forall(\mathcal{R})$ to determine an interval $I = [c, d] \subseteq \mathbb{R}$ with $|I| \leq \frac{\epsilon}{2}$ such that $\sigma^{\mathcal{R}} \in I$. By CEP, $\sigma^{\mathcal{R}} = \sigma^A$. We claim that there is an N such that $c - \frac{\epsilon}{2} \leq \varphi(a_N)^A$. Indeed, there is N such that $\sigma^A - \frac{\epsilon}{2} \leq \varphi(a_N)^A$. For such an N, we have that $c - \frac{\epsilon}{2} \leq \varphi(a_N)^A \leq \sigma^A \leq d$ and $d - (c - \frac{\epsilon}{2}) \leq \epsilon$, whence N is good for (A, X, σ, ϵ). Now we just start computing $\varphi(a_i)^A$ (which we can do since (A, X) is recursively presented) and wait until we reach N with $c - \frac{\epsilon}{2} \leq \varphi(a_N)^A$. □

Note that there is a countable $X \subseteq \mathcal{R}$ such that $(\mathcal{R}, X)$ is recursively presented. In the rest of this paper, we fix such an X and let $T_{\mathcal{R}} := T_{(\mathcal{R},X)}$ and let $\mathcal{R}_X$ denote the obvious expansion of $\mathcal{R}$ to an $\mathcal{L}_X$-structure.

In the next proof, we will need the following fact (see [7, Lemma 3.1]):

Fact 6.2. For any nonprincipal ultrafilter ω on $\mathbb{N}$, any embedding $h : \mathcal{R} \to \mathcal{R}^\omega$ is *elementary*, that is, for any formula $\varphi(\vec{x})$, and any tuple $\vec{a} \in \mathcal{R}$, we have $\varphi^{\mathcal{R}}(\vec{a}) = \varphi^{\mathcal{R}^\omega}(h(\vec{a}))$.

Lemma 6.3. *Suppose that σ is a universal or existential $\mathcal{L}_X$-sentence. Then $\sigma_{T_{\mathcal{R}}} = \sigma^{\mathcal{R}_X}$.*

Proof. As in Section 4, we need only show that $\sigma^M \leq \sigma^{\mathcal{R}_X}$ for every $M \models T_{\mathcal{R}}$. First suppose that σ is existential, say $\sigma = \inf_x \varphi(c_a, x)$, where a is a tuple from X and c_a is the corresponding tuple of constants. Let $i : \mathcal{R} \to M$ be the embedding of $\mathcal{R}$ into M determined by setting $i(a) := c_a^M$ for every $a \in X$. Then

$$\sigma^M = \inf\{\varphi(i(a), b)^M \ : \ b \in M\} \leq \inf\{\varphi(i(a), i(d))^M \ : \ d \in \mathcal{R}\} = \sigma^{\mathcal{R}_X}.$$

Now suppose that σ is universal, say $\sigma = \sup_x \varphi(c_a, x)$. Fix an embedding $j : M \to \mathcal{R}^\omega$. Then

$$\sigma^M = \sup\{\varphi(i(a), b)^M \ : \ b \in M\} \leq \sup\{\varphi(ji(a), d)^{\mathcal{R}^{\mathcal{U}}} \ : \ d \in \mathcal{R}^{\mathcal{U}}\} = \sigma^{\mathcal{R}_X},$$

since $ji : \mathcal{R} \to \mathcal{R}^\omega$ is elementary. □

Corollary 6.4. $\mathrm{Th}_\forall(\mathcal{R}_X)$ *and* $\mathrm{Th}_\exists(\mathcal{R}_X)$ *are computable.*

Proof. This follows from the previous lemma just as in Section 4. □

Define $\mathrm{Th}_{\exists\forall}(\mathcal{R})$ to be the restriction of $\mathrm{Th}(\mathcal{R})$ to the set of formulae of the form

$$Q^1_{x_1 \in A_1} \cdots Q^k_{x_k \in A_1} \varphi(x_1, \ldots, x_n),$$

where φ is quantifier-free, $k \le n$, and such that there is $l \in \{1, \ldots, k\}$ such that $Q^i = \inf$ for $i \in \{1, \ldots, l\}$ and $Q^i = \sup$ for $i \in \{l+1, \cdots, k\}$.

We say that $\mathrm{Th}_{\exists\forall}(\mathcal{R})$ is *upper computably enumerable* if there is an algorithm that enumerates all sentences of the form $\sigma \dot{-} s$, where σ is an $\exists\forall$-sentence and s is a dyadic rational with $\sigma^{\mathcal{R}} < s$.

Corollary 6.5. $\mathrm{Th}_{\exists\forall}(\mathcal{R})$ *is upper computably enumerable.*

Proof. Consider (for simplicity) the sentence $\inf_x \sup_y \varphi(x, y)$. For each $a \in X$ and $\epsilon \in \mathbb{D}^{>0}$, use the previous corollary to find an interval $I = [r, s]$ with dyadic endpoints of length $\le \epsilon$ such that $\sup_y \varphi(a, y)^{\mathcal{R}} \in I$. We then add the condition $\inf_x \sup_y \varphi(a, y) \le s$ to our enumeration. We claim that this algorithm shows that $\mathrm{Th}_{\exists\forall}(\mathcal{R})$ is upper computably enumerable. Indeed, suppose that $\inf_x \sup_y \varphi(x, y) = s$. Fix $s' \in \mathbb{D}$, $s < s'$. Fix $\delta \in \mathbb{D}^{>0}$ such that $s + 2\delta < s'$. We claim that when the algorithm encounters $a \in X$ such that $\sup_y \varphi(a, y)^{\mathcal{R}} \le s + \delta$, our algorithm will let us know that $\inf_x \sup_y \varphi(x, y) \le s'$. Indeed, our algorithm will tell us that $\inf_x \sup_y \varphi(x, y) \le d$, where $d \in \mathbb{D}^{>0}$ and $d \le \sup_y \varphi(a, y)^{\mathcal{R}} + \delta$. $\square$

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