APPROXIMATE POLYNOMIAL STRUCTURE IN ADDITIVELY LARGE SETS

MAURO DI NASSO, ISAAC GOLDBRING, RENLING JIN, STEVEN LETH, MARTINO LUPINI, KARL MAHLBURG

ABSTRACT. We show that any subset of the natural numbers with positive logarithmic Banach density contains a set that is within a factor of two of a geometric progression, improving the bound on a previous result of the authors. Density conditions on subsets of the natural numbers that imply the existence of approximate powers of arithmetic progressions are developed and explored.

1. INTRODUCTION

In [1], the authors introduced a measure space, obtained by taking a quotient of a Loeb measure space, that has the property that multiplication is measure-preserving and for which standard sets of positive logarithmic density have positive measure. The log Banach density of a standard set (see Section 2 below for the definition) was also introduced, and this measure space framework was used, in conjunction with Furstenberg's Recurrence Theorem, to obtain a standard result about the existence of approximate geometric progressions in sets of positive log Banach density. In this paper, we improve the bounds of approximation of this result by using Szemerédi's Theorem together with a "logarithmic change of coordinates." More specifically, in Proposition 3.1, we show that if A is a standard subset of the natural numbers, then the Banach density of $\{\lceil \log_2(x) \rceil : x \in A\}$ is greater than or equal to the log Banach density of A. This allows us to use Szemerédi's Theorem to show that every set of positive Banach log density contains a set which is "within a factor of 2" of being a geometric sequence; Theorem 3.3 provides a precise version of this statement. We also explore a family of densities on the natural numbers, the (upper) r-Banach densities for $0 < r \leq 1$, which have the property that positive $1/m$-Banach density implies the existence of approximate mth powers of arithmetic progressions, in a sense made precise in Theorem 3.7. (This family of densities was introduced in [1], although $BD_m(A)$ in that paper corresponds to $BD_{1/m}(A)$ here.)

In Section 2 we establish some properties of the log Banach density and the r-Banach densities, most notably that the log Banach density of a set A is always less than or

equal to every r-Banach density of A, and that if $r < s$ then the r-Banach density of A is less than or equal to the s-Banach density of A (Theorem 2.12). These inequalities can both be strict. In fact, it is easy to see that the log Banach density of a set A could be 0 while every r-Banach density of A is 1, and in Example 2.13 we see that if $r < s$ then it is possible to have the r-Banach density of a set A be 0 while the s-Banach density of A is 1.

In Section 3 we establish the aforementioned approximation results and provide examples to show that the level of approximation is optimal.

We use nonstandard methods, which simplifies a number of the arguments. For an introduction to nonstandard methods aimed specifically toward applications to combinatorial number theory, see [3].

1.1. **Acknowledgements.** This work was initiated during a week-long meeting at the American Institute for Mathematics on August 4-8, 2014 as part of the SQuaRE (Structured Quartet Research Ensemble) project "Nonstandard Methods in Number Theory." The authors would like to thank the Institute for the opportunity and for the Institute's hospitality during their stay.

2. r-DENSITY AND LOGARITHMIC DENSITY

Convention 2.1. In this paper, $\mathbb{N}$ denotes the set of *positive* natural numbers. For any real numbers $a \leq b$, we set $[a, b] := \{x \in \mathbb{N} \; : \; a \leq x \leq b\}$. We make a similar convention for the intervals $(a, b]$, $[a, b)$, and (a, b).

We recall some well-known densities on $\mathbb{N}$.

Definition 2.2. Suppose that $A \subseteq \mathbb{N}$ and $0 < r \leq 1$.

- The *upper r-density of A* is defined to be

$$\overline{d}_r(A) := \limsup_{n \to \infty} \frac{r}{n^r} \sum_{x \in A \cap [1,n]} \frac{1}{x^{1-r}}.$$

- The *lower r-density of A* is defined to be

$$\underline{d}_r(A) := \liminf_{n \to \infty} \frac{r}{n^r} \sum_{x \in A \cap [1,n]} \frac{1}{x^{1-r}}.$$

Note that $\overline{d}_1(A)$ and $\underline{d}_1(A)$ are simply the usual upper and lower asymptotic densities of A, respectively. For that reason, we omit the subscript r when $r = 1$.

Definition 2.3. Suppose that $A \subseteq \mathbb{N}$. Then:

- The *upper logarithmic density of A* is defined to be

$$\overline{ld}(A) := \limsup_{n \to \infty} \frac{1}{\ln n} \sum_{x \in A \cap [1,n]} \frac{1}{x}.$$

- The *lower logarithmic density of A* is defined to be

$$\underline{ld}(A) := \liminf_{n\to\infty} \frac{1}{\ln n} \sum_{x \in A \cap [1,n]} \frac{1}{x}.$$

The following result establishing relationships amongst the above densities was proven in [2].

Fact 2.4. *For $A \subseteq \mathbb{N}$ and $0 < r < s \leq 1$, we have*

$$\underline{d}_s(A) \leq \underline{d}_r(A) \leq \underline{ld}(A) \leq \overline{ld}(A) \leq \overline{d}_r(A) \leq \overline{d}_s(A).$$

In working with these densities, we often use the following elementary estimates (established using an integral approximation): for any $a < b$ in $\mathbb{N}$, we have

$$\sum_{x=a}^{b-1} \frac{1}{x^{1-r}} \leq \frac{b^r - a^r}{r} \leq \sum_{x=a+1}^{b} \frac{1}{x^{1-r}}.$$

Theorem 2.5. *For $A \subseteq \mathbb{N}$ and $0 < r \leq 1$, we have*

$$\overline{d}_r(A) \geq 1 - \left(1 - \overline{d}(A)\right)^r.$$

Proof. Set $\alpha := \overline{d}(A)$ and take $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\frac{N}{H} \approx \alpha$, where $N := |{}^*A \cap [1, H]|$. Set $\epsilon := \frac{N}{H} - \alpha$, so ϵ is a (possibly negative) infinitesimal. We now have

$$
\begin{aligned}
\overline{d}_r(A) &\geq \mathrm{st}\left(\frac{r}{H^r} \sum_{x \in {}^*A \cap [1,H]} \frac{1}{x^{1-r}} \right) \\
&\geq \mathrm{st}\left(\frac{r}{H^r} \sum_{x \in (H-N,H]} \frac{1}{x^{1-r}} \right) \\
&\geq \mathrm{st}\left(\frac{r}{H^r} \cdot \frac{H^r - (H-N)^r}{r} \right) \\
&= \mathrm{st}(1 - (1 - (\alpha + \epsilon))^r) \\
&= 1 - (1 - \alpha)^r. \quad \square
\end{aligned}
$$

Corollary 2.6. *If $\overline{d}(A) > 0$, then $\overline{d}_r(A) > 0$ for all $0 < r \leq 1$.*

Remark 2.7. It is easy to construct a set $A \subseteq \mathbb{N}$ with $\overline{d}(A) = 1$ and $\overline{ld}(A) = 0$. As a consequence of the theorem above, we also have $\overline{d}_r(A) = 1$ for any $0 < r \leq 1$.

We now introduce the corresponding uniform versions of the above densities.

Definition 2.8. For $A \subseteq \mathbb{N}$ and $0 < r \leq 1$, the *(upper) r-Banach density* of A is defined to be

$$\mathrm{BD}_r(A) := \limsup_{n\to\infty} \sup_{k \in \mathbb{N}} \frac{r}{n} \sum_{x \in A \cap [k, (k^r+n)^{1/r}]} \frac{1}{x^{1-r}}.$$

Note that $\mathrm{BD}(A) = \mathrm{BD}_1(A)$ is the usual upper Banach density of A. Note also that $\overline{d}_r(A) \leq \mathrm{BD}_r(A)$ from definition.

Definition 2.9. For $A \subseteq \mathbb{N}$, the *(upper) log Banach density* of A is

$$\ell\mathrm{BD}(A) := \lim_{n \to \infty} \sup_{k \geq 1} \frac{1}{\ln n} \sum_{x \in A \cap [k, nk]} \frac{1}{x}.$$

Of course one could also define the lower r-Banach density and the lower log Banach density, but in this paper we only focus on the upper r-Banach density and upper log Banach density.

The following nonstandard formulation of r-Banach density and log Banach density follows immediately from the nonstandard characterization of limit.

Proposition 2.10. *Let $A \subseteq \mathbb{N}$, $0 < r \leq 1$, and $0 \leq \alpha \leq 1$.*

(1) $\mathrm{BD}_r(A) \geq \alpha$ *if and only if there are $k, N \in {}^*\mathbb{N}$ with $N > \mathbb{N}$ such that*

$$\mathrm{st}\left(\frac{r}{N} \sum_{x \in A \cap [k, (k^r + N)^{1/r}]} \frac{1}{x^{1-r}} \right) \geq \alpha.$$

(2) $\ell\mathrm{BD}(A) \geq \alpha$ *if and only if there are $k, N \in {}^*\mathbb{N}$ with $N > \mathbb{N}$ such that*

$$\mathrm{st}\left(\frac{1}{\ln N} \sum_{x \in {}^*A \cap [k, Nk]} \frac{1}{x} \right) \geq \alpha.$$

We now establish the uniform version of Fact 2.4 above. The results in [2] do not immediately apply in the uniform setting. Nevertheless, our proof is inspired by the arguments from [2], although we argue in the nonstandard model to make the idea more transparent.

Theorem 2.11. *For any $A \subseteq \mathbb{N}$ and $0 < r < s \leq 1$, we have*

$$\ell\mathrm{BD}(A) \leq \mathrm{BD}_r(A) \leq \mathrm{BD}_s(A).$$

Proof. We first prove that $\mathrm{BD}_r(A) \leq \mathrm{BD}_s(A)$.

Let $0 < \alpha < 1$ be such that $\mathrm{BD}_s(A) < \alpha$. It suffices to show that $\beta := \mathrm{BD}_r(A) \leq \alpha$. By Proposition 2.10, we can find $a, b \in {}^*\mathbb{N}$ such that $b^r - a^r > \mathbb{N}$ and

$$\mathrm{BD}_r(A) = \mathrm{st}\left(\left(\sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-r}} \right) \left(\sum_{n=a}^{b} \frac{1}{n^{1-r}} \right)^{-1} \right).$$

Here, χ_A denotes the characteristic function of (the nonstandard extension of) A. Next note that if $c, d \in {}^*\mathbb{N}$ are such that $d^s - c^s > \mathbb{N}$, then Proposition 2.10 once again implies that

$$\mathrm{st}\left(\left(\sum_{i=c}^{d} \frac{\chi_A(i)}{i^{1-s}} \right) \left(\sum_{i=c}^{d} \frac{1}{i^{1-s}} \right)^{-1} \right) \leq \mathrm{BD}_s(A) < \alpha.$$

Choose $m \in [a, b]$ such that $m^r - a^r > \mathbb{N}$ and

$$\left(\sum_{n=a}^{m} \frac{1}{n^{1-r}} \right) \left(\sum_{n=a}^{b} \frac{1}{n^{1-r}} \right)^{-1} \approx 0.$$

(For example, let $m = \lceil ((b^r - a^r)^{1/2} + a^r)^{1/r} \rceil$.) Since $x \mapsto x^s - x^r$ is an increasing function, we have that $m^s - a^s \geq m^r - a^r$. Hence $\sum_{n=a}^{i} \frac{\chi_A(n)}{n^{1-s}} < \alpha \sum_{n=a}^{i} \frac{1}{n^{1-s}}$ for any $i > m$. Now we have

$$\sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-r}} = \sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-s}} \frac{1}{n^{s-r}}$$

$$= \sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-s}} \left(\sum_{i=n}^{b} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right) + \frac{1}{(b+1)^{s-r}} \right)$$

$$= \sum_{n=a}^{b} \sum_{i=n}^{b} \frac{\chi_A(n)}{n^{1-s}} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right) + \sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-s}} \frac{1}{(b+1)^{s-r}}$$

$$= \sum_{i=a}^{b} \sum_{n=a}^{i} \frac{\chi_A(n)}{n^{1-s}} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right) + \sum_{n=a}^{b} \frac{\chi_A(n)}{n^{1-s}} \frac{1}{(b+1)^{s-r}}$$

$$< \alpha \sum_{i=a}^{b} \sum_{n=a}^{i} \frac{1}{n^{1-s}} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right) + \alpha \sum_{n=a}^{b} \frac{1}{n^{1-s}} \frac{1}{(b+1)^{s-r}}$$

$$+ (1-\alpha) \sum_{i=a}^{m} \sum_{n=a}^{i} \frac{1}{n^{1-s}} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right)$$

$$= \alpha \sum_{n=a}^{b} \frac{1}{n^{1-s}} \sum_{i=n}^{b} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right) + \alpha \sum_{n=a}^{b} \frac{1}{n^{1-s}} \frac{1}{(b+1)^{s-r}}$$

$$+ (1-\alpha) \sum_{n=a}^{m} \frac{1}{n^{1-s}} \sum_{i=n}^{m} \left(\frac{1}{i^{s-r}} - \frac{1}{(i+1)^{s-r}} \right)$$

$$= \alpha \sum_{n=a}^{b} \frac{1}{n^{1-s}} \frac{1}{n^{s-r}}$$

$$+ (1-\alpha) \sum_{n=a}^{m} \frac{1}{n^{1-s}} \frac{1}{n^{s-r}} - (1-\alpha) \sum_{n=a}^{m} \frac{1}{n^{1-s}} \frac{1}{(m+1)^{s-r}}$$

$$\leq \alpha \sum_{n=a}^{b} \frac{1}{n^{1-r}} + (1-\alpha) \sum_{n=a}^{m} \frac{1}{n^{1-r}}.$$

We conclude that $\beta \leq \alpha$ since $\left(\sum_{n=a}^{m} \frac{1}{n^{1-r}} \right) \left(\sum_{n=a}^{b} \frac{1}{n^{1-r}} \right)^{-1} \approx 0$.

In the arguments above, if we let $r = 0$ and instead require that $\ln(m) - \ln(a) > \mathbb{N}$, then we get $\ell BD(A) \leq BD_s(A)$. $\qquad\square$

It is easy to see that the set $A = \bigcup_{n=1}^{\infty} [n!, 2n!]$ has the property that $\ell BD(A) = 0$ while $BD_r(A) = 1$ for every $r \in (0, 1]$. The following example shows that the r-Banach densities can also disagree to this extent.

Example 2.12. *For any $0 < r < s \leq 1$ there is a set $A \subseteq \mathbb{N}$ such that $\mathrm{BD}_r(A) = 0$ and $\mathrm{BD}_s(A) = 1$.*

Proof. Let (a_n) be a sequence of positive integers defined by setting a_1 to be any integer larger than 1 and $a_{n+1} := a_n^2$. Let

$$A = \bigcup_{n=1}^{\infty} \left[a_n^{1/(rs)}, \left(a_n^{1/s} + 1 \right)^{1/r} \right].$$

We show that $\mathrm{BD}_r(A) = 0$ and $\mathrm{BD}_s(A) = 1$.

Suppose that $k, N \in {}^*\mathbb{N}$ with $N > \mathbb{N}$ are such that

$$\mathrm{BD}_r(A) = \mathrm{st}\left(\frac{r}{N} \sum_{x \in {}^*A \cap [k, (k^r+N)^{1/r}]} \frac{1}{x^{1-r}} \right).$$

Let ν be the maximal $m \in {}^*\mathbb{N}$ such that

$$[a_m^{1/rs}, (a_m^{1/s} + 1)^{1/r} \cap [k, (k^r + N)^{1/r}] \neq \emptyset.$$

Note then that

$${}^*A \cap [k, (k^r + N)^{1/r}] \subseteq [k, (\sqrt{a_\nu}^{-1/s} + 1)^{1/r}] \cup [a_\nu^{1/rs}, (a_\nu^{1/s} + 1)^{1/r}].$$

The latter interval is negligble:

$$\mathrm{st}\left(\frac{r}{N} \sum_{x \in [a_\nu^{1/(rs)}, (a_\nu^{1/s}+1)^{1/r}]} \frac{1}{x^{1-r}} \right]$$

$$= \mathrm{st}\left(\frac{r}{N} \left(\left(\frac{\left(\left(a_\nu^{1/s} + 1 \right)^{1/r} \right)^r}{r} \right) - \left(\frac{\left(\left(a_\nu^{1/s} \right)^{1/r} \right)^r}{r} \right) \right) \right)$$

$$= \mathrm{st}\left(\frac{1}{N} \right)$$

$$= 0.$$

Next observe that $(\sqrt{a_\nu}^{-1/s} + 1)^{1/r} < 2(\sqrt{a_\nu})^{1/rs} \leq 2\sqrt{(k^r + N)^{1/r}}$. If $2\sqrt{(k^r + N)^{1/r}} < k$, then the above computation shows that $\mathrm{BD}_r(A) = 0$. Thus, we may assume that $2\sqrt{(k^r + N)^{1/r}} \geq k$, from which it is readily verified that $N > k^r$. It follows that

$$\mathrm{BD}_r(A) \leq \mathrm{st}\left(\frac{r}{N} \sum_{x \in [k, 2(\sqrt{k^r+N})^{1/r}]} \frac{1}{x^{1-r}} \right)$$

$$= \mathrm{st}\left(\frac{r}{N} \left(\frac{2^r(\sqrt{k^r + N})}{r} - \frac{k^r}{r} \right) \right)$$

$$\leq \mathrm{st}\left(\frac{r}{N} \left(\frac{2^r(\sqrt{N + N})}{r} \right) \right)$$

$$= 0.$$

For showing $\mathrm{BD}_s(A) = 1$, it suffices to show that $\left(\left(a_n^{1/s} + 1\right)^{1/r}\right)^s - a_n^{1/r} > \mathbb{N}$ when $n > \mathbb{N}$. Indeed, if $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ is such that $\left(\left(a_n^{1/s} + 1\right)^{1/r}\right)^s - a_n^{1/r} > N$, then ${}^*A$ contains the interval

$$[a_n^{1/rs}, ((a_n^{1/rs})^s + N)^{1/s}].$$

Note that

$$\left(\left(a_n^{1/s} + 1\right)^{1/r}\right)^s - a_n^{1/r}$$

$$= \left(a_n^{1/s} + 1\right)\left(a_n^{1/s} + 1\right)^{(s-r)/r} - a_n^{1/r}$$

$$\geq \left(a_n^{1/s} + 1\right) a_n^{(s-r)/(rs)} - a_n^{1/r}$$

$$= a_n^{\frac{1}{s}+\frac{s-r}{rs}} + a_n^{\frac{s-r}{rs}} - a_n^{\frac{1}{r}} = a_n^{\frac{s-r}{rs}}.$$

It remains to observe that $a_n^{\frac{s-r}{rs}} > \mathbb{N}$ because $a_n > \mathbb{N}$ and $(s-r)/(rs)$ is a positive standard real number. $\square$

3. Polynomial structure and multiplicative structure

In what follows, $\log$ denotes $\log_2$. For $A \subseteq \mathbb{N}$, set

$$\log A := \{\lceil \log x \rceil : x \in A\}.$$

We also introduce some convenient notation: for $k, N \in {}^*\mathbb{N}$ and $E \subseteq {}^*\mathbb{N}$, set $L_{k,N}(E) = \frac{1}{\ln N} \sum_{x \in E \cap [k, Nk]} 1/x$.

Proposition 3.1. *If $A \subseteq \mathbb{N}$, we have $\mathrm{BD}(\log A) \geq \ell\mathrm{BD}(A)$.*

Proof. Without loss of generality, we can assume that $\ell\mathrm{BD}(A) = \alpha > 0$. Take $k, N \in {}^*\mathbb{N}$ with $N > \mathbb{N}$ so that $\mathrm{st}(L_{k,N}({}^*A)) = \alpha$. We first claim that we can assume that k and kN are integer powers of 2. Indeed, choose integers a, b such that $2^{a-1} < k \leqslant 2^a$ and $2^b \leqslant kN < 2^{b+1}$. Note that $b - a > \mathbb{N}$. Observe now that

$$\sum_{x=k}^{2^a-1} 1/x, \sum_{x=2^b+1}^{Nk} 1/x \leq \ln 2,$$

so

$$L_{k,N}({}^*A) \approx \frac{1}{\ln N} \sum_{x \in {}^*A \cap [2^a, 2^b]} 1/x.$$

It remains now to notice that $\ln(2^{b-1}) \leq \ln N \leq \ln(2^{b-a}) + \ln 2$, whence

$$\frac{1}{\ln N} \sum_{x \in {}^*A \cap [2^a, 2^b]} 1/x \approx \frac{1}{\ln 2^{b-a}} \sum_{x \in {}^*A \cap [2^a, 2^b]} 1/x.$$

In light of the previous paragraph, we may take $a < b$ in ${}^*\mathbb{N}$ so that $\mathrm{st}(L_{2^a, 2^{b-a}}({}^*A)) = \alpha$. For $a \leq i < b$, set $I_i := [2^i + 1, 2^{i+1}]$. Observe that $\lceil \log(x) \rceil = i + 1$ for all $x \in I_i$. Set $\mathcal{I} := \{i : I_i \cap {}^*A \neq \emptyset\}$. We then have:

$$|\log(^*A) \cap (a, b)| = |\mathcal{I}|$$

$$= \sum_{i \in \mathcal{I}} \log(2^{i+1}) - \log(2^i)$$

$$\geq \log(e) \sum_{i \in \mathcal{I}} \sum_{x \in ^*A \cap [2^i, 2^{i+1})} \frac{1}{x}.$$

Recalling that $\ln(2^{b-a}) = \frac{b-a}{\log(e)}$, it follows that

$$\mathrm{BD}(\log(A)) \gtrapprox \frac{|\log(^*A) \cap (a, b)|}{b - a} \gtrapprox \frac{1}{\ln 2^{b-a}} \sum_{x \in ^*A \cap [2^a, 2^b - 1)} \frac{1}{x} = \alpha.$$

$\square$

We now come to the central notion of this paper.

Definition 3.2. Fix $c, r \in \mathbb{R}^{>0}$.

(1) For $a, x \in \mathbb{R}$, we say that a is a (c, r)-*approximation of* x if $a \in [x, x + cx^r)$.

(2) For $A, X \subseteq \mathbb{R}$, we say that A is a (c, r)-*approximate subset of* X if every $a \in A$ is an (c, r)-approximation of some $x \in X$.

Theorem 3.3. *Suppose that $A \subseteq \mathbb{N}$ is such that $\ell\mathrm{BD}(A) > 0$. Then for any $l \in \mathbb{N}$, there exist arbitrarily large $a, d \in \mathbb{N}$ such that the geometric sequence $G := \{2^a(2^d)^n : n = 0, 1, \ldots, l - 1\}$ is a $(1, 1)$-approximate subset of A.*

Proof. Fix $m \in \mathbb{N}$. Since $\mathrm{BD}(\log A) \geq \ell\mathrm{BD}(A) > 0$, the set $\log A$ contains an arithmetic progression $\{a + nd : n = 0, 1, \ldots, l - 1\}$ with $a, d > m$. Fix $n \in \{0, 1, \ldots, l - 1\}$ and take $x \in A$ and $\theta \in [0, 1)$ such that $a + nd = \log x + \theta$. Then $x \leq 2^{a+nd} = 2^\theta x < 2x$. $\square$

The following example shows that we cannot improve upon the level of approximation in the previous theorem.

Example 3.4. *For each $\epsilon > 0$, there is $A \subseteq \mathbb{N}$ such that $\underline{ld}(A) = \overline{ld}(A) > 0$ and no positive integer power of 2 is a $(1 - \epsilon, 1)$-approximation of any element of A.*

Proof. Choose $\delta > 0$ such that $(2 - \epsilon)2^\delta < 2$. Set

$$A := \bigcup_{n=1}^{\infty} [2^n + 1, 2^{n+\delta}].$$

Note that the interval $[2^{n+\delta}, (2 - \epsilon)2^{n+\delta}]$ does not contain any positive integer power of 2 as $2^{n+1} \leqslant (2 - \epsilon)2^{n+\delta}$ implies that $2 \leq (2 - \epsilon)2^\delta$. It follows that no power of 2 is a $(1 - \epsilon, 1)$ approximation of any element of A. We leave it to the reader to show that $\underline{ld}(A) = \overline{ld}(A) \geq \delta$. $\square$

Our next example shows that one cannot prove Theorem 3.3 under the weaker assumption of positive Banach density.

Example 3.5. *Let $\alpha < 1$. Fix a j such that $(j-1)/j > \alpha$. Let $u_0 = 2$, $u_{i+1} > (ju_i)^3$, and set*

$$A = \bigcup_{i=1}^{\infty} [u_i, ju_i].$$

Then $\overline{d}(A) > \alpha$. For any $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that there does not exist 3-term geometric progression $G = \{a, ar, ar^2\}$ with $a, r > m$ and G is an $(n-1, 1)$-approximate subset of A.

For a proof of the claim in the previous example, one can consult our paper [1].

For $A \subseteq \mathbb{N}$ and $0 < r \leq 1$, set

$$A^r := \{\lceil x^r \rceil : x \in A\}.$$

One proves the following proposition in a manner similar to the proof of Proposition 3.1

Proposition 3.6. *For any $A \subseteq \mathbb{N}$, we have $\mathrm{BD}(A^r) \geq \mathrm{BD}_r(A)$.*

Theorem 3.7. *Suppose $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$ are such that $\mathrm{BD}_{1/m}(A) > 0$. Then for any $\epsilon > 0$ and $l \in \mathbb{N}$, there exist arbitrarily large $a, d \in \mathbb{N}$ such that $\{(a + nd)^m : n = 0, 1, \ldots, l-1\}$ is an $(m + \epsilon, \frac{m-1}{m})$-approximate subset of A.*

Proof. Fix $p \in \mathbb{N}$. Since $\mathrm{BD}(A^{1/m}) \geq \mathrm{BD}_m(A) > 0$, there are $a, d > p$ such that $\{a + nd : n = 0, 1, \ldots, l-1\} \subseteq A^{1/m}$. Choose a sufficiently large so that, for any $z \geq a - 1$, we have

$$\epsilon z^{m-1} > \binom{m}{m-2} z^{m-2} + \binom{m}{m-3} x^{m-3} + \cdots + mx + 1.$$

Fix $n \in \{0, 1, \ldots, l-1\}$ and take $x \in A$ and $\theta \in [0, 1)$ such that $a + nd = x^{1/m} + \theta$. Since $x^{1/m} > a - 1$, we have

$$x \leqslant (a + nd)^m = (x^{\frac{1}{m}} + \theta)^m = x + (m + \epsilon)x^{\frac{m-1}{m}}\theta$$
$$+ \theta\left(-\epsilon x^{\frac{m-1}{m}} + \binom{m}{m-2} x^{\frac{m-2}{m}}\theta + \binom{m}{m-3} x^{\frac{m-3}{m}}\theta^2 + \cdots + \theta^{m-1}\right)$$
$$< x + (m + \epsilon)x^{\frac{m-1}{m}}.$$

Hence $(a + nd)^m$ is an $(m + \epsilon, \frac{m-1}{m})$-approximation of $x \in A$. $\square$

The next example shows that there is not much room left to improve upon the level of approximation in the previous theorem.

Example 3.8. *For any $\epsilon > 0$, there exists a $\delta > 0$ and there exists a set $A \subseteq \mathbb{N}$ such that $\underline{d}_{1/m}(A) = \overline{d}_{1/m}(A) = \delta$ and such that, for any $a \in \mathbb{N}$, a^m is not an $(m - \epsilon, \frac{m-1}{m})$-approximation of any element in A.*

Proof. Fix $0 < \delta < \epsilon/m$ and set $A := \bigcup_{n=1}^{\infty} [n^m + 1, (n + \delta)^m)$. Suppose, towards a contradiction, that $a^m \in [x, x + (m - \epsilon)x^{(m-1)/m})$ for some $x \in [n^m + 1, (n + \delta)^m)$. It follows that $n + 1 \leq a$, whence

$$((n + \delta) + (1 - \delta))^m = (n + 1)^m \leqslant (n + \delta)^m + (m - \epsilon)(n + \delta)^{m-1}.$$

Hence $m(n + \delta)^{m-1}(1 - \delta) \leqslant (m - \epsilon)(n + \delta)^{m-1}$, which implies that $\delta m \geq \epsilon$, a contradiction. We leave it to the reader to check that $\underline{d}_{1/m}(A) = \overline{d}_{1/m}(A) = \delta$. $\qquad\square$

Corollary 2.6 and Theorem 3.7 immediately imply:

Corollary 3.9. *Suppose that $A \subseteq \mathbb{N}$ is such that $\overline{d}(A) > 0$. Then for any $l, m \in \mathbb{N}$ and $\epsilon > 0$, there exists arbitrarily large $a, d \in \mathbb{N}$ such that $\{(a + nd)^m : n = 0, 1, \ldots, l\}$ is an $\left(m + \epsilon, \frac{m-1}{m}\right)$-approximate subset of A.*

We should remark that the conclusions of approximate structure really are necessary. For example, if A is the set of all square-free numbers, then $\underline{d}(A) > 0$ but A does not contain any 3-term geometric progression or any m-th power of an integer greater than 1 with $m \geq 2$.

References

[1] Mauro Di Nasso, Isaac Goldbring, Renling Jin, Steven Leth, Martino Lupini, and Karl Mahlburg, *A monad measure space for logarithmic density*, arXiv:1503.03810 (2015).

[2] Cadambattur T. Rajagopal, *Some limit theorems*, American Journal of Mathematics **70** (1948), 157–166.

[3] Renling Jin, *Introduction of nonstandard methods for number theorists*, Integers **8** (2008), no. 2.

Dipartimento di Matematica, Universita' di Pisa, Largo Bruno Pontecorvo 5, Pisa 56127, Italy
E-mail address: `dinasso@dm.unipi.it`

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Science and Engineering Offices M/C 249, 851 S. Morgan St., Chicago, IL, 60607-7045
E-mail address: `isaac@math.uic.edu`

Department of Mathematics, College of Charleston, Charleston, SC, 29424
E-mail address: `JinR@cofc.edu`

School of Mathematical Sciences, University of Northern Colorado, Campus Box 122, 510 20th Street, Greeley, CO 80639
E-mail address: `Steven.Leth@unco.edu`

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, Room 02.126, 1090 Wien, Austria.
E-mail address: `martino.lupini@univie.ac.at`

Department of Mathematics, Louisiana State University, 228 Lockett Hall, Baton Rouge, LA 70803
E-mail address: `mahlburg@math.lsu.edu`