$\pi_1(|\Gamma|)$: a hyperfinite approach

Isaac Goldbring
(joint work with Alessandro Sisto)

University of Illinois at Chicago

University of Pennsylvania Logic Seminar
October 16, 2012

$\pi_1(X)$

- Suppose that X is a space and $p \in X$.
- Recall that $\pi_1(X;p)$ is the set of (continuous) loops based at p modulo the relation of two loops being homotopic.
- The operation of concatenating loops based at p induces a group operation on $\pi_1(X;p)$ (with identity being the homotopy class of the constant loop at p).
- If X is pathconnected, then this group is independent of p and is denoted by $\pi_1(X)$, referred to as the *fundamental group* of X.
- The typical example is $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$.
- This construction is functorial: if $f : X \to Y$ is continuous, then there is an induced map $f_* : \pi_1(X) \to \pi_1(Y)$ given by $f_*([\alpha]) := [f \circ \alpha]$.
- X is called *simply connected* if it is pathconnected and $\pi_1(X) = \{1\}$.

$\pi_1(\Gamma)$ when Γ is finite

Theorem

Suppose that Γ is a connected, finite graph. Then $\pi_1(\Gamma)$ is a finitely generated free group.

Proof.

- Let T be a spanning tree of Γ.
- Let $\vec{e}_1, \ldots, \vec{e}_n$ be *oriented chords* of T, that is, edges of Γ not in T, given a fixed orientation.
- Given $[\alpha] \in \pi_1(\Gamma)$, let r_α be the reduced word on $\{e_1^{\pm 1}, \ldots, e_n^{\pm 1}\}$ obtained by recording which chords α traverses fully and in which direction.
- The map $[\alpha] \mapsto r_\alpha : \pi_1(\Gamma) \to F_n$ is an isomorphism.

$\square$

End compactifications of finite graphs

- We now consider infinite, locally finite, connected graphs.
- Many results from finite graph theory are plain false for infinite graphs.
- However, by compactifying an infinite graph by adding its "ends," one can obtain topological analogues of theorems from finite graph theory.

Ends

Definition

Let X be a metric space and $p \in X$.

1. For $x, y \in X$, we write $x \propto_n y$ to indicate that x and y are in the same path component of $X \setminus B(p; n)$.
2. For $r_1, r_2 : [0, \infty) \to X$ *proper* rays with $r_1(0) = r_2(0) = p$, we say $\mathrm{end}(r_1) = \mathrm{end}(r_2)$ if and only if:
$$(\forall n \in \mathbb{N})(\exists m_0 \in \mathbb{N})(\forall m \geq m_0)(r_1(m) \propto_n r_2(m)).$$
3. $\mathrm{Ends}(X) := \{\mathrm{end}(r) \mid r \text{ a proper ray starting at } p\}$.
4. $|X| := X \cup \mathrm{Ends}(X)$ is the *end compactification of* X, topologized in such a way so that proper rays converge to their ends.

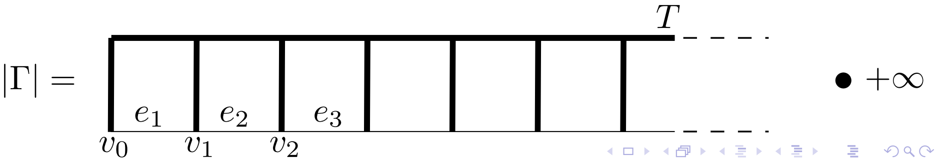
The main problem

Question (Diestel/Sprüssel)

Is there a nice combinatorial characterization of the fundamental group of the end compactification of a locally finite, connected graph in the spirit of the result in the second slide?

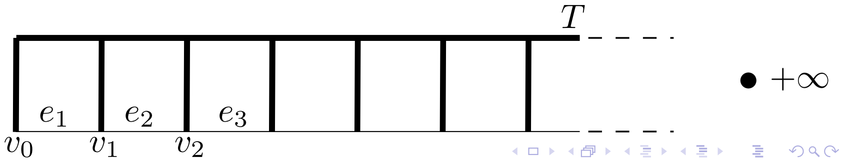
An example: the infinite sideways ladder

- Consider the loop α beginning at v_0, going along the bottom rung of the ladder to the end at $+\infty$, and then back again along the bottom rung of the ladder. α is certainly *nullhomotopic* (i.e. homotopic to the constant loop at v_0).
- If we consider the *topological spanning tree* T for Γ pictured below in bold with oriented edges $\vec{e}_1, \vec{e}_2, \ldots$, then the "word" α induces is $(\vec{e}_1\vec{e}_2\cdots)^\frown(\cdots\overleftarrow{e_2}\overleftarrow{e_1})$.
- This word is of order type $\omega + \omega^*$ with no consecutive appearances of $\overrightarrow{e_i}$ and $\overleftarrow{e_i}$. So we cannot combinatorially tell that this loop is nullhomotopic.



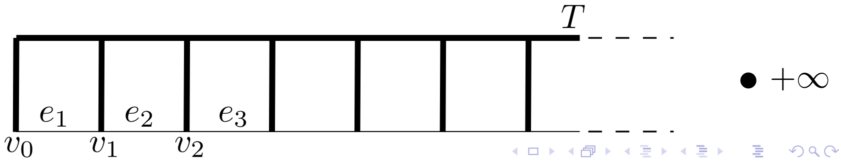
An example: the infinite sideways ladder

- Consider the loop α beginning at v_0, going along the bottom rung of the ladder to the end at $+\infty$, and then back again along the bottom rung of the ladder. α is certainly *nullhomotopic* (i.e. homotopic to the constant loop at v_0).
- If we consider the *topological spanning tree* T for Γ pictured below in bold with oriented edges $\vec{e}_1, \vec{e}_2, \ldots$, then the "word" α induces is $(\vec{e}_1\vec{e}_2\cdots)^\frown(\cdots\overleftarrow{e_2}\overleftarrow{e_1})$.
- This word is of order type $\omega+\omega^*$ with no consecutive appearances of $\overrightarrow{e_i}$ and $\overleftarrow{e_i}$. So we cannot combinatorially tell that this loop is nullhomotopic.

$|\Gamma| =$ 

An example: the infinite sideways ladder

- Consider the loop α beginning at v_0, going along the bottom rung of the ladder to the end at $+\infty$, and then back again along the bottom rung of the ladder. α is certainly *nullhomotopic* (i.e. homotopic to the constant loop at v_0).
- If we consider the *topological spanning tree* T for Γ pictured below in bold with oriented edges $\vec{e}_1, \vec{e}_2, \ldots$, then the "word" α induces is $(\vec{e}_1\vec{e}_2\cdots)^\frown(\cdots\overleftarrow{e_2}\overleftarrow{e_1})$.
- This word is of order type $\omega + \omega^*$ with no consecutive appearances of $\overrightarrow{e_i}$ and $\overleftarrow{e_i}$. So we cannot combinatorially tell that this loop is nullhomotopic.

$|\Gamma| =$ 

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes ≥ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes ≥ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes ≥ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes ≥ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes $\geq$ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

Can nonstandard analysis help?

After seeing my nonstandard treatment on ends, Diestel asked me the following question:

Question (Diestel)

Can nonstandard analysis make any of this simpler?

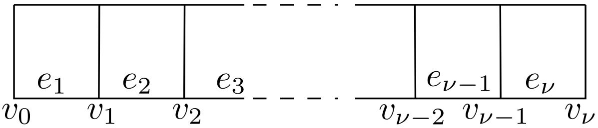
Answer (G., Sisto)

Yes!

The infinite sideways ladder revisited

Let ν be an *infinite natural number*. We can then consider the following *hyperfinite* extension of Γ:

$\Gamma_\nu =$



Our loop α from before "clearly" induces the *hyperfinite word*

$$\overrightarrow{e_1}\overrightarrow{e_2}\cdots\overrightarrow{e_\nu}\overleftarrow{e_\nu}\cdots\overleftarrow{e_2}\overleftarrow{e_1},$$

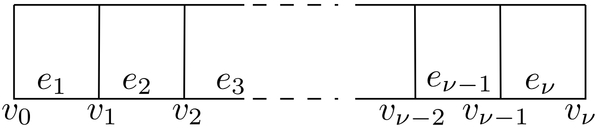
which "clearly" *internally* reduces to the empty word, exhibiting that α is nullhomotopic.

In this way, we get an injective group morphism $\pi_1(|\Gamma|) \hookrightarrow \pi_1(\Gamma_\nu)$, where $\pi_1(\Gamma_\nu)$ is the *internal fundamental group of* Γ_ν, which is a hyperfinitely generated internally free group on ν generators.

The infinite sideways ladder revisited

Let ν be an *infinite natural number*. We can then consider the following *hyperfinite* extension of Γ:

$\Gamma_\nu =$



Our loop α from before "clearly" induces the *hyperfinite word*

$$\overrightarrow{e_1}\overrightarrow{e_2}\cdots\overrightarrow{e_\nu}\overleftarrow{e_\nu}\cdots\overleftarrow{e_2}\overleftarrow{e_1},$$

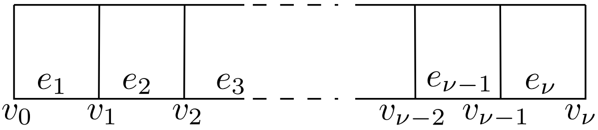
which "clearly" *internally* reduces to the empty word, exhibiting that α is nullhomotopic.

In this way, we get an injective group morphism $\pi_1(|\Gamma|) \hookrightarrow \pi_1(\Gamma_\nu)$, where $\pi_1(\Gamma_\nu)$ is the *internal fundamental group of* Γ_ν, which is a hyperfinitely generated internally free group on ν generators.

The infinite sideways ladder revisited

Let ν be an *infinite natural number*. We can then consider the following *hyperfinite* extension of Γ:

$\Gamma_\nu =$



Our loop α from before "clearly" induces the *hyperfinite word*

$$\overrightarrow{e_1}\overrightarrow{e_2}\cdots\overrightarrow{e_\nu}\overleftarrow{e_\nu}\cdots\overleftarrow{e_2}\overleftarrow{e_1},$$

which "clearly" *internally* reduces to the empty word, exhibiting that α is nullhomotopic.

In this way, we get an injective group morphism $\pi_1(|\Gamma|) \hookrightarrow \pi_1(\Gamma_\nu)$, where $\pi_1(\Gamma_\nu)$ is the *internal fundamental group of* Γ_ν, which is a hyperfinitely generated internally free group on ν generators.

NSA in a nutshell

- Every set X gets enlarged, in a functorial fashion, to a set X^*, the *nonstandard extension of X*.
- X^* "logically behaves" like X (Transfer Principle), but contains new "ideal" elements, e.g. $\mathbb{R}^*$ contains infinitesimal and infinite numbers.
- In a natural way, $\mathcal{P}(X)^*$ embeds into $\mathcal{P}(X^*)$. The subsets of $\mathcal{P}(X^*)$ that belong to $\mathcal{P}(X)^*$ are called the *internal* subsets of X^*; noninternal subsets of X^* are called *external*.
- The similarity in logical behavior applies only to *internal subsets* of X^*. For example, internal subsets of $\mathbb{R}^*$ that are bounded above have suprema; it follows that the set of infinitesimal numbers is external.

The ultraproduct approach

- Suppose that $\mathcal{U}$ is a *nonprincipal ultrafilter* on $\mathbb{N}$, that is, $\mathcal{U}$ is a $\{0,1\}$-valued measure on $\mathcal{P}(\mathbb{N})$ such that finite sets get measure 0.
- For $f, g : \mathbb{N} \to X$, write $f \sim_{\mathcal{U}} g$ to mean $f = g$ a.e.
- Set $X^{\mathcal{U}} := X^{\mathbb{N}} / \sim_{\mathcal{U}}$, the *ultrapower of* X *with respect to* $\mathcal{U}$.
- This construction is easily seen to be functorial and the fact that $X^{\mathcal{U}}$ behaves "logically" like X is known to model theorists as *Łos' theorem*.
- In this setting, $A \subseteq X^{\mathcal{U}}$ is internal if there are $A_n \subseteq X$ such that $A = \prod_{\mathcal{U}} A_n := (\prod_n A_n)/ \sim_{\mathcal{U}}$.
- $N := [(1,2,3,\ldots)]_{\mathcal{U}} \in \mathbb{N}^*$ is a *positive infinite number* whose reciprocal $\frac{1}{N} = [(1, \frac{1}{2}, \frac{1}{3}, \ldots] \in \mathbb{R}^*$ is a *positive infinitesimal*.
- If $A := \prod_{\mathcal{U}} A_n$ with each A_n finite, then we say that A is *hyperfinite* with *internal cardinality* $[(|A_n|)] \in \mathbb{N}^*$.

Nonstandard metric spaces

- If (X, d) is a metric space, then (X^*, d) is almost a metric space except for the fact that the metric takes values in $\mathbb{R}^*$ rather than in $\mathbb{R}$.
- There are two important subsets of X^* to consider:
  - $X_{ns} := \{a \in X^* \mid \text{there is } b \in X \text{ with } d(a, b) \text{ infinitesimal}\}$.
  - $X_{fin} := \{a \in X^* \mid \text{there is } b \in X \text{ with } d(a, b) \text{ finite}\}$.
- Clearly $X_{ns} \subseteq X_{fin}$ with equality holding if and only if X is a *proper* metric space, that is, closed balls are compact.
- If X is proper, then a ray $r : [0, \infty) \to X$ is proper if and only if $r(\sigma) \in X_{inf}$ for all infinite elements σ of $\mathbb{R}^*$.

The nonstandard approach to ends

- Suppose that (X, d) is a proper, *geodesic* metric space and $p \in X$.
- For $x, y \in X^*$, write $x \propto y$ to mean there is $\alpha \in C([0,1], X)^*$ (an *internal* path in X^*) such that $\alpha(0) = x$, $\alpha(1) = y$, and $\alpha(t) \in X_{\text{inf}} := X^* \setminus X_{\text{fin}}$ for all $t \in [0,1]^*$.
  "x and y are in the same path component at infinity."

Theorem (G.)

1. $\text{end}(r_1) = \text{end}(r_2)$ *if and only if for all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^{>0}_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$.
2. *Set* $\text{IPC}(X) := \{[x] \mid x \in X_{\text{inf}}\}$, *where* $[x]$ *denotes the equivalence class of* x *with respect to* $\propto$. *Fix* $\sigma \in \mathbb{R}^{>0}_{\text{inf}}$. *Then the map* $\text{end}(r) \mapsto [r(\sigma)] : \text{Ends}(X) \to \text{IPC}(X)$ *is a bijection.*

The nonstandard approach to ends

- Suppose that (X, d) is a proper, *geodesic* metric space and $p \in X$.
- For $x, y \in X^*$, write $x \propto y$ to mean there is $\alpha \in C([0,1], X)^*$ (an *internal* path in X^*) such that $\alpha(0) = x$, $\alpha(1) = y$, and $\alpha(t) \in X_{\text{inf}} := X^* \setminus X_{\text{fin}}$ for all $t \in [0,1]^*$.
  "x and y are in the same path component at infinity."

Theorem (G.)

1. $\text{end}(r_1) = \text{end}(r_2)$ *if and only if for all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^{>0}_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$.
2. *Set* $\text{IPC}(X) := \{[x] \mid x \in X_{\text{inf}}\}$, *where* $[x]$ *denotes the equivalence class of* x *with respect to* $\propto$. *Fix* $\sigma \in \mathbb{R}^{>0}_{\text{inf}}$. *Then the map* $\text{end}(r) \mapsto [r(\sigma)] : \text{Ends}(X) \to \text{IPC}(X)$ *is a bijection.*

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\text{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\text{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\mathrm{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\mathrm{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\mathrm{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\mathrm{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\mathrm{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\mathrm{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\text{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\text{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p;n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\text{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\text{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p;N)$ to points, where $N := [(1,2,3,\ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\text{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\text{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(\Gamma)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \upharpoonright \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ is injective.

Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(\Gamma)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ is injective.

Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(\Gamma)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ is injective.

Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(\Gamma)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ *is injective.*

Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(\Gamma)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ *is injective.*

About the theorem

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\mathrm{hyp}})$ *is injective.*

Remarks

1. Θ is generally not injective.
2. The result does not imply that $\pi_1(|\Gamma|)$ is free. (This happens if and only if every end is contractible.) Indeed, internally free groups (e.g. $\pi_1(\Gamma_{\mathrm{hyp}})$) need not be free. For example, $\mathbb{Z}^*$ is internally free on one generator, while, for infinite $M, N \in \mathbb{N}^*$ with $\frac{M}{N}$ infinite, we have the map $(a, b) \mapsto aM + bN : \mathbb{Z}^2 \to \mathbb{Z}^*$ is injective. If $\mathbb{Z}^*$ were free, then $\mathbb{Z}^2$ would be free.
3. $\pi_1(|\Gamma|)$ has the same universal theory as the theory of free groups. (If $\pi_1(|\Gamma|)$ were finitely generated, we would say it is a *limit group*.)

About the theorem

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ *is injective.*

Remarks

1. Θ is generally not injective.
2. The result does not imply that $\pi_1(|\Gamma|)$ is free. (This happens if and only if every end is contractible.) Indeed, internally free groups (e.g. $\pi_1(\Gamma_{\text{hyp}})$) need not be free. For example, $\mathbb{Z}^*$ is internally free on one generator, while, for infinite $M, N \in \mathbb{N}^*$ with $\frac{M}{N}$ infinite, we have the map $(a, b) \mapsto aM + bN : \mathbb{Z}^2 \to \mathbb{Z}^*$ is injective. If $\mathbb{Z}^*$ were free, then $\mathbb{Z}^2$ would be free.
3. $\pi_1(|\Gamma|)$ has the same universal theory as the theory of free groups. (If $\pi_1(|\Gamma|)$ were finitely generated, we would say it is a *limit group*.)

About the theorem

Theorem (G., Sisto)

$\Theta \upharpoonright \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{hyp})$ *is injective.*

Remarks

1. Θ is generally not injective.
2. The result does not imply that $\pi_1(|\Gamma|)$ is free. (This happens if and only if every end is contractible.) Indeed, internally free groups (e.g. $\pi_1(\Gamma_{hyp})$) need not be free. For example, $\mathbb{Z}^*$ is internally free on one generator, while, for infinite $M, N \in \mathbb{N}^*$ with $\frac{M}{N}$ infinite, we have the map $(a, b) \mapsto aM + bN : \mathbb{Z}^2 \to \mathbb{Z}^*$ is injective. If $\mathbb{Z}^*$ were free, then $\mathbb{Z}^2$ would be free.
3. $\pi_1(|\Gamma|)$ has the same universal theory as the theory of free groups. (If $\pi_1(|\Gamma|)$ were finitely generated, we would say it is a *limit group*.)

Universal covers

- Given a topological space X, a *cover of X* is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of X* is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

Universal covers

- Given a topological space X, a *cover of X* is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of X* is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

Universal covers

- Given a topological space X, a *cover of X* is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of X* is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

Universal covers

- Given a topological space X, a *cover of* X is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of* X is a cover of X whose associated topological space is simply connected. “Nice” spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

Universal covers

- Given a topological space X, a *cover of* X is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of* X is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

Universal covers

- Given a topological space X, a *cover of X* is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of X* is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

About the proof

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\mathrm{hyp}})$ *is injective.*

Idea of Proof

- Suppose $\theta(\alpha)$ is internally nullhomotopic.
- Since Γ_{hyp} is hyperfinite, its internal universal cover $\widetilde{\Gamma_{\mathrm{hyp}}}$ is an internal tree. This passage to universal covering tree is not possible in the standard approach!
- $\theta(\alpha)$ lifts to an internal loop in $\widetilde{\Gamma_{\mathrm{hyp}}}$.
- Using nice geodesic paths in $\widetilde{\Gamma_{\mathrm{hyp}}}$, we can project back onto Γ_{hyp} to construct a homotopy witnessing that α is nullhomotopic.
- We do not need to consider topological spanning trees in Γ, an added bonus since their existence is nontrivial.

Connection with the standard result

- With more effort, we can completely recover the Diestel-Sprüssel result.
- However, we can easily recover the embedding of $\pi_1(|\Gamma|)$ into an inverse limit of f.g. free groups.
- The maps $\theta_n : |\Gamma| \to \Gamma_n$ yield a homomorphism

$$\Psi : \pi_1(|\Gamma|) \to \varprojlim \pi_1(\Gamma_n).$$

- Define $\Phi : \varprojlim \pi_1(\Gamma_n) \to \prod_{\mathcal{U}} \pi_1(\Gamma_n) = \pi_1(\Gamma_{\text{hyp}})$ by $\Phi((x_n)) := [(x_n)]$.
- Check that $\Theta \restriction \pi_1(|\Gamma|) = \Phi \circ \Psi$. Since $\Theta \restriction \pi_1(|\Gamma|)$ is injective, so is Ψ.
- As a result, we see that $\pi_1(|\Gamma|)$ is ω*-residually free*, a property known to be equivalent to being a limit group for finitely generated groups.

The first homology group

Definition

1. If G is a group, its *first homology group* is the group $H_1(G) := G/[G,G]$.
2. If X is a pathconnected space, its first singular homology group is $H_1(X) := H_1(\pi_1(X))$.

- For finite graphs, the first singular homology group coincides with a familiar combinatorial object, the so-called cycle space $\mathcal{C}(\Gamma)$.
- For infinite graphs, Diestel and Sprüssel devised an ad hoc homology theory for $|\Gamma|$, the *topological cycle space* $\mathcal{C}^{\text{top}}(\Gamma)$.
- They wondered if the topological cycle space coincides with the first singular homology.

The first homology group

Definition

1. If G is a group, its *first homology group* is the group $H_1(G) := G/[G,G]$.
2. If X is a pathconnected space, its first singular homology group is $H_1(X) := H_1(\pi_1(X))$.

- For finite graphs, the first singular homology group coincides with a familiar combinatorial object, the so-called cycle space $\mathcal{C}(\Gamma)$.
- For infinite graphs, Diestel and Sprüssel devised an ad hoc homology theory for $|\Gamma|$, the *topological cycle space* $\mathcal{C}^{\text{top}}(\Gamma)$.
- They wondered if the topological cycle space coincides with the first singular homology.

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\mathrm{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\mathrm{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\mathrm{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\mathrm{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\mathrm{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\mathrm{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\mathrm{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\mathrm{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\text{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\text{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\text{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\text{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\mathrm{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\mathrm{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\mathrm{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\mathrm{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\mathrm{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\mathrm{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

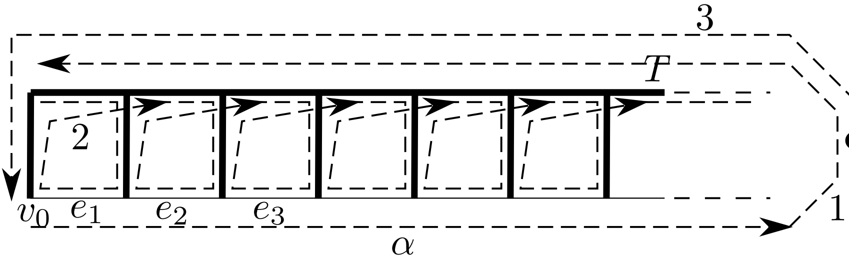
There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\mathrm{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\mathrm{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

Two different homology theories

Theorem (Diestel/Sprüssel)

The loop α depicted below is trivial in $\mathcal{C}^{\text{top}}(\Gamma)$ but not in $H_1(|\Gamma|)$.

$|\Gamma| =$ 

Why is α not nullhomologous?

- A finite version of α would trace the word

$$\overrightarrow{e_1} \cdots \overrightarrow{e_n} \overleftarrow{e_1} \cdots \overleftarrow{e_n} \in [\pi_1(\Gamma), \pi_1(\Gamma)],$$

  whence the finite version of α would be nullhomologous.

- Diestel and Sprüssel give a topological proof that the finite version of the loop α is nullhomologous. They then remark "But we cannot imitate this proof for α and our infinite ladder, because homology classes in $H_1(|\Gamma|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide α infinitely often." (I.e. Wouldn't it be great if nonstandard analysis existed?)

- Instead, Diestel and Sprüssel use their analysis of the fundamental group to attach a complicated invariant to loops which vanish on nullhomologous loops. They then show that the invariant for α is nonzero.

Why is α not nullhomologous?

- A finite version of α would trace the word

$$\overrightarrow{e_1}\cdots\overrightarrow{e_n}\overleftarrow{e_1}\cdots\overleftarrow{e_n}\in[\pi_1(\Gamma),\pi_1(\Gamma)],$$

whence the finite version of α would be nullhomologous.

- Diestel and Sprüssel give a topological proof that the finite version of the loop α is nullhomologous. They then remark "But we cannot imitate this proof for α and our infinite ladder, because homology classes in $H_1(|\Gamma|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide α infinitely often." (I.e. Wouldn't it be great if nonstandard analysis existed?)
- Instead, Diestel and Sprüssel use their analysis of the fundamental group to attach a complicated invariant to loops which vanish on nullhomologous loops. They then show that the invariant for α is nonzero.

Why is α not nullhomologous?

- A finite version of α would trace the word

$$\overrightarrow{e_1} \cdots \overrightarrow{e_n} \overleftarrow{e_1} \cdots \overleftarrow{e_n} \in [\pi_1(\Gamma), \pi_1(\Gamma)],$$

whence the finite version of α would be nullhomologous.

- Diestel and Sprüssel give a topological proof that the finite version of the loop α is nullhomologous. They then remark "But we cannot imitate this proof for α and our infinite ladder, because homology classes in $H_1(|\Gamma|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide α infinitely often." (I.e. Wouldn't it be great if nonstandard analysis existed?)
- Instead, Diestel and Sprüssel use their analysis of the fundamental group to attach a complicated invariant to loops which vanish on nullhomologous loops. They then show that the invariant for α is nonzero.

Why is α not nullhomologous?

- A finite version of α would trace the word

$$\overrightarrow{e_1} \cdots \overrightarrow{e_n}\overleftarrow{e_1} \cdots \overleftarrow{e_n} \in [\pi_1(\Gamma), \pi_1(\Gamma)],$$

  whence the finite version of α would be nullhomologous.
- Diestel and Sprüssel give a topological proof that the finite version of the loop α is nullhomologous. They then remark "But we cannot imitate this proof for α and our infinite ladder, because homology classes in $H_1(|\Gamma|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide α infinitely often." (I.e. Wouldn't it be great if nonstandard analysis existed?)
- Instead, Diestel and Sprüssel use their analysis of the fundamental group to attach a complicated invariant to loops which vanish on nullhomologous loops. They then show that the invariant for α is nonzero.

A simple lemma

Recall our maps $\theta : \Gamma \to \Gamma_{\text{hyp}}$ and $\Theta : \pi_1(|\Gamma|)^* \to \pi_1(\Gamma_{\text{hyp}})$.

Lemma

If the loop α is null-homologous, then $\theta(\alpha)$ has finite commutator length as an element of $\pi_1(\Gamma_{\text{hyp}})$.

Proof.

Let $g \in \pi_1(|\Gamma|)$ be the element represented by α and let $f : G \to H_1(G)$ be the natural map. If $f(g) = 0$, then we can write g as the product of, say, n commutators. As Θ is a group homomorphism, we have that $\Theta(g)$ can be written as a product of n commutators as well. $\square$

A simple proof that α is not nullhomologous

Fact (Goldstein-Turner; G.-Sisto)

If $e_1, \ldots, e_n$ generate a free group, then $e_1 \cdots e_n e_1^{-1} \cdots e_n^{-1}$ has commutator length $\lfloor \frac{n}{2} \rfloor$.

- By transfer, $\theta(\alpha)$ induces the word $e_1 \cdots e_\nu e_1^{-1} \cdots e_\nu^{-1}$ in Γ_{hyp}.
- By transfer of the above fact, $\theta(\alpha)$ has commutator length $\lfloor \frac{\nu}{2} \rfloor > \mathbb{N}$.
- By the previous lemma, α is not nullhomologous.

Remarks about homology

- We have seen that $\theta(\alpha)$ internally nullhomotopic implies α nullhomotopic.
- The preceding example shows that $\theta(\alpha)$ internally nullhomologous does not necessarily imply α nullhomologous.
- In fact, one can check that $\theta(\alpha)$ is internally nullhomologous if and only if α is trivial in $\mathcal{C}^{\text{top}}(\Gamma)$. So, in some sense, their ad hoc homology theory is really the internal version of the ordinary homology theory.

References

- R. Diestel, *Locally finite graphs with ends: a topological approach, I. Basic theory*, Discrete Math. **311** (2011), 1423-1447.
- R. Diestel and P. Sprüssel, *The fundamental group of a locally finite graph with ends*, Adv. Math. **226** (2011), 2643-2675.
- I. Goldbring and A. Sisto, *The fundamental group of a locally finite graph with ends: a hyperfinite approach*, submitted.
- I. Goldbring, *Ends of groups: a nonstandard perspective*, J. Log. Anal. **3** (2011), Paper 7, 1-28.