Nonstandard Hulls of Locally Exponential Lie Groups and Algebras

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Outline

Introduction

Nonstandard Analysis

Nonstandard Smoothness Conditions on Locally Convex Spaces

Proof of the Main Theorem

A Theorem of Pestov

In the early 1990s, Pestov proved the following theorem using nonstandard analysis.

Theorem (Pestov)

Let $\mathfrak{g}$ be a Banach-Lie algebra. Suppose that there exists a family $\mathcal{H}$ of closed Lie subalgebras of $\mathfrak{g}$ and a neighborhood V of 0 in $\mathfrak{g}$ such that

- *For each $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}$, there is an $\mathfrak{h}_3 \in \mathcal{H}$ such that $\mathfrak{h}_1 \cup \mathfrak{h}_2 \subseteq \mathfrak{h}_3$ ($\mathcal{H}$ is directed upwards);*
- *$\bigcup \mathcal{H}$ is dense in $\mathfrak{g}$;*
- *Every $\mathfrak{h} \in \mathcal{H}$ is enlargeable and if H is a corresponding connected, simply connected Banach-Lie group, then the restriction $\exp_H | V \cap \mathfrak{h}$ is injective.*

Then $\mathfrak{g}$ is enlargeable.

Idea of the Proof

- Use the fact that $\mathcal{H}$ is directed and has a dense union to find an internal subalgebra $\mathfrak{h} \in \mathcal{H}^*$ such that $\mathfrak{g} \subseteq \mathfrak{h}$.
- Construct the *nonstandard hull* $\hat{\mathfrak{h}}$ of $\mathfrak{h}$, which is a standard Banach-Lie algebra. $\mathfrak{g}$ will be a closed subalgebra of $\hat{\mathfrak{h}}$.
- If H was an internal Banach-Lie group whose Lie algebra was $\mathfrak{h}$, use the BCH-series to construct the nonstandard hull $\hat{H}$ of H, which will be a standard Banach-Lie group whose Lie algebra is $\hat{\mathfrak{h}}$. It follows that $\hat{\mathfrak{h}}$ is an enlargeable Banach-Lie algebra.
- Since $\mathfrak{g}$ is a closed subalgebra of an enlargeable Banach-Lie algebra, it is also enlargeable.

Locally Exponential Lie Groups and Algebras

Definition

A locally convex Lie group G is called **locally exponential** if there is a smooth exponential map $\exp : \mathrm{Lie}(G) \to G$ which is a diffeomorphism between a neighborhood of 0 in $\mathrm{Lie}(G)$ and a neighborhood of 1 in G.

Definition

A locally convex Lie algebra $\mathfrak{g}$ is called **locally exponential** if there exists a circular, convex open 0-neighborhood $U \subseteq \mathfrak{g}$, an open subset $D \subseteq U \times U$, and a smooth map $m_U : D \to U$ such that $(U, D, m_U, 0)$ is a local Lie group satisfying:

1. For $x \in U$ and $|t|, |s|, |t+s| \leq 1$, we have $(tx, sx) \in D$ and $m_U(tx, sx) = (t+s)x$;
2. $\mathrm{Lie}(U, D, m_U, 0) \cong \mathfrak{g}$.

Pestov's Theorem for Locally Exponential Lie Algebras??

If G is a locally exponential Lie group, then $\mathrm{Lie}(G)$ is a locally exponential Lie algebra (use exponential coordinates!).

Definition

If $\mathfrak{g}$ is a locally exponential Lie algebra, then we say that $\mathfrak{g}$ is **enlargeable** if there is a locally exponential Lie group G such that $\mathrm{Lie}(G) \cong \mathfrak{g}$.

Due to the existence of an Implicit Function Theorem, Banach-Lie groups are locally exponential. Due to the BCH series, Banach-Lie algebras are locally exponential. It thus makes sense to ask for an analogue of Pestov's theorem for locally exponential Lie algebras!

Difficulties in Extending Pestov's Theorem

In trying to adapt Pestov's proof, one runs into a few problems.

- One can still find an internal subalgebra $\mathfrak{h} \in \mathcal{H}^*$ for which $\hat{\mathfrak{h}}$ is a locally convex Lie algebra and $\mathfrak{g}$ is a closed subalgebra of $\hat{\mathfrak{h}}$. However, there is no guarantee that $\hat{\mathfrak{h}}$ will be a locally exponential Lie algebra.
- Suppose H was an internal locally exponential Lie group such that $\mathrm{Lie}(H) \cong \mathfrak{h}$. The construction of $\hat{H}$ is much harder due to the lack of a BCH series. Furthermore, once $\hat{H}$ has been constructed, proving that the exponential map from $\hat{\mathfrak{h}}$ to $\hat{H}$ is a local diffeomorphism is not immediate either.

The Main Theorem

Theorem

Suppose $\mathfrak{g}$ is a locally exponential Lie algebra, $\mathcal{H}$ is a family of closed subalgebras of $\mathfrak{g}$, V is a neighborhood of 0 in $\mathfrak{g}$ and p is a continuous seminorm on $\mathfrak{g}$ satisfying:

1. $\bigcup \mathcal{H}$ *is dense in* $\mathfrak{g}$;
2. *for each* $\mathfrak{h} \in \mathcal{H}$, *there is a locally exponential Lie group* H *such that* $L(H) \cong \mathfrak{h}$;
3. *for each* $\mathfrak{h} \in \mathcal{H}$, *if* H *is a connected locally exponential Lie group such that* $L(H) \cong \mathfrak{h}$, *then* $\exp_H | V \cap \mathfrak{h} : V \cap \mathfrak{h} \to H$ *is injective;*
4. $(\exp_H(\{x \in \mathfrak{h} | \, p(x) < 1\}))^2 \subseteq W_{\mathfrak{h}}$, *where* $W_{\mathfrak{h}}$ *is an open neighborhood of* 1 *contained in* $\exp_H(V)$;
5. m_U *is uniformly continuous on* $\{p < 1\}^2$;
6. m_U *is uniformly smooth at finite points.*

Then $\mathfrak{g}$ *is enlargeable.*

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Proof of the Main Theorem

Nonstandard Extensions

Start with a mathematical universe V containing all relevant mathematical objects, e.g.

- $\mathbb{N}$, $\mathbb{R}$, a locally convex Lie algebra $\mathfrak{g}$ (basic sets);
- various cartesian products of the above sets;
- the elements of the above sets and the power sets of the above sets;

Then extend to a *nonstandard* mathematical universe V^*:

- To every $A \in V$, there is a corresponding $A^* \in V^*$, e.g. we have $\mathbb{N}^*$, $\mathbb{R}^*$, $\mathfrak{g}^*$, π^*, $\sin^*(x)$ (which is a function $\mathbb{R}^* \to \mathbb{R}^*$);
- For simplicity, we write a for a^* when a is an element of a basic set.
- An object in V^* which is not in V is called *nonstandard*.

The Transfer Principle

We want V^* to behave logically like V, so we assume the

Transfer Principle

If S is a bounded first-order statement about objects in V, then it is true in V if and only if it is true in V^*.

Example

By transfer, for any distinct $a, b \in \mathbb{R}$, we have that $a^*, b^* \in \mathbb{R}^*$ are distinct. Since we have agreed to identify a with a^*, this allows us to view $\mathbb{R}$ as a subset of $\mathbb{R}^*$. Now suppose that $f : \mathbb{R} \to \mathbb{R}$. Then we have $f^* : \mathbb{R}^* \to \mathbb{R}^*$ and transfer shows that $f^*|\mathbb{R} = f$, so we write f for both the original function and its nonstandard extension. We do this for arbitrary functions in our nonstandard universe. Since the axioms for being an ordered field are first-order, we see that $\mathbb{R}^*$ is an ordered field containing (an isomorphic copy of) $\mathbb{R}$ as a subfield.

Extensions of Lie Algebras

Example

Consider our locally convex Lie algebra $\mathfrak{g}$. We then have the extension of the bracket

$$[\cdot, \cdot] : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*.$$

Since the axioms for being a Lie algebra are first-order, we see that $\mathfrak{g}^*$ becomes a Lie algebra (over the field $\mathbb{R}^*$). Also, each seminorm p on $\mathfrak{g}$ extends to a seminorm $p : \mathfrak{g}^* \to \mathfrak{g}^*$. However, if the set of seminorms Γ defining $\mathfrak{g}$ is *infinite*, we will have elements of Γ^* which are not the extension of a standard seminorm to $\mathfrak{g}^*$ (a consequence of *saturation*, to be defined in the next slide).

Internal Sets and Saturation

If X is a basic set of our universe V, the logical apparatus of our nonstandard framework only applies to certain subsets of X^*, namely the *internal* subsets of X, which are the elements of $\mathcal{P}(X)^*$. The richness of nonstandard extensions come from the following notion.

Definition

Let κ be an infinite cardinal. We say that V^* is κ-**saturated** if whenever $\{\mathcal{O}_i \mid i < \kappa\}$ is a family of *internal* sets such that any intersection of a finite number of them is nonempty, then the intersection of all them is nonempty.

We will assume our V^* is κ-saturated for a suitably large κ.

An Example of Saturation: Infinitesimals

For $n > 0$, let $\mathcal{O}_n := \{x \in \mathbb{R}^* \mid 0 < |x| < \frac{1}{n}\}$. It can be shown that each $\mathcal{O}_n$ is internal. Clearly any finite intersection of the $(\mathcal{O}_n)$ is nonempty and so saturation yields that there is $\alpha \in \bigcap_{n>0} \mathcal{O}_n$. Such an α is positive but smaller than any *standard* real number, i.e. α is an infinitesimal. Moreover, $\frac{1}{\alpha}$ is an element of $\mathbb{R}^*$ bigger than any *standard* real number, i.e. $\frac{1}{\alpha}$ is an *infinite* element of $\mathbb{R}$.

Nonstandard Hulls of Internal Subspaces

Suppose that $\mathfrak{g}$ is a locally convex space with Γ the set of all continuous seminorms on $\mathfrak{g}$. Suppose that $\mathfrak{h}$ is an *internal subspace* of $\mathfrak{g}^*$, that is $\mathfrak{h} \subseteq \mathfrak{g}^*$ is internal, $\mathfrak{h} + \mathfrak{h} \subseteq \mathfrak{h}$ and $\mathbb{R}^* \cdot \mathfrak{h} \subseteq \mathfrak{h}$.

Consider the following sets:

$$\mathfrak{h}_{\text{fin}} := \{x \in \mathfrak{h}^* \mid p(x) \text{ is finite for all } p \in \Gamma\}$$

$$\mu_{\mathfrak{h}} := \{x \in \mathfrak{h}^* \mid p(x) \text{ is infinitesimal for all } p \in \Gamma\}.$$

We call $\mathfrak{h}_f$ the set of **finite vectors of** $\mathfrak{h}$ and $\mu_{\mathfrak{h}}$ the set of **infinitesimal vectors of** $\mathfrak{h}$. For $x, y \in \mathfrak{h}^*$, we write $x \sim y$ if $x - y \in \mu_{\mathfrak{h}}$.

Lemma

$\mathfrak{h}_{\text{fin}}$ *is a real vector space and* $\mu_{\mathfrak{h}}$ *is a subspace of* $\mathfrak{h}_{\text{fin}}$*. We denote the quotient* $\mathfrak{h}_{\text{fin}}/\mu_{\mathfrak{h}}$ *by* $\hat{\mathfrak{h}}$ *and call it the* ***nonstandard hull of*** $\mathfrak{h}$*.*

Nonstandard Hulls of Internal Subspaces (cont'd)

For each $p \in \Gamma$, define $\hat{p} : \hat{\mathfrak{h}} \to \mathbb{R}$ by $p(x + \mu_{\mathfrak{h}}) := \text{st}(p(x))$. (Here, if $r \in \mathbb{R}^*$ is finite, then $\text{st}(r)$ denotes the unique standard real number s such that $|r - s|$ is infinitesimal.) Let $\hat{\Gamma} := \{\hat{p} \mid p \in \Gamma\}$. It is then straightforward to show that $\hat{\Gamma}$ is a separating family of seminorms rendering $\hat{\mathfrak{h}}$ a locally convex space.

One can show that $\hat{\mathfrak{g}^*}$ is complete and $\hat{\mathfrak{h}}$ is a closed subspace of $\hat{\mathfrak{g}^*}$. Moreover, the map $\iota : \mathfrak{g} \to \hat{\mathfrak{g}^*}$ defined by $\iota(x) := x + \mu$ is such that $p(x) = \hat{p}(\iota(x))$. In this way, we can identify $\mathfrak{g}$ with a closed subspace of $\hat{\mathfrak{g}^*}$.

An Example

Consider the vector space $\mathfrak{g} := C(\mathbb{R}, \mathbb{R})$. $\mathfrak{g}$ becomes a locally convex space when equipped with the family of seminorms (p_n), where $p_n(f) := \sup_{x \in [-n,n]} |f(x)|$. Now $\mathfrak{g}^* = C(\mathbb{R}, \mathbb{R})^*$, which consists of the internally continuous functions $\mathbb{R}^* \to \mathbb{R}^*$. (Think ϵ-δ definition of continuity for $\epsilon, \delta \in \mathbb{R}^*$)

Note that

$$\mathfrak{g}^*_{\text{fin}} = \{f \in \mathfrak{g}^* \mid f(\mathbb{R}_{\text{fin}}) \subseteq \mathbb{R}_{\text{fin}}\}$$

and

$$\mu_{\mathfrak{g}^*} = \{f \in \mathfrak{g}^* \mid f(\mathbb{R}_{\text{fin}}) \subseteq \mu_{\mathbb{R}^*}\}.$$

Note that $\hat{\mathfrak{g}^*}$ is a proper extension of of $\mathfrak{g}$; indeed, an element of $\mathfrak{g}_{\text{fin}}$ which makes an finite, noninfinitesimal jump on an interval of infinitesimal radius cannot be infinitely close to an element of $\mathfrak{g}$.

Nonstandard Hulls of Internal Lie Algebras

Now suppose that $\mathfrak{g}$ is a locally convex Lie algebra and $\mathfrak{h}$ is an *internal subalgebra* of $\mathfrak{g}^*$, i.e. $\mathfrak{h}$ is an internal subspace of $\mathfrak{g}^*$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Lemma

$\mathfrak{h}_f$ is a real Lie algebra and $\mu_{\mathfrak{h}}$ is a Lie ideal of $\mathfrak{h}_f$.

Part of the Proof

- Fix $x, y \in \mathfrak{h}_{\text{fin}}$ and p a continuous seminorm on $\mathfrak{g}$. Choose a continuous seminorm q on $\mathfrak{g}$ and $r \in \mathbb{R}^{>0}$ so that for all $a, b \in \mathfrak{g}$, if $q(a), q(b) < r$, then $p([a, b]) < 1$. Since $x, y \in \mathfrak{h}_{\text{fin}}$, we can choose $\alpha \in \mathbb{R}^{>0}$ such that $q(\alpha x), q(\alpha y) < r$. Then $p([\alpha x, \alpha y]) < 1$, whence $p([x, y]) < \frac{1}{\alpha^2}$. This shows that $[\mathfrak{h}_{\text{fin}}, \mathfrak{h}_{\text{fin}}] \subseteq \mathfrak{h}_{\text{fin}}$.
- Similar reasoning shows that $[\mathfrak{h}_{\text{fin}}, \mu_{\mathfrak{h}}] \subseteq \mu_{\mathfrak{h}}$.

Nonstandard Hulls of Internal Lie Algebras (cont'd)

We let $\hat{\mathfrak{h}} := \mathfrak{h}_f/\mu_{\mathfrak{h}}$. Then $\hat{\mathfrak{h}}$ is a real Lie algebra. Moreover, one can show that $\hat{\mathfrak{h}}$ is a locally convex Lie algebra with respect to the set of seminorms $\hat{\Gamma}$ defined above. As before, $\hat{\mathfrak{h}}$ is a closed subalgebra of $\hat{\mathfrak{g}^*}$ and we can identify $\mathfrak{g}$ with a closed subalgebra of $\hat{\mathfrak{g}^*}$.

Internal Linear Maps

For the rest of this section, E and F denote locally convex spaces. Let $\text{Lin}^k(E, F)$ denote the set of k-linear maps from E to F. Then $\text{Lin}^k(E, F)^*$ denotes the set of *internal* k-linear maps from E^* to F^*. Note that such maps are k-linear maps from the $\mathbb{R}^*$-vector space E^* to the $\mathbb{R}^*$-vector space F^*.

Definition

$\text{FLin}^k(E^*, F^*) := \{T \in \text{Lin}^k(E^*, F^*) \mid T((E_f)^k) \subseteq F_f\}.$

Example

Let $\lambda \in \mathbb{R}^*$. Then the internal linear map $x \mapsto \lambda x : E^* \to E^*$ is in $\text{FLin}^1(E^*, E^*)$ if and only if λ is a finite element of $\mathbb{R}^*$.

Uniformly Smooth at Finite Points

Fix $f : U \to F$, where $U \subseteq E$ is open. Let

$$\text{in}(U^*) = \{a \in U^* \mid \text{ for all } b \in E^*, \text{ if } b \sim a, \text{ then } b \in U^*\}.$$

Definition

We define what it means for f to be **uniformly C^k at finite points** by recursion. Say that f is uniformly C^1 at finite points if there is a map $df : U \to \text{Lin}(E, F)$ such that for every $a \in \text{in}(U^*) \cap E_f$, every $h \in E_f$, and every positive infinitesimal δ, we have

$$df(a) \in \text{FLin}^1(E^*, F^*)$$

and

$$\frac{1}{\delta}(f(a+\delta h) - f(a)) \sim df(a)(h).$$

Uniformly Smooth at Finite Points (cont'd)

Definition (Cont'd)

Suppose, inductively, that f is uniformly C^k at finite points. Then f is uniformly C^{k+1} at finite points if there is a map $d^{k+1}f : U \to \text{Lin}^{k+1}(E, F)$ such that for every $a \in \text{in}(U^*) \cap E_f$, every $x \in E_f$, every $h \in (E_f)^k$, and every positive infinitesimal δ, we have

$$d^{k+1}f(a) \in \text{FLin}^{k+1}(E^*, F^*)$$

and

$$\frac{1}{\delta}(d^k f(a + \delta x)(h) - d^k f(a)(h)) \sim d^{k+1} f(a)(h, x).$$

We say that f is **uniformly smooth at finite points** if f is uniformly C^k at finite points for every k.

Reason for the Definition

Write

$$U = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} \{x \in E \mid p_{ij}(x - x_{ij}) < \epsilon_{ij}\}$$

and define

$$\hat{U} = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} \{x + \mu_E \in \hat{E} \mid \hat{p}_{ij}((x - x_{ij}) + \mu_E) < \epsilon_{ij}\}.$$

Theorem

Suppose that f is uniformly smooth at finite points and $f(U^* \cap E_f) \subseteq F_f$. Then the map $\hat{f} : \hat{U} \to \hat{F}$ given by $f(x + \mu_E) := f(x) + \mu_F$ is well-defined and smooth. Moreover,

$$d^k(\hat{f})(a + \mu_E, h + \mu_E) = d^k f(a, h) + \mu_F.$$

Another Way of Defining Smoothness

Suppose f is C^1. Let
$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{R} \mid x + ty \in U\}$ and define
$f^{[1]} : U^{[1]} \to F$ by

$$f^{[1]}(x, y, t) = \begin{cases} \frac{1}{t}(f(x + ty) - f(x)) & \text{if } t \neq 0 \\ df(x)(y) & \text{if } t = 0 \end{cases}$$

Then the Mean Value Theorem shows that $f^{[1]}$ is continuous.
More generally, define $U^{[k]}$ and $f^{[k]}$ recursively by

$$U^{[k+1]} := (U^{[k]})^{[1]} \qquad \text{and } f^{[k+1]} := (f^{[k]})^{[1]}.$$

Fact

[Bertram, Glöckner, Neeb] Suppose f is C^k. Then f is C^{k+1} if and only if $f^{[k]}$ is C^1.

Strong Smoothness

Definition

We define the notion f is **strongly** C^k by recursion. We say that f is strongly C^1 if f is continuous and $f^{[1]}$ is *uniformly* continuous. Supposing that f is strongly C^k, we say that f is strongly C^{k+1} is $f^{[k]}$ is strongly C^1. We say that f is **strongly smooth** if f is strongly C^k for all k.

Lemma

If f is strongly C^k, then f is uniformly C^k at finite points.

The Case of Complete (HM)-spaces

Definition

E is an **(HM)-space** if whenever $\mathcal{F}$ is an ultrafilter on E with the property that for every U from a fixed neighborhood base of 0, there is n such that $nU \in \mathcal{F}$, then $\mathcal{F}$ is a Cauchy filter.

- In nonstandard terms, this means that the standard points are "dense" in the finite points.
- For metrizable E, E is an (HM)-space if and only if every bounded set is totally bounded.
- Examples of (HM)-spaces are the (FM)-spaces, the nuclear spaces, and the Schwarz spaces (e.g. Silva spaces).

Lemma

For complete (HM)-spaces, "uniformly smooth at finite points" is the same notion as "smooth".

Finite Functions

Definition

$f : U \to F$ is *finite* if $f(U^* \cap E_{\text{fin}}) \subseteq F_{\text{fin}}$.

Recall that we needed f to be finite in order for it to induce a map $\hat{f} : \hat{U} \to \hat{F}$.

Theorem

- *If $f(U)$ is bounded, then f is finite.*
- *If f is uniformly continuous and U is convex, then f is finite.*
- *If f is Lipshitz, then f is finite.*
- *If E is an (HM)-space, then f has a restriction which is finite.*

Proof of the Main Theorem

Main Theorem Recalled

Theorem

Suppose $\mathfrak{g}$ *is a locally exponential Lie algebra,* $\mathcal{H}$ *is a family of closed subalgebras of* $\mathfrak{g}$*,* V *is a neighborhood of* 0 *in* $\mathfrak{g}$ *and* p *is a continuous seminorm on* $\mathfrak{g}$ *satisfying:*

1. $\bigcup \mathcal{H}$ *is dense in* $\mathfrak{g}$*;*
2. *for each* $\mathfrak{h} \in \mathcal{H}$*, there is a locally exponential Lie group* H *such that* $L(H) \cong \mathfrak{h}$*;*
3. *for each* $\mathfrak{h} \in \mathcal{H}$*, if* H *is a connected locally exponential Lie group such that* $L(H) \cong \mathfrak{h}$*, then* $\exp_H | V \cap \mathfrak{h} : V \cap \mathfrak{h} \to H$ *is injective;*
4. $(\exp_H(\{x \in \mathfrak{h} | \, p(x) < 1\}))^2 \subseteq W_{\mathfrak{h}}$*, where* $W_{\mathfrak{h}}$ *is an open neighborhood of* 1 *contained in* $\exp_H(V)$*;*
5. m_U *is uniformly continuous on* $\{p < 1\}^2$*;*
6. m_U *is uniformly smooth at finite points.*

Then $\mathfrak{g}$ *is enlargeable.*

Main Theorem Recalled

Theorem

Suppose $\mathfrak{g}$ is a locally exponential Lie algebra, $\mathcal{H}$ is a family of closed subalgebras of $\mathfrak{g}$, V is a neighborhood of 0 in $\mathfrak{g}$ and p is a continuous seminorm on $\mathfrak{g}$ satisfying:

1. $\bigcup \mathcal{H}$ *is dense in* $\mathfrak{g}$;
2. *for each* $\mathfrak{h} \in \mathcal{H}$, *there is a locally exponential Lie group* H *such that* $L(H) \cong \mathfrak{h}$;
3. *for each* $\mathfrak{h} \in \mathcal{H}$, *if* H *is a connected locally exponential Lie group such that* $L(H) \cong \mathfrak{h}$, *then* $\exp_H | V \cap \mathfrak{h} : V \cap \mathfrak{h} \to H$ *is injective;*
4. $(\exp_H(\{x \in \mathfrak{h} | \ p(x) < 1\}))^2 \subseteq W_{\mathfrak{h}}$, *where* $W_{\mathfrak{h}}$ *is an open neighborhood of* 1 *contained in* $\exp_H(V)$;
5. m_U *is finite;*
6. m_U *is uniformly smooth at finite points.*

Then $\mathfrak{g}$ *is enlargeable.*

Sketch of The Proof

- As in the original Pestov Theorem, find $\mathfrak{h} \in \mathcal{H}^*$ such that the map $\iota : \mathfrak{g} \to \hat{g}^*$ given by $\iota(x) = x + \mu$ actually takes values in $\hat{\mathfrak{h}}$.
- Let H be an internal locally exponential Lie group such that $\text{Lie}(H) \cong \mathfrak{h}$. Define

$$H_f := \bigcup_n \exp_H(\mathfrak{h}_f)^n$$

and

$$\mu_H := \exp(\mu_{\mathfrak{h}}).$$

- H_f is clearly a group. Using our extra assumptions, we show that μ_H is a normal subgroup of H_f.
- Let $\hat{H} := H_f/\mu_H$ and let $\pi_H : H_f \to \hat{H}$ be the quotient map.

Sketch of The Proof (Cont'd)

- One shows that for all $x, y \in \mathfrak{h}$, if $x + \mu_{\mathfrak{h}} = y + \mu_{\mathfrak{h}}$, then $\pi_H(\exp(x)) = \pi_H(\exp(y))$.
- This allows us to define $\hat{\exp} : \hat{\mathfrak{h}} \to \hat{H}$ by $\hat{\exp}(x + \mu_{\mathfrak{h}}) = \pi_H(\exp x)$.
- Let $\hat{W} := \{x + \mu_{\mathfrak{h}} \mid \hat{p}(x + \mu_{\mathfrak{h}}) < 1\}$. Then $\hat{\exp}$ is injective on $\hat{W}$.
- Let $m_{\hat{\mathfrak{h}}} : \hat{W} \times \hat{W} \to \hat{\mathfrak{h}}$ be given by $m_{\hat{\mathfrak{h}}}(x + \mu_{\mathfrak{h}}, y + \mu_{\mathfrak{h}}) := (x * y) + \mu_{\mathfrak{h}}$. By uniform smoothness at finite points, this is a smooth map and suitably restricted, this witnesses that $\hat{W}$ is a local Lie group satisfying the necessary hypotheses to show that $\hat{\mathfrak{h}}$ is a locally exponential Lie algebra.

Sketch of The Proof (Cont'd)

- Equip $\hat{H}_1$, the subgroup of $\hat{H}$ generated by $\exp(\hat{W})$, with the structure of a locally exponential Lie group such that $\mathrm{Lie}(\hat{H}_1) \cong \hat{\mathfrak{h}}$, whence $\hat{\mathfrak{h}}$ is enlargeable.
- Since $\iota : \mathfrak{g} \to \hat{\mathfrak{h}}$ is an injective morphism of locally convex Lie algebras, it follows that $\mathfrak{g}$ is enlargeable.

A Standard Formulation

Theorem

Suppose $\mathfrak{g}$ is a locally exponential Lie algebra, $\mathcal{H}$ is a family of closed subalgebras of $\mathfrak{g}$, V is a neighborhood of 0 in $\mathfrak{g}$ and p is a continuous seminorm on $\mathfrak{g}$ satisfying:

1. *$\bigcup \mathcal{H}$ is dense in $\mathfrak{g}$;*
2. *for each $\mathfrak{h} \in \mathcal{H}$, there is a locally exponential Lie group H such that $L(H) \cong \mathfrak{h}$;*
3. *for each $\mathfrak{h} \in \mathcal{H}$, if H is a connected locally exponential Lie group such that $L(H) \cong \mathfrak{h}$, then $\exp_H | V \cap \mathfrak{h} : V \cap \mathfrak{h} \to H$ is injective;*
4. *$\exp_H(\{x \in \mathfrak{h} | \ p(x) < 1\})^2 \subseteq W_{\mathfrak{h}}$, where $W_{\mathfrak{h}}$ is an open neighborhood of 1 contained in $\exp_H(V)$.*

Moreover, assume that either the local group operation on $\mathfrak{g}$ is strongly smooth or that $\mathfrak{g}$ is modeled on a complete (HM)-space. Then $\mathfrak{g}$ is enlargeable.

A Question

Even for locally exponential Lie algebras with strongly smooth local group operation or those modeled on complete (HM)-spaces, we have the extra assumption (4) that does not appear in the original Pestov theorem.

(4) $(\exp_H(\{x \in \mathfrak{h} | \, p(x) < 1\}))^2 \subseteq W_{\mathfrak{h}}$, where $W_{\mathfrak{h}}$ is an open neighborhood of 1 contained in $\exp_H(V)$.

The real import of this assumption is that the local group operation on each $\mathfrak{h} \in H$ given by exponential coordinates agrees with the local group operation m_U on $\mathfrak{g}$ on the set $\{(x, y) \in \mathfrak{h} \times \mathfrak{h} \mid p(x), p(y) < 1\}$.

Question

Can assumption (4) be removed?

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