Euler-Poincaré Characteristic, Todd genus and signature of singular varieties

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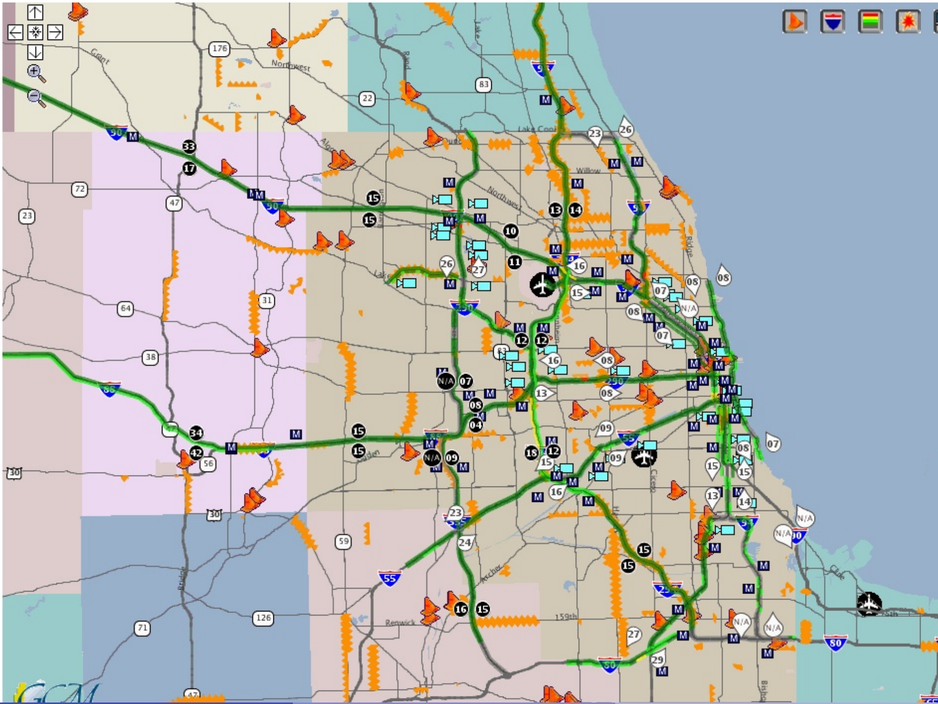

Preamble : 1. Euler - Poincaré Characteristic

Definition (Poincaré)

Let X be a triangulated compact (smooth or singular) variety, the Euler - Poincaré characteristic of X is defined as

$$e(X) = \sum_{i=0}^{m} (-1)^i k_i$$

where $m = \dim_{\mathbb{R}} X$ and k_i is the number of i-dimensional simplexes.

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Example 1 (Lhuilier)

Let X be a complex algebraic curve, *i.e.* a compact Riemann surface.
X is homeomorphic to a sphere with g handles.
The Euler - Poincaré characteristic of X is

$$e(X) = 2 - 2g.$$

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Example 2

The Euler - Poincaré characteristic of the pinched torus

is $e(P) = 1$

Preamble : 1. Euler - Poincaré Characteristic

Theorem (Poincaré-Hopf)

Let X be a compact manifold and let v be a (continuous) vector field with (finitely many) isolated singularities $(a_j)_{j \in J}$ of index $I(v, a_j)$, then

$$e(X) = \sum_{j \in J} I(v, a_j).$$

Preamble : 2. The arithmetic genus

Let X be a complex algebraic manifold, $n = \dim_{\mathbb{C}} X$.
Let g_i be the number of $\mathbb{C}$-linearly independent holomorphic differential i-forms on X.

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Preamble : 3. The arithmetic genus (an example)

Example

Let X be a complex algebraic curve, *i.e.* a compact Riemann surface. X is homeomorphic to a sphere with g handles. Then $g_0 = 1$ and $g_1 = g_n = g$.
The arithmetic genus of X is:

$$\chi(X) = 1 - g$$

Preamble : 4. The Todd genus

The Todd genus $T(X)$ has been defined (by Todd) in terms of Eger-Todd fundamental classes (polar varieties), using Severi results. The Eger-Todd classes are homological Chern classes of X.

Todd "proved" that

$$T(X) = \chi(X).$$

In fact, the Todd proof uses a Severi Lemma which has never been completely proved. The Todd result has been proved by Hirzebruch.

Preamble : 5. The signature

Definition (Thom-Hirzebruch)

Let M be a (real) compact oriented $4k$-dimensional manifold. Let x and y two elements of $H^{2k}(M;\mathbb{R})$, then

$$\langle x \cup y, [M] \rangle \in \mathbb{R}$$

defines a bilinear form on the vector space $H^{2k}(M;\mathbb{R})$.
The index (or signature) of M, denoted by $\mathrm{sign}(M)$, is defined as the index of this form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

What shall we do ?

X manifold		
number		
$e(X)$		
—	—	—
$\chi(X)$		
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X manifold	X manifold	
number	cohomology classes	
$e(X)$	Chern	
—	—	—
$\chi(X)$	Todd	
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$\mathrm{sign}(X)$	Thom-Hirzebruch	

Hirzebruch Theory

What shall we do ?

X manifold	X manifold	X singular variety
number	cohomology classes	homology classes
$e(X)$	Chern	Schwartz-MacPherson
—	—	—
$\chi(X)$	Todd	Baum-Fulton-MacPherson
—	—	—
$\mathrm{sign}(X)$	Thom-Hirzebruch	Cappell-Shaneson

Hirzebruch Theory   Motivic Theory (BSY)

Hirzebruch Series

$$Q_y(\alpha) := \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]]$$

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- $Q_1(\alpha) = \frac{\alpha}{\tanh \alpha} \qquad y = 1$

Characteristic Classes of Manifolds.

Let X be a complex manifold with dimension $\dim_{\mathbb{C}} X = n$, let us denote by

$$c^*(TX) = \sum_{j=0}^{n} c^j(TX), \qquad c^j(TX) \in H^{2j}(X;\mathbb{Z})$$

the total Chern class of the (complex) tangent bundle TX.

Definition

The Chern roots α_i of TX are defined by:

$$\sum_{j=0}^{n} c^j(TX)\, t^j = \prod_{i=1}^{n} (1 + \alpha_i t)$$

$\alpha_i \in H^2(X;\mathbb{Z})$.

One defines the Todd-Hirzebruch class: $\widetilde{\mathbf{td}_{(\mathbf{y})}}(\mathbf{TX}) := \prod_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{Q}_{\mathbf{y}}(\alpha_{\mathbf{i}})$

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The χ_y-characteristic

Let X be a complex projective manifold.

Definition

One defines the χ_y*-characteristic of* X *by*

$$\chi_y(X) := \sum_{p=0}^{<\infty} \left(\sum_{i=0}^{<\infty} (-1)^i \dim_{\mathbb{C}} H^i(X, \wedge^p T^* X) \right) \cdot y^p$$

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Hirzebruch Riemann-Roch Theorem

One has:

$$\chi_y(X) = \int_X \widetilde{td_{(y)}}(TX) \cap [X] \qquad \in \mathbb{Q}[y].$$

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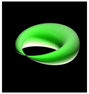
- $\mathrm{sign}(X) = \int_X L^*(TX) \cap [X]$ $\qquad y = 1$
  signature of X
  Hirzebruch signature Theorem

Question

What can we do for singular varieties?



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Chern Transformation (MacPherson)

$\mathbb{F}(X)$: Group of constructible functions (ex. $\mathbf{1}_X$)

$$c_* : \mathbb{F}(X) \to H_*(X)$$

One defines $c_*(X) := c_*(\mathbf{1}_X)$: Schwartz-MacPherson class of X.

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$G_0(X)$: Grothendieck Group of coherent sheaves (ex. $\mathcal{O}_X$)

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L-Transformation (Cappell-Shaneson)

$\Omega(X)$: Group of constructible self-dual sheaves (ex. $\mathcal{IC}_X$)

$$L_* : \Omega(X) \to H_{2*}(X; \mathbb{Q})$$

One defines $L_*(X) := L_*([\mathcal{IC}_X])$

Problem:

The three transformations are defined on different spaces:

$$\mathbb{F}(X), \qquad G_0(X) \qquad \textit{and} \qquad \Omega(X)$$

Where the "motivic" arrives...

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Definition

The Grothendieck relative group of algebraic varieties over X

$$K_0(var/X)$$

is the quotient of the free abelian group of isomorphy classes of algebraic maps $Y \longrightarrow X$, modulo the "additivity relation":

$$[Y \longrightarrow X] = [Z \longrightarrow Y \longrightarrow X] + [Y \setminus Z \longrightarrow Y \longrightarrow X]$$

for closed algebraic sub-spaces Z in Y.

Three results...

Theorem

The map $e : K_0(var/X) \longrightarrow \mathbb{F}(X)$ *defined by* $e([f : Y \to X]) := f_! \mathbf{1}_Y$ *is the unique group morphism which commutes with direct images for proper maps and such that* $e([id_X]) = \mathbf{1}_X$ *for* X *smooth and pure dimensional.*

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The morphism $sd : K_0(var/X) \longrightarrow \Omega(X)$ *defined by*

$$sd([f : Y \to X]) := [Rf_* \mathbb{Q}_Y[\dim_{\mathbb{C}}(Y) + \dim_{\mathbb{C}}(X)]]$$

is the unique group morphism which commutes with direct images for proper maps and such that $sd([id_X]) = [\mathbb{Q}_X[2\dim_{\mathbb{C}}(X)]] = [\mathcal{IC}_X]$ *for* X *smooth and pure dimensional.*

...plus one...

Theorem

There is an unique group morphism

$$T_y : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]$$

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Remark

If a complex algebraic variety X has only rational singularities (for example if X is a toric variety), then:

$$mC([id_X]) = [\mathcal{O}_X] \in G_0(X) \quad \text{and in this case} \quad T_0([id_X]) = td_*(X).$$

That is not true in general !

Only for specialists...

Verdier Riemann-Roch Formula

Let $f : X' \to X$ be a smooth map (or a map with constant relative dimension), then one has

$$\widetilde{td_{(y)}}(T_f) \cap f^* T_y([Z \longrightarrow X]) = T_y f^*([Z \longrightarrow X]).$$

Here T_f is the bundle over X' of tangent spaces to fibres of f.

Still for specialists...

Let us define $td_{(1+y)}([\mathcal{F}]) := \sum_{i=0}^{<\infty} \widetilde{td}_i([\mathcal{F}]) \cdot (1+y)^{-i}$.

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Then one has:

Factorisation of T_y

$$T_y = td_{(1+y)} \circ mC : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y].$$

The main result

The following diagrams commute:

$$
\begin{array}{ccccccc}
\mathbb{F}(X) & \xleftarrow{e} & K_0(var/X) & & & \xrightarrow{mC} & G_0(X) \\
 & & & sd \searrow & & & \\
c_* \downarrow & & T_y \downarrow & & \Omega(X) & & \downarrow td_* \\
 & & & & & & \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] & & L_* \downarrow & \xrightarrow{y=0} & H_*(X) \otimes \mathbb{Q} \\
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Thanks for your attention

Happy birthday Anatoly