Contact manifolds and (non-)isolated singularities

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Jaca, June 24, 2009 – Lib60ber

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Example (Brieskorn)

There is an infinite family of non conjugated hypersurface germs $X_p = f_p^{-1}(0) \subset (\mathbb{C}^{2n+2}, 0)$ such that $M(X_p) \simeq S^{4n+1}$ for all p.

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- in dimension 3 : yes (Neumann)
- in higher dimensions : not yet in general. But there are necessary conditions on the cohomology ring (Hain&Durfee, Popescu-Pampu).

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Proposition (Scherk)

The CR-manifold $(M(X), \xi(X), J_X)$ is NOT invariant w.r.t the choices, BUT completely determines the analytical type of (X, x).

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The pair $(M(X), \xi(X))$ is a contact manifold, independant on the choices up to isotopy. It is the contact boundary of (X, x).

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For all $n \geq 2$ the infinite family of contact boundaries of the hypersurface singularities $(X_p, 0) \subset (\mathbb{C}^{2n+2}, 0)$ are pairwise non isomorphic, thus giving infinitely many different contact structures on the sphere S^{4n+1}.

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Remark: In μ-constant deformations, the homotopy invariants of $\xi(X_t)$ remain constant (C)

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If (X, x) is a normal surface singularity, the isomorphism type of the contact boundary $(M(X), \xi(X))$ only depends on that of the boundary $M(X)$.

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One has to find another way to construct contact manifolds from surface singularities

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Example (Lê)

For $f : (x, y, z, t) \mapsto (xy + z^2, x)$, one has $Disc(f) = \{(0, 0)\}$, but $M_{\varepsilon,(\alpha,0)} \simeq S^3 \coprod S^3$ and $M_{\varepsilon,(0,\alpha)} \simeq S^3$

The Milnor boundary of a germ of application (continued)

Proposition

If the germ $f : (\mathbb{C}^{p+2}, 0) \to (\mathbb{C}^p, 0)$ satisfies Thom's a_f-condition, then the Milnor fibre $F(f)$ and the Milnor boundary $M(f)$ are well defined oriented compact manifolds up to diffeomorphism.

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The complex tangencies give a contact structure $\xi(f)$ on the Milnor boundary $M(f)$, well defined up to contact isotopy. This is the contact Milnor boundary $(M(f), \xi(f))$ of f

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- $\mathbb{T}^3$ is (smoothly) Milnor fillable : one has $\mathbb{T}^3 \simeq M(xyz)$.

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- In higher dimensions, the three notions are pairwise distinct: there are uncompatible (in general) necessary (co)homological conditions for these three fillabilities to hold (Bungart, Durfee&Hain, Popescu-Pampu)

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The contact boundaries $(M(X_{2k,1}), \xi(X_{2k,1}))$ and $(M(f_k), \xi(f_k))$ are both diffeomorphic to the lens space $L(2k, 1)$, but are not contactomorphic.

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  $$c_1(\xi(f_k)) = 0 \in H^2(L(2k,1))$$

  whatever identification $L(2k,1) \simeq M(f_k)$ is made.

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$\widetilde{X}$: disk bundle over $E \simeq S^2$, with Chern class $-2k \in H^2(E) \simeq H^2(\widetilde{X}) \simeq \mathbb{Z}$.

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Proof

We will distinguish $\xi(X_{2k,1})$ and $\xi(f_k)$ by their first Chern class.

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Remark

The same result is true for any normal surface singularity X with $K.K \notin \mathbb{Z}$: its contact boundary cannot be isomorphic to any contact Milnor boundary.

Open books and contact manifolds

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How can one handle contact structures?

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Theorem (Giroux)

Any two contact structures supported by the same open book are isotopic.

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For any germ (X, x) of isolated singularity and for any function $g : X \to \mathbb{C}$ having an isolated singularity at x, the pair $(N(f), \theta(f))$ where

$$N(f) := M(X) \cap g^{-1}(0) \quad \text{and} \quad \theta(f) := g/|g| : M(X) \setminus N(g) \to S^1$$

is an open book in the boundary $M(X)$ which supports the natural contact structure $\xi(X)$. It is the Milnor open book of g in the boundary of X

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If the germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defines with f an isolated complete intersection singularity in $\mathbb{C}^3$, then for $1 \gg \varepsilon \gg |s| > 0$ the restriction

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Example

The restriction of z to the Milnor boundary $M(x^{2l+1} + y^2)$ defines the open book $(\Sigma_{l,1}, Id)$. Unfortunately, $\underset{k}{\#}(S^2 \times S^1)$ admits only one tight contact structure up to isotopy (Eliashberg, Colin). So this is the one carried by this open book.