Self-linking projective algebraic knots

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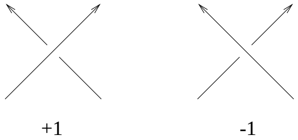
Linking of two curves in $\mathbb{R}^3$

Let $\alpha, \beta : \mathbf{S^1} \to \mathbb{R}^\mathbf{3}$ be smooth maps (oriented curves) with disjoint images.

Define the sign of a double point:



Project the link to the *xy*-plane so that only double points. The linking number $\text{lk}(\alpha, \beta)$ is one-half the number of signed double points where one curves crosses the other.

The linking number is invariant under Reidemeister moves, hence an isotopy invariant:

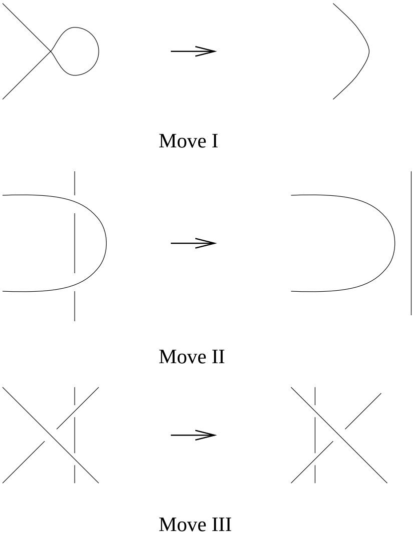
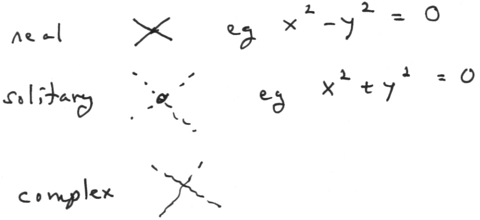


Figure: The three Reidemeister moves

Can't use this method to define the self-linking of a curve.
Problem: Double points disappear under the first Reidemeister move.
Solution: Oleg Viro, *Encomplexing the writhe*, AG/0005162.
Suppose that an algebraic curve $C \subset \mathbb{RP}^3$ is smoothly embedded over $\mathbb{C}$.
Project to $\mathbb{RP}^2$ such that has only double points. Three types:



Under the first algebraic Reidemeister move (I*) a real double point becomes a solitary double point.

Easier to explain in this situation:
A *projective knot* is a rational map

$$\alpha : \mathbb{RP} \to \mathbb{RP}^3$$

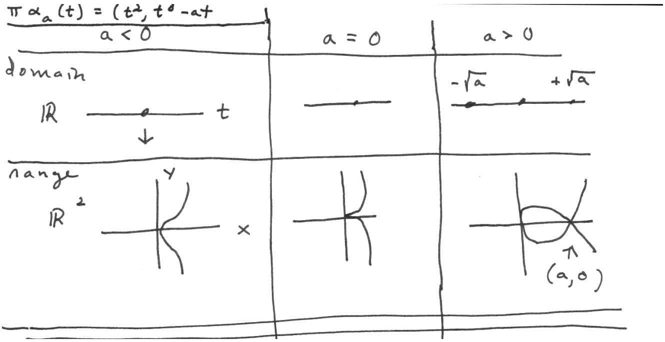
which is a smooth embedding over the complex numbers.
Will use these from now on. Usually give in affine coordinates.

Example:

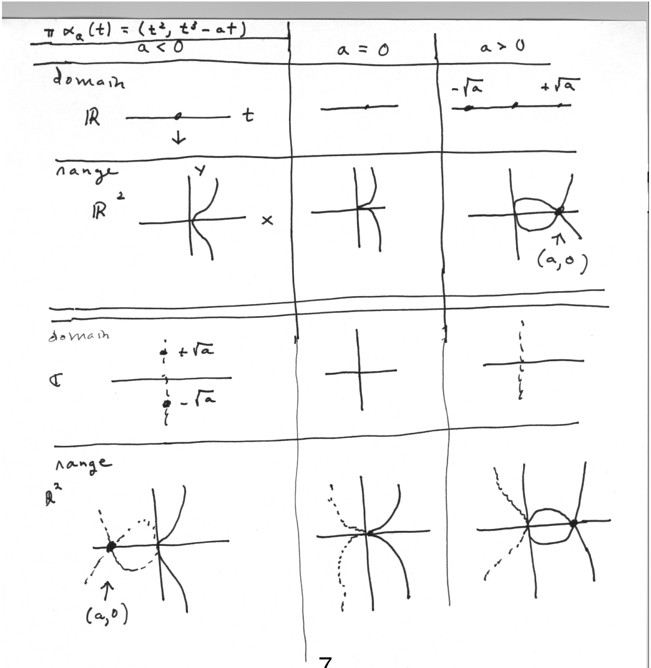
$$\alpha_a : \mathbb{R} \to \mathbb{R}^3$$

$$\alpha_a(t) = (t^2, t^3 - at, t)$$

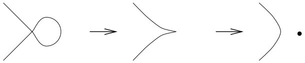
where $a \in \mathbb{R}$. Project to xy-plane.
This is the usual first Reidemeister move:



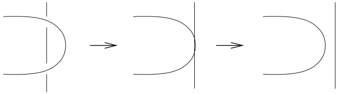
Now add image imaginary axis, get Reidemeister move I*:



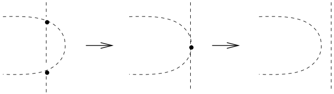
Two projective knots are equivalent iff their projections are equivalent under the five algebraic Reidemeister moves:



Move I*



Move II



Move II*

Move III

Move III*

The sign of a real or isolated double point at $(\alpha(r_0), \alpha(s_0))$ is the sign of the determinant:

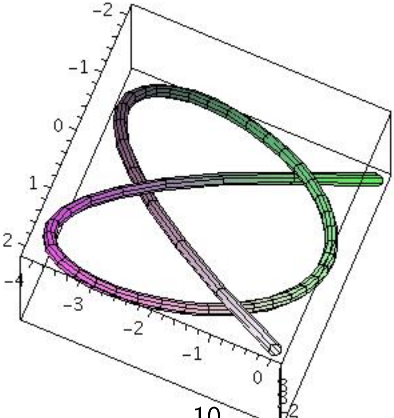
$$\det \begin{bmatrix} \alpha(r_0) - \alpha(s_0) \\ \alpha'(r_0) \\ \alpha'(s_0) \end{bmatrix}$$

By computation Reidemeister I* preserves the sign
(Above example is the general case...)
Other algebraic Reidemeister moves preserve the sign.
The *self-linking* $SL(\kappa)$ of a projective knot κ is the sum over all signed double points of a projection.

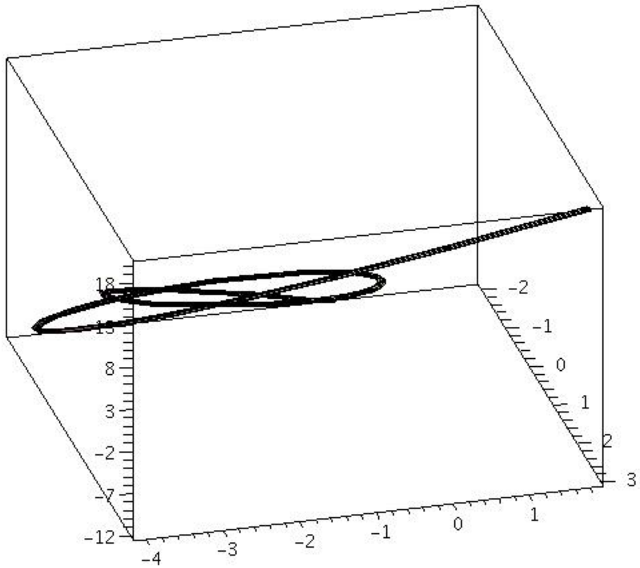
Example: Shastri's trefoil

$$\kappa(t) = (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$$

Six double points:
Three real double points, signs -1, -1, -1
Three solitary double points, signs -1, -1, $+1$
So the self-linking is -4.



Another rotated view with 6 real double points



Remark: The number of double points of a complex plane curve in generic position of degree d and genus zero is

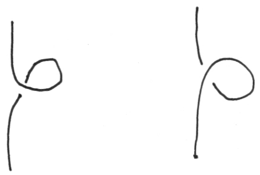
$$D = \frac{1}{2}(d-1)(d-2).$$

Thus
d = 3: D = 1
d = 5: D = 6
d = 7: D = 15
Also the self-linking of the mirror image of a knot is the negative of the SL of the original knot:

$$\mathrm{S}L(-\kappa) = -SL(\kappa)$$

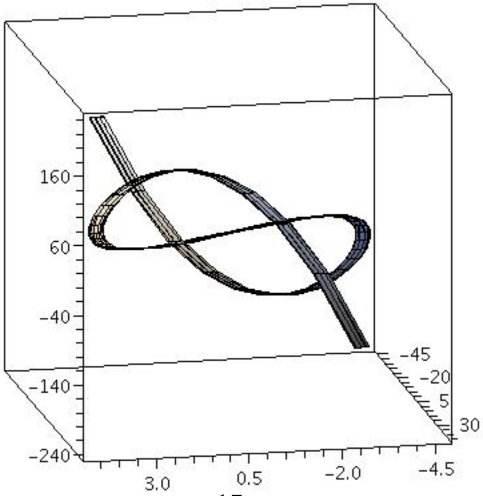
Space of projective knots:
Let $\mathcal{M}_d$ be the space of all algebraic maps $\mathbb{RP} \to \mathbb{RP}^3$.
Let $\mathcal{K}_d$ be the subspace of projective knots (smooth and embedded over $\mathbb{C}$).
For $d \leq 4$ these are topologically unknotted.
Remark: The knot κ is null homotopic iff d is even.

The space $\mathcal{K}_3$ has at least two components, since
$\mathsf{S}L\big((t^3 - t, t^2, -t)\big) = +1$
$\mathsf{S}L\big((t^3 - t, t^2, t)\big) = -1$



Equations of degree 7 give the figure-8 knot (REU):

$$\kappa(t) = (t^3 - 5t, t^5 - 28t, t^7 - 32t^3)$$



A projection of this figure-8 knot has 15 double points.
Since complex double points occur in pairs,
(# real dp) + (# solitary dp) is an odd number, in particular $\neq 0$.
Thus the algebraic figure 8 knot and its mirror image lie in distinct components of $\mathcal{K}_7$.
The topological figure-8 knot is amphicheiral (Colin Adams, *The knot book*)

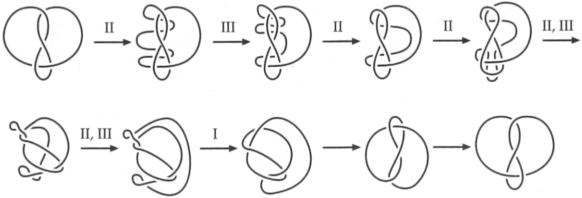


Figure 1.27 The figure-eight knot is equivalent to its mirror image.
(Colin Adams, The Knot Book)

Abortive Gauss map: