# Representations of arrangement groups

Michael Falk

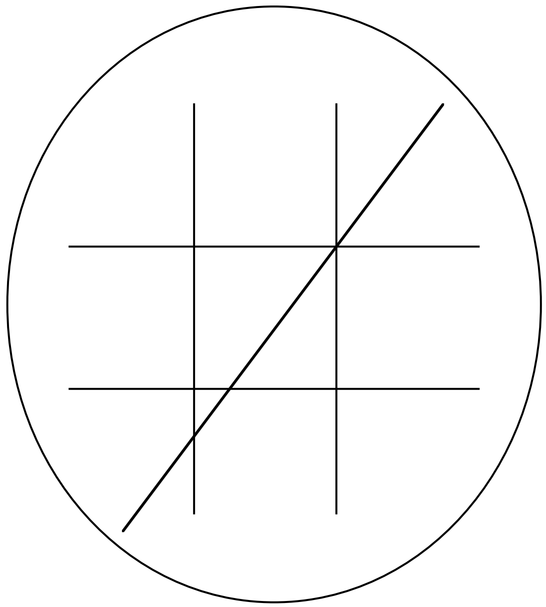
Northern Arizona University

(joint work w/ Dick Randell and Dan Cohen)

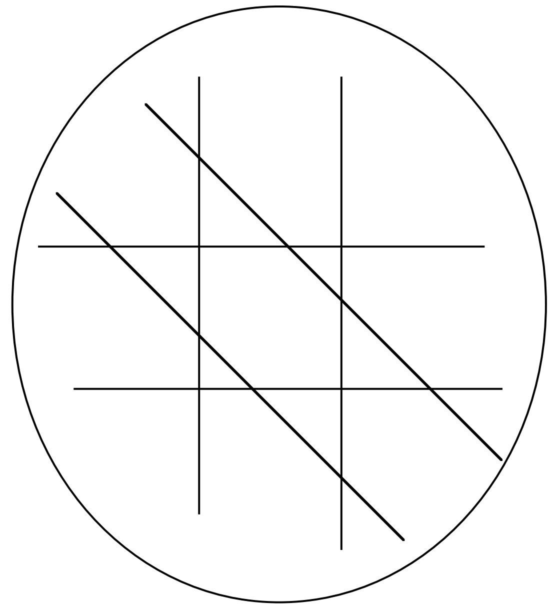
Lib60ber: Topology of algebraic varieties

Jaca, Huesca, Spain. June 26, 2009

## The motivating example



$A_1$



$A_2$

$\mathbb{P}^2 - \bigcup \mathcal{A}_1$ is an infinite cyclic cover of $\mathbb{P}^2 - \bigcup \mathcal{A}_2$.
(Matei-Suciu, 2004, private communication)

**Notation**

$\mathcal{A}$ : an arrangement of lines in $\mathbb{P}^2$

$\overline{M} := \mathbb{P}^2 - \bigcup \mathcal{A}$

$\overline{G} := \pi_1(\overline{M})$

$\mathcal{X}$ : the set of intersection points of multiplicity $\geq 3$

For $X \in \mathcal{X}$:

$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \in H\}$ Assume $\bigcup_{X \in \mathcal{X}} \mathcal{A}_X = \mathcal{A}$

$\overline{G}_X := \pi_1(\mathbb{P}^2 - \bigcup \mathcal{A}_X)$ $\overline{G}_X$ is free of rank $|\mathcal{A}_X| - 1$

$G_X := \pi_1(B(X, \epsilon) - \bigcup \mathcal{A}_X)$ $G_X \cong \overline{G}_X \times \mathbb{Z}$

## A split homomorphism

For $X \in \mathcal{X}$, we have an inclusion-induced homomorphism

$$\phi_X \, : \, \overline{G} \to \overline{G}_X,$$

which splits because $\overline{G}_X$ is free.

Consider the product homomorphism

$$\phi \, : \, \overline{G} \to \prod_{X \in \mathcal{X}} \overline{G}_X$$

**Theorem** *The image of $\phi$ is normal, and the cokernel is free abelian of rank*

$$\sum_{X \in \mathcal{X}} (|\mathcal{A}_X| - 1) - |\mathcal{A}| + 1.$$

Can $\phi$ be injective?

## Consequences of injectivity

If $\phi$ is injective:

- $\overline{G}$ is linear, hence residually finite.
- $\overline{G}$ is residually free, hence residually nilpotent.
- $\overline{G}$ has solvable word and conjugacy problems.
- $\overline{G}$ is determined up to isomorphism by the combinatorics (e.g., incidence graph) of $\mathcal{A}$.
- $\overline{G}$ is of type $F_{k-1}$ but not $F_k$[1], hence $\overline{M}$ is not aspherical.

[1]Meier, Meinert, Van Wyk, Commentarii Math. Helvetici, 73:2244, 1998.

## Decomposable arrangements

Let $\overline{G} = \overline{G}^1 \supseteq \overline{G}^2 \supseteq \cdots \overline{G}^n \supseteq \cdots$ be the lower central series[2] and $\overline{G}^\infty = \bigcap_{n\geq 1} \overline{G}^n$ the nilpotent residue of $\overline{G}$.

**Definition** *$\mathcal{A}$ is decomposable if*

$$\phi_*\colon (\overline{G}^n/\overline{G}^{n+1}) \otimes \mathbb{Q} \to \oplus_{X\in\mathcal{X}}[(\overline{G}_X)^n/(\overline{G}_X)^{n+1}] \otimes \mathbb{Q}$$

*is an isomorphism for every $n \geq 2$.*

[2] $\overline{G}^{n+1} = [\overline{G}, \overline{G}^n]$ for $n \geq 1$

Since $\overline{G}$ and $\overline{G}_X$ are 1-formal, decomposability of $\mathcal{A}$ is determined by the cohomology ring of $\overline{M}$. A result of S. Papadima and A. Suciu[3] makes it easy to check for decomposability: it is enough to check the $n = 3$ stage. This amounts to a straightforward matrix calculation. (Note: If $\mathcal{A}$ is decomposable then "all resonance is local," but not conversely.)

**Theorem** *If $\mathcal{A}$ is decomposable, then* $\ker(\phi) = \overline{G}^{\infty}$.

**Corollary** *If $\mathcal{A}$ is decomposable and $\phi$ is not injective, then $\overline{G}$ is not residually nilpotent.*

[3] Commentarii Math. Helvetici, 81(4):859875, 2006

**Brunnian elements**

$\overline{G}$ is generated by meridians $g_H, \quad H \in \mathcal{A}$.

$\overline{G}_X$ is isomorphic to the quotient

$$\overline{G}/\langle\langle g_H \mid H \notin \mathcal{A}_X \rangle\rangle$$

and $\phi_X$ is the canonical projection.

Ted Stanford gave a description of Brunnian[4] braids in terms of the standard generators of the braid group[5] . His argument applies also to our situation.

[4]deleting any strand yields the trivial braid

[5]`arxiv.org/math.GT/9907072`

**Definition** *An element* $g \in \overline{G}$ *is a monic commutator if* $g = g_H$ *for some* $H \in \mathcal{A}$, *or* $g$ *can be expressed as a commutator with entries from* $\{g_H \mid H \in \mathcal{A}\}$. *The support of a monic commutator* $g$ *is*

$$\{H \in \mathcal{A} \mid g \text{ can be expressed as a commutator involving } g_H\}.$$

**Theorem** *The kernel of* $\phi$ *is generated by* monic commutators *whose support meets* $\mathcal{A} - \mathcal{A}_X$ *for every* $X \in \mathcal{X}$.

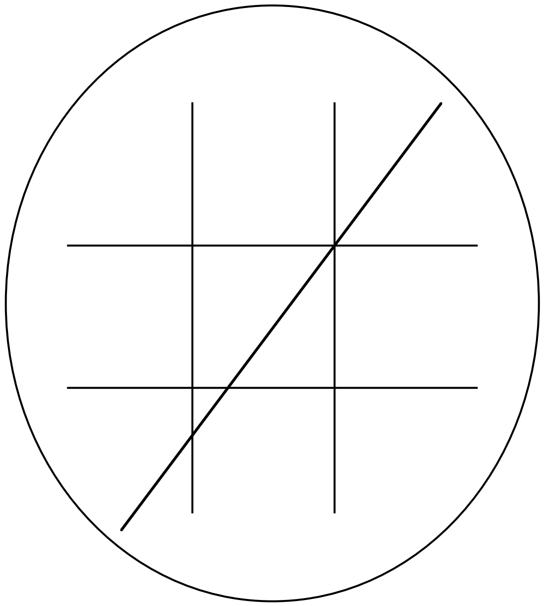
## A simple criterion for injectivity

**Theorem** *$\phi$ is injective if and only if, for every $X \in \mathcal{X}$, for every $H \notin \mathcal{A}_X$, $[g_H, (G_X)^2] = 1$.*

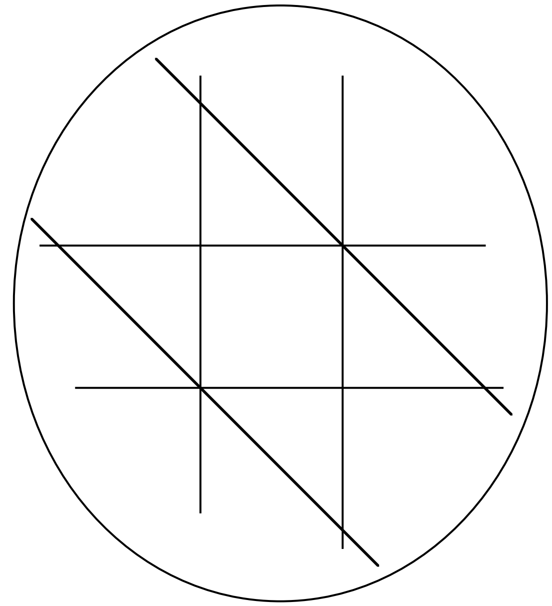
Since $G_X \cong \overline{G}_X \times \mathbb{Z}$, $(G_X)^2$ is generated by commutators of $|\mathcal{A}_X| - 1$ of the standard generators $g_H$, $H \in \mathcal{A}_X$.

**Corollary** *Suppose, for every $X \in \mathcal{X}$, for every $H \notin \mathcal{A}_X$, $H \cap K$ is a double point for all but one $K \in \mathcal{A}_X$. Then $\phi$ is injective.*

## Examples



$A_1$



$A_3$

The corollary applies to the family of examples (generalizing $\mathcal{A}_1$) constructed by E. Artal-Bartolo, J. Cogolludo-Augustin, and D. Matei.[6] $\mathcal{A}_3$ is a "new" example.

[6] in preparation