# objects: real elliptic Lefschetz fibrations 


aim: classification (up to equivariant diff.)
too $\frac{1}{3}$ necklace- diagrams

## (A) Lefschetz fibrations

$$
\left(p: X^{4} \rightarrow B^{2}\right)
$$

"complex morse functions"
around critical points $p$ looks like : $\mathbb{C}^{2} \rightarrow \mathbb{C}$

$$
\left(z_{1}, z_{2}\right) \rightarrow z_{1}^{2}+z_{2}^{2}
$$



## (B) real structure

"smooth version of complex conjugation"
$c_{X}: X^{4} \rightarrow X^{4}$
.orientation preserving involution
.dimension of fixed point set (if not empty) $=2$
$c_{B}: B^{2} \rightarrow B^{2}$
.orientation reversing involution
( $X$, c): real manifold, Fix(c): real part

## (C) real Lefschetz fibrations


(D) elliptic: regular fiber


## (E) some properties

1)critical sets are invariant under the action of real structures.
2) over real points of $B$, fibers inherit real structure from the real structure of $X$.
3)monodromy decomposes into product of two real structures.


# (F) main theorem 

*RELFs over sphere
. have only real critical values
. admit a real section

## 1-1

*RELFs over sphere
have only real critical values
necklace diag. .monodromy=id

## REFINED

 necklace diag.up to symmetry
.monodromy=id

## (Moishezon \& Livné, 1977)

1-1
*ELFs over sphere<----> \# of critical values=12n

$$
\begin{aligned}
& E(1)=\mathbb{C} P^{2} \# 9 \mathbb{C} \bar{P}^{2} \\
& E(n)=E(n-1) \# E(1)
\end{aligned}
$$

$(G)$ necklace diagrams "from 4 to 2"
look at the real locus:
(assume for the moment that there exists a real section)

(H)monodromy of necklace diagrams
idea: $f=c_{0}^{\prime} c \leadsto f_{x}=c_{x}^{\prime} \circ c_{*}=P^{-1}\left[\begin{array}{c}10 \\ 0-1\end{array}\right] P\left[\begin{array}{c}1 \\ 0 \\ 0-1\end{array}\right]$ mood differ
isomorphism in homdogy Monodromy
of necklace bass of of neck dial
teigospales

$$
\begin{aligned}
c: T^{2} & \rightarrow T^{2} \Rightarrow c_{*}: H_{1}\left(T^{2}, \mathbb{Z}\right) \rightarrow H_{1}\left(T^{2}, \mathbb{Z}\right) \\
& H_{ \pm}^{c}
\end{aligned}=\left\{a: c_{*}(a)= \pm a\right\}
$$

Around a critical valve:

$$
\begin{aligned}
& c \\
& \left.a_{b} b\right\rangle 0 \\
& \langle a\rangle=H_{+}^{c} \\
& \left\langle c^{\prime}\right. \\
& \left\langle b^{\prime}, b^{\prime}\right\rangle=H^{c} \\
& \left\langle a^{\prime}\right\rangle \\
& \left\langle b^{\prime}\right\rangle=H_{+}^{c}
\end{aligned}
$$

defined up to sian since $a \cdot b>0 \Leftrightarrow-a \cdot-b>0$
$\ell \therefore \in P S(2, Z)$
$P_{-x=}$ - base charge matrice from $\left(a^{\prime}, b^{\prime}\right)$ to $(a, b)$

# $\operatorname{PSL}(2, \mathbb{Z})=\left\{x, y: x^{2}=y^{3}=i d\right\}$ 

$$
\begin{aligned}
& P_{-0<} P_{>0-}=x y x y x \\
& P_{-0<} P_{>\times-}=x y^{2} \\
& P_{-\times<} P_{>0-}=y^{2} x \\
& P_{-\times<} P_{>\times-}=y x y
\end{aligned}
$$

Necklace diagrams of real E(1) - having only cal crit. valves

- adiritting a real section


