Order 1 local invariants of maps between 3-manifolds

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History

Vassiliev	finite order invariants of knots
Arnold	semi-local invariants of order 1 of plane curves and fronts
VG, Houston Nowik	local order 1 invariants of maps of surfaces into $\mathbb{R}^3$
Ohmoto Aicardi	local order 1 invariants of maps of surfaces into $\mathbb{R}^2$
Oset Sinha Romero Fuster	local order 1 invariants of maps of 3-manifolds into $\mathbb{R}^3$, non-oriented

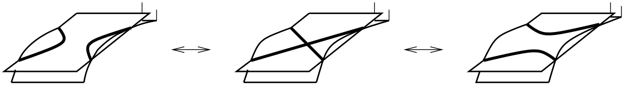
Example: maps of oriented surfaces into $\mathbb{R}^3$

3 **integer** invariants:
numbers of triple and pinch points,
and a self-linking number of a lifting of the image to $ST^*\mathbb{R}^3$

The latter counts a generalised number of inverse self-tangencies of the image in generic homotopies between maps

mod2:
4th invariant, counting similar number of direct self-tangencies

Non-coorientable direct self-tangency stratum:



Today

Local order 1 invariants of maps between oriented 3-manifolds, of any rank

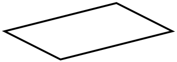
Main result

There are 8 linearly independent invariants over $\mathbb{Z}$ *and 12 over* $\mathbb{Z}_2$

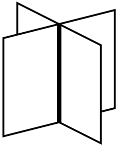
Generic critical value sets

$f : M^3 \to N^3$ Critical values: $\mathcal{C} \subset N$

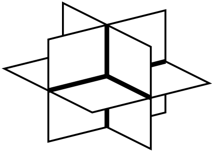
Smooth sheets of $\mathcal{C}$ and their transversal intersections



A_1



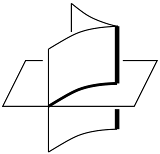
A_1^2



A_1^3

Co-orientation of the regular part of $\mathcal{C}$:

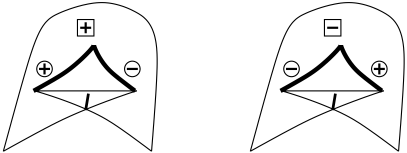
towards its side with more local preimages



A_2 A_2A_1

Cuspidal edges: positive and negative
according to the local degree of the map being ± 1

Hence signs for swallowtails:



A_3^+ A_3^-

Classification of integer-valued invariants

Theorem

The space of all integer-valued order 1 local invariants of maps of a compact 3-manifold to $\mathbb{R}^3$ is 8-dimensional. It is generated by:

I_s, half the number of all swallowtails;

I_s^-, half the difference between the numbers of positive and negative swallowtails;

I_t, the number of triple points A_1^3;

$I_{A_2A_1}$, half the number of intersection points of the cuspidal edge with the regular part of $\mathcal{C}$;

$I^-_{A_2A_1}$, half the difference between the numbers of intersections of the regular part of $\mathcal{C}$ with positive and negative cuspidal edges;

I_χ, the Euler characteristic of $\mathcal{C}$;

I_ℓ, the linking invariant (Oset Sinha, Romero Fuster);

I_{D_4}, the linking number of the image of the 1-jet extension of a map with $\Sigma^2 \subset J^1(M, \mathbb{R}^3)$.

Definition of I_{D_4}

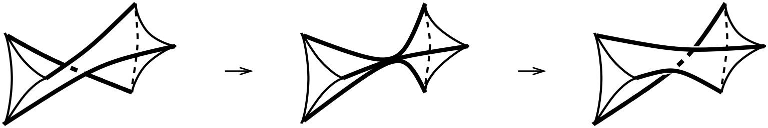
In $J^1(M, \mathbb{R}^3)$, take the set Σ^2 of all jets with corank 2 linear parts.
Fix a generic map $f_0 : M \to \mathbb{R}^3$.
For any other generic f_1, consider its generic homotopy $\{f_t\}_{0 \le t \le 1}$.
The images of the extensions $j^1 f_t$ define a 4-dimensional film $\varphi \subset J^1(M, N)$ which meets Σ^2 at isolated points.
We have the intersection index which may be interpreted as a result of integration:

$$\langle \varphi, \Sigma^2 \rangle = I_{D_4}(f_1) - I_{D_4}(f_0)\,,$$

where the last term is an arbitrary fixed number.

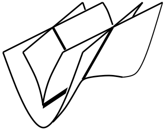
Corank 2 bifurcations in 1-parameter families

$D_4^{-\pm}: (\pm(x^2 - y^2) + zx - \lambda y, xy, z)$, of local degree ± 2

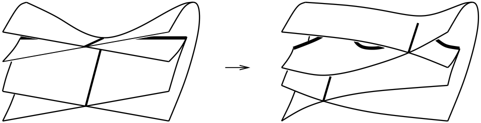


By this transition we co-orient the D_4^{-+} stratum.
The co-orientation of D_4^{--} is in the opposite direction.
Both co-orientations correspond to the increase of the deformation parameter λ.

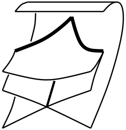
$D_4^{+\pm}$: $(x^2 + y^2 + zy + \lambda x, \pm xy, z)$, where $\pm$ is the edge sign for $\lambda = 0$:



Deformation:



Half of the right surface:



Co-orientation:

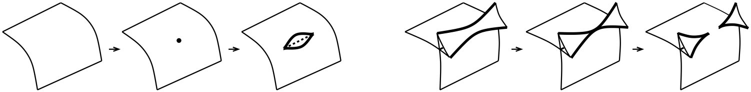
by the sign of the swallowtails, equivalently by the increase of λ

Classification of mod2 invariants

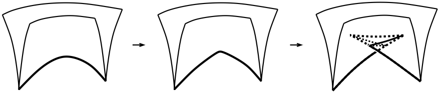
Theorem

The space of all mod2 order 1 local invariants of maps of a compact 3-manifold to $\mathbb{R}^3$ has rank 12. It is generated by:

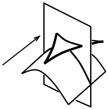
I_s, the number of positive swallowtails (same as negative)



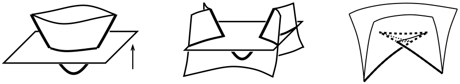
I_s^-, half the difference between the numbers of negative and positive swallowtails



I_t, the number of triple points

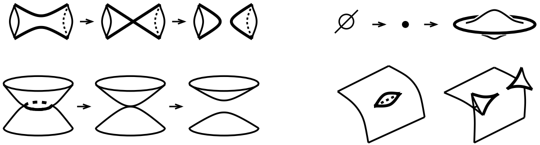


$I^-_{A_2A_1}$, half the difference between the numbers of intersections of the regular part of $\mathcal{C}$ with positive and negative cuspidal edges



I_{D_4}, the linking number as earlier (dual to D_4)

I_e, the number of connected components of the edge

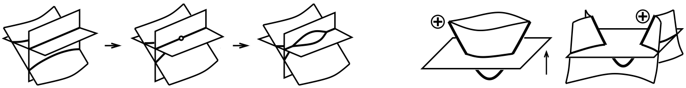


I_i, counts the number of inverse self-tangencies of the regular part of $\mathcal{C}$ in homotopies

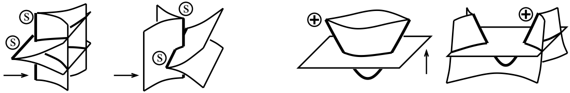
I_d, similar *generalised* count of direct self-tangencies

I_q, generalised count of quadruple points in homotopies

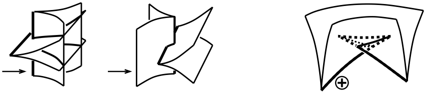
I_{10}, dual to



I_{11}, dual to



I_{12}, dual to



We have mod2

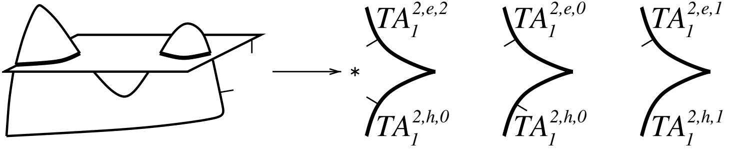
$$I_{A_2A_1} = I^-_{A_2A_1} + I_t$$

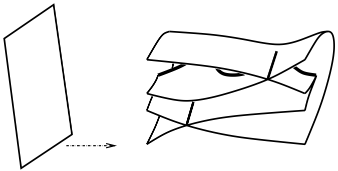
$$I_\chi = I_s + I^-_s + I_t$$

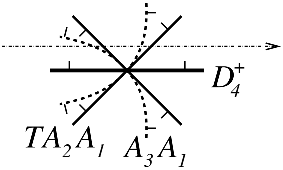
$$I_\ell = I_e + I_{A_2A_1} + I^-_s$$

Relations coming from codimension 2 singularities

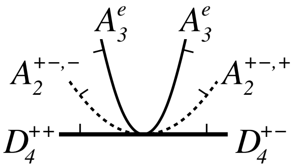
Degenerate tangency of two smooth sheets:






D_4^+
TA_2A_1
A_3A_1

Degenerate D_4^+:
$(x, y, z) \mapsto (x^2 + zy, y^2 + x(\pm x^2 \pm z^2 + \lambda_1 z + \lambda_2), z)$



$$D_5^{\pm}\colon\ (x,y,z)\mapsto(\pm x^2+y^3+y^2(\pm z+\alpha xy+\lambda_1)+\lambda_2 x+zy,\, xy,\, z)$$