Lefschetz theorems for vector bundles with a connection

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1. Introduction and main results

In SGA2, A.Grothendieck developed different kinds of Lefschetz theorems, in particular for the Picard group.

Question: Let X be a closed subvariety of some projective space, H a hyperplane. Under which hypothesis do we get $Pic\, X \simeq Pic\, X \cap H$?

Grothendieck: algebraic methods, so he admitted any field as ground field.

His strategy: use intermediate steps:

$$Pic\, X \simeq \lim_{\rightarrow} Pic\, U \simeq Pic\, \hat{X} \simeq Pic\, X \cap H$$

Here, U runs through the open neighbourhoods of $X \cap H$ in X, $\hat{X}$:= formal completion of X along $X \cap H$.

Note: $\hat{X}$ algebraic substitute for a tubular neighbourhood of $X \cap H$ in X.

It turned out that the middle isomorphism holds under reasonable assumptions, even with vector bundles instead of line bundles,
whereas we have the first and third isomorphism only under severe hypotheses.

Note that in the case of the first and third isomorphism it is hopeless to work with vector bundles.

As a result, Grothendieck obtained the desired Lefschetz theorem just for complete intersections.

Project with Lê Dũng Tráng: replace X by a quasi-projective variety $X \setminus Y$,
restrict to complex case and use transcendental methods, too.

In this talk, however, we will look at (complex) vector bundles: $Vect\, X \setminus Y$ instead of $Pic\, X \setminus Y$.

As said before: hopeless to get even the first isomorphism:

$$Vect\, X \setminus Y \simeq \lim_{\rightarrow} Vect\, U$$

where U runs through the Zariski-open neighbourhoods of $X \cap H \setminus Y$ in $X \setminus Y$.

So look at vector bundles with additional structure: **connection**. Then write $Vect_c$ instead of $Vect$. Note that we should have a vector bundle on a smooth variety.

Furthermore we can work in the algebraic and analytic category.

Theorem 1: *Suppose that $X \setminus Y$ is smooth of dimension ≥ 3, H intersects $X \setminus Y$ transversally,* $\operatorname{codim}_X Y \cap H \geq 4$. *Then one has a commutative diagram:*

$$\begin{array}{ccccc} Vect_c(X \setminus Y) & & \simeq & & Vect_c(\hat{X} \setminus \hat{Y}) \\ \downarrow\simeq & & & & \downarrow\simeq \\ Vect_c(X^{an} \setminus Y^{an}) & \simeq & \lim\limits_{\rightarrow} Vect_c(V - Y^{an}) & \simeq & Vect_c(\hat{X}^{an} \setminus \hat{Y}^{an}) \end{array}$$

where V runs through the (ordinary) open neighbourhoods of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$.

As for the third third isomorphism, it is not clear how to pass from $Vect\, \hat{X} \setminus \hat{Y}$ to $Vect\, X \cap H \setminus Y$.

Better: look at integrable connections. Then one can prove a Lefschetz theorem in the original sense:

Theorem 2: *Let $X \subset \mathbb{P}_N(\mathbb{C})$ be a Zariski-closed subspace, Y a Zariski-closed subset of X, $X \setminus Y$ smooth, everywhere of dimension ≥ 3, H a hyperplane. Let us fix a Whitney stratification of (X, Y). Suppose that there is a Zariski-closed subset Σ of $Y \cap H$ such that* $\mathrm{codim}_X \Sigma \geq 3$ *and that H intersects all strata of X transversally outside Σ. Then one has a commutative diagram of isomorphisms:*

$$\begin{array}{ccc} Vect_{cir}(X \setminus Y) & \simeq & Vect_{cir}(X \cap H \setminus Y) \\ \downarrow\simeq & & \downarrow\simeq \\ Vect_{ci}(X^{an} \setminus Y^{an}) & \simeq & Vect_{ci}(X^{an} \cap H^{an} \setminus Y^{an}) \end{array}$$

Here *cir* means that we impose the condition that the integrable condition is regular.

Note that we may apply Theorem 2 in particular under the hypothesis of Theorem 1.

It is relatively easy to prove this theorem - but by transcendental methods, not the one of Grothendieck. On the other hand, one cannot treat in the same way connections which are not integrable.

The situation is easier for the Picard group:

Theorem 3: *Suppose that $X \setminus Y$ is smooth of dimension ≥ 4, H intersects $X \setminus Y$ transversally,* $\mathrm{codim}_X Y \geq 4$. *Then one has a commutative diagram:*

$$\begin{array}{ccc} Pic_c(X \setminus Y) & \simeq & Pic_c(X \cap H \setminus Y) \\ \downarrow\simeq & & \downarrow\simeq \\ Pic_c(X^{an} \setminus Y^{an}) & \simeq & Pic_c(X^{an} \cap H^{an} \setminus Y^{an}) \end{array}$$

2. Definitions

Completion of X along $X \cap H$: formal scheme.

Underlying topological space: same as for the scheme $X \cap H$.
Structure sheaf: $\mathcal{O}_{\hat{X}} := \varprojlim(\mathcal{O}_X/\mathcal{I}^n)|X \cap H$, where $\mathcal{I} :=$ ideal sheaf of $X \cap H$ in X.

Now let X be a complex analytic manifold, $\mathcal{E}$ a holomorphic vector bundle on X (i.e. locally free analytic sheaf on X).

Connection on $\mathcal{E}$: defined by $\mathbb{C}_X$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ such that $\nabla(fs) = f\nabla s + s \otimes df$.

∇ induces $\nabla^p : \mathcal{E} \otimes \Omega^p_X \to \mathcal{E} \otimes \Omega^{p+1}_X$.

∇ integrable iff $\nabla^1 \circ \nabla^0 = 0$.

In this case, $\mathcal{L} := ker\, \nabla$ is a locally constant sheaf such that $\mathcal{E} \simeq \mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X$.

It is a technical disadvantage that ∇ fails to be $\mathcal{O}_X$-linear. But there is an equivalent description: Let $P^1(\mathcal{E})$ be the sheaf of 1-jets in $\mathcal{E}$. Then one has an exact sequence

$$0 \to \mathcal{E} \otimes \Omega^1_X \to P^1(\mathcal{E}) \to \mathcal{E} \to 0$$

and a connection on $\mathcal{E}$ corresponds to a splitting of this sequence, given e.g. by some map $\mathcal{E} \to P^1(\mathcal{E})$.

Similarly in the algebraic context (without $\mathcal{L}$). Here it makes sense to ask if an integrable connection is regular:

For curves it corresponds to the classical notion of regular singular points, in general it means that the pull-back to each curve is regular.

See P.Deligne:
Equations différentielles à points singuliers réguliers.

Regularity may be expected for connections which arise in a geometric situation (Gauß-Manin connection).

Note that the theory of connections has been largely superseded by the theory of D-modules. However D-modules correspond to connections which are automatically integrable!

3. Proofs

Start with Theorem 2 since it is easier.

Bijectivity of the lower horizontal: Recall that vector bundles with an integrable connection correspond to locally constant sheaves, so on each connected component of X it is given by a linear representation of its fundamental group.

So it is sufficient to quote a Lefschetz theorem for homotopy groups:

Under the hypothesis of Theorem 2,

$$\pi_j(X^{an} \cap H^{an} \setminus Y^{an}, x) \simeq \pi_j(X^{an} \setminus Y^{an}, x), \ x \in X^{an} \cap H^{an}, \ j = 0, 1.$$

Vertical arrows: use existence theorem of P.Deligne.

Before proving Theorem 1, ignore the connections and prove the following, where $S_k(X) := \{x \in X \text{ closed} \mid \text{depth } \mathcal{O}_{X,x} \leq k\}$:

Theorem 4: *Let $X \subset \mathbb{P}_N(\mathbb{C})$ be a Zariski-closed subspace, Y a Zariski-closed subset of X, H a hyperplane which is defined by an ideal sheaf $\mathcal{I}$ such that $\mathcal{I} \otimes \mathcal{O}_X \simeq \mathcal{I}\mathcal{O}_X$. Suppose that $\dim S_{l+2}(X \cap H \setminus Y) \leq l$ for $l \leq \dim Y \cap H$. Then one has a commutative diagram:*

$$\begin{array}{ccccc}
\lim\limits_{\rightarrow} Vect\, U & & \simeq & & Vect(\hat{X} \setminus \hat{Y}) \\
\downarrow\simeq & & & & \downarrow\simeq \\
\lim\limits_{\rightarrow} Vect\, U^{an} & \simeq & \lim\limits_{\rightarrow} Vect_c(V - Y^{an}) & \simeq & Vect(\hat{X}^{an} \setminus \hat{Y}^{an})
\end{array}$$

where U runs through the Zariski open neighbourhoods of $X \cap H \setminus Y$ in $X \setminus H \cap Y$ and V through the (ordinary) open neighbourhoods of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$.

In particular, Theorem 4 can be applied under the hypothesis of Theorem 1.

Proof: We ignore the middle term in the lower horizontal, as we will do in the proof of Theorem 1, too.

Surjectivity of the left vertical: Let U be a Zariski open neighbourhood of $X \cap H \setminus Y$ in $X \setminus Y$ and $j : U \to X$ be the inclusion. We can achieve that $\dim S_{l+2}(U) \leq l, l \leq \dim X \setminus U$. Let $\mathcal{E}$ be an (analytic) vector bundle on U^{an}. Under our hypothesis, one knows that $j_*^{an}\mathcal{E}$ is coherent. By GAGA it comes from an (algebraic) coherent sheaf on X. Its restriction to U must be locally free and represents an inverse image of $\mathcal{E}$.

Injectivity: Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be locally free on U such that $\mathcal{E}_1^{an} \simeq \mathcal{E}_2^{an}$. Then $(j_*\mathcal{E}_1)^{an} \simeq j_*^{an}\mathcal{E}_1^{an} \simeq j_*^{an}\mathcal{E}_2^{an} \simeq (j_*\mathcal{E}_1)^{an}$, so $j_*\mathcal{E}_1 \simeq j_*\mathcal{E}_2$, therefore $\mathcal{E}_1 \simeq \mathcal{E}_2$.

Right vertical: first show that

$$Vect\,(\hat{X} \setminus \hat{Y}) \simeq \lim_{\leftarrow} Vect\,(X_n \setminus Y_n)$$

where X_n is the n-th infinitesimal neighbourhood of $X \cap H$ in X.

Similarly in the analytic context. So it suffices to show:

$$Vect\,(X_n \setminus Y_n) \simeq Vect\,(X_n^{an} \setminus Y_n^{an})$$

This is shown as in the case of the left vertical, i.e. X_n, Y_n instead of X, Y.

Upper horizontal: The main difficulty is to prove surjectivity.

Here one shows that every vector bundle on $\hat{X} \setminus \hat{Y}$ admits a coherent extension to X, in particular to $X \setminus Y$. Now it follows that the latter is locally free on some neighbourhood of $X \cap H \setminus Y$ in $X \setminus Y$.

Injectivity: In fact, $Vect\,U \to \lim_{\leftarrow} Vect\,U_n$ is injective. Let $\mathcal{E}_1, \mathcal{E}_2$ be vector bundles on U, $\mathcal{S} := Hom(\mathcal{E}_1, \mathcal{E}_2)$, then $\mathcal{S}_n = Hom(\mathcal{E}_1|U_n, \mathcal{E}_2|U_n)$. An isomorphism between $\mathcal{E}_1|U_n$ and $\mathcal{E}_2|U_n$ gives an element of $H^0(U_n, \mathcal{S}_n)$, but $H^0(U, \mathcal{S}) \simeq H^0(U_n, \mathcal{S}_n)$ for $n \gg 0$, so it comes from an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$.

Now pass to the proof of Theorem 1.

As before with $U = X \setminus Y$, one has $Vect\,(X \setminus Y) \simeq Vect\,(X^{an} \setminus Y^{an})$.

In order to prove surjectivity of the left vertical, show that every analytic connection on an algebraic vector bundle is already algebraic. This follows by a GAGA argument. Notice that a connection is only $\mathbb{C}_{X\setminus Y}$-linear but it can be also expressed by a mapping which is $\mathcal{O}_{X\setminus Y}$-linear, see above.
More precisely: Let $\mathcal{E}$ be an analytic vector bundle on $X \setminus Y$ together with a connection given by $D : \mathcal{E} \to P^1(\mathcal{E})$, i.e. a global section of $\mathcal{S} := Hom(\mathcal{E}, P^1(\mathcal{E}))$. Let $j : X \setminus Y \to Y$ be the inclusion. Then $j_*^{an}\mathcal{S}$ is coherent, and we have an extension of the section of $\mathcal{S}$. By GAGA we can conclude that the vector bundle and the connection must both be algebraic.

The injectivity is easier.

As for bijectivity of the upper horizontal the delicate point is again to show surjectivity. Let $\mathcal{E}$ be a vector bundle on $\hat{X} \setminus \hat{Y}$ with a connection. It comes from a vector bundle $\mathcal{F}$ on some U, the connection can be extended. If j is as above, $j_*\mathcal{F}$ is coherent, and the connection extends to $j_*\mathcal{F}$.

Now the crucial fact is that a coherent sheaf with a connection is automatically locally free! So we get an inverse image.

Similarly for the lower horizontal.

Note that in Theorem 1 and 2 the most interesting case is the one where $Y = Sing\, X$:

Theorem 5: *Suppose that $X \setminus Y$ is smooth, so $Sing\, X \subset Y$.*
a) If Y is of codimension ≥ 2 we have a commutative diagram

$$\begin{array}{ccc} Vect_{ci}(X \setminus Y) & \simeq & Vect_{ci}(X \setminus Sing\, X) \\ \downarrow\simeq & & \downarrow\simeq \\ Vect_{ci}(X \setminus Y) & \simeq & Vect_{ci}(X \setminus Sing\, X) \end{array}$$

b) If Y is of codimension ≥ 3 we have a commutative diagram

$$\begin{array}{ccc} Vect_{c}(X \setminus Y) & \simeq & Vect_{c}(X \setminus Sing\, X) \\ \downarrow\simeq & & \downarrow\simeq \\ Vect_{c}(X \setminus Y) & \simeq & Vect_{c}(X \setminus Sing\, X) \end{array}$$

Proof: a) The main point is to show that

$$\pi_k(X^{an} \setminus Y^{an}) \simeq \pi_k(X^{an} \setminus Sing\, X^{an}),\ \ k = 0, 1$$

It would be conceptually simpler to prove the analogue for cohomology:
$H^k(X^{an} \setminus Sing\, X^{an}, Y^{an} \setminus Y^{an}; \mathbb{Z}) = 0, k \le 2$,
i.e. $H^k_{Y^{an} \setminus Sing\, X^{an}}(X^{an} \setminus Sing\, X^{an}, \mathbb{Z}_{X^{an}}) = 0, k \le 2$.
It is sufficient to show that $\mathcal{H}^k_{Y^{an} \setminus Sing\, X^{an}}(\mathbb{Z}_{X^{an}}) = 0, k \le 2$.
So a local consideration is sufficient.
Similarly for homotopy. The easiest seems to take a filtration of the total space which comes from a stratification. Then we are reduced to showing that

$$\pi_k(X^{an} \setminus Y'^{an}) \simeq \pi_k(X^{an} \setminus Y''^{an}),\ \ k = 0, 1,$$

if $Y' \setminus Y''$ is smooth of codimension ≥ 2.

b) Here we argue as in the case of the horizontal mappings in Theorem 1.

For the proof of Theorem 3 we use (with $\dim \emptyset := -1$):

Theorem 6: *a) Let $\mathcal{L}$ be an ample sheaf on X. Then*

$$H^q(X \setminus Y, \Omega^p_{X \setminus Y} \otimes \mathcal{L}^{-1}) = 0, p + q < \dim\, X - \dim\, Y - 1$$

b) $H^q(X \setminus Y, \Omega^p_{X \setminus Y}) \to H^q(X \cap H \setminus Y, \Omega^p_{X \cap H \setminus Y})$
is bijective for $p + q < \dim\, X - \dim\, Y - 2$
and injective for $p + q = \dim\, X - \dim\, Y - 2$.

Note that a) is a generalization of the theorem of Akizuki-Nakano, where $Y = \emptyset$.

Proof: a) We may suppose $\mathcal{L} = \mathcal{O}_X(1)$. We use the exact sequences
$0 \to \Omega^p_{X \setminus Y}(-1) \to \Omega^p_{X \setminus Y} \to i_*(\Omega^p_{X \setminus Y} | X \cap H \setminus Y) \to 0$ and
$0 \to \Omega^{p-1}_{X \setminus Y}(-1) | X \cap H \setminus Y \to \Omega^p_{X \setminus Y} | X \cap H \setminus Y \to \Omega^p_{X \cap H \setminus Y} \to 0$.
Furthermore we need that $H^q(X \setminus Y, \Omega^p_{X \setminus Y}(-k)) = 0, k \gg 0$, and the classical Akizuki-Nakano theorem.

b) follows from a).

Corollary: $H^0(X \setminus Y, P^1(\mathcal{O}_{X\setminus Y})) \simeq H^0(X \cap H \setminus Y, P^1(\mathcal{O}_{X\cap H\setminus Y}))$ *for* $\dim Y \leq \dim X - 4$.

Proof of Theorem 3:
In order to get $Pic(X \setminus Y) \simeq Pic(X \cap H \setminus Y)$ it is sufficient to have $H^k(X^{an} \setminus Y^{an}; \mathbb{Z}) \simeq H^k(X^{an} \cap H^{an} \setminus Y^{an}; \mathbb{Z})$, $k = 1, 2$.
Surjectivity of the upper horizontal: Let us start from a line bundle on $X \cap H \setminus Y$ with a connection. We saw that it comes from a line bundle $\mathcal{E}$ on $X \setminus Y$. The connection on $\mathcal{E}|X \cap H \setminus Y$ corresponds to an element of $H^0(X \cap H \setminus Y, \mathcal{S}|X \cap H \setminus Y)$ with $\mathcal{S} := Hom(\mathcal{E}, P^1(\mathcal{E})) \simeq P^1(\mathcal{O}_{X\setminus Y})$. We conclude by the corollary that the connection comes from a unique connection on $\mathcal{E}$. The rest is easy.

Corollary: *Let X be a complete intersection in $\mathbb{P}_r$,* $\mathrm{codim}_X Y \geq 4$*, $X \setminus Y$ smooth. Then $Pic_c(X \setminus Y)$ and $Pic_{ci}(X \setminus Y)$ are trivial.*

Proof: Suppose that we have a line bundle $\mathcal{E}$ with a connection. The curvature is an element of $H^0(X \setminus Y, \mathcal{S})$ with $\mathcal{S} := Hom(\mathcal{E}, \Omega^2(\mathcal{E})) \simeq \Omega^2_{X\setminus Y}$. But $H^0(X \setminus Y, \Omega^2_{X\setminus Y}) = 0$.
So $Pic_c(X \setminus Y) \simeq Pic_{ci}(X \setminus Y)$.
But $Pic_{ci}(X \setminus Y) = 0$ because $\pi_k(X^{an} \setminus Y^{an}, x) = 0$, $k = 0, 1$, $x \in X^{an} \cap H^{an} \setminus Y^{an}$.

4. Remarks

It is easy to prove the analogue of Theorem 1 with $Vect_{ci}$ instead of $Vect_c$.

In particular we get: $Vect_{ci}(X \setminus Y) \simeq Vect_{ci}(X^{an} \setminus Y^{an})$.

On the other hand, $Vect_{cir}(X \setminus Y) \simeq Vect_{ci}(X^{an} \setminus Y^{an})$.

So we can conclude that every integral connection on a vector bundle on $X \setminus Y$ is automatically regular! This is a priori not at all clear.

Similarly for $X \cap H \setminus Y$.

In particular, the corresponding question for regularity in the non-integrable case is open. So we have the following questions:

a) Is every connection on a vector bundle on $X \setminus Y$ (integrable or not) regular?

b) If not: is $Vect_{cr}(X \setminus Y) \simeq Vect_{cr}(\hat{X} \setminus \hat{Y})$?