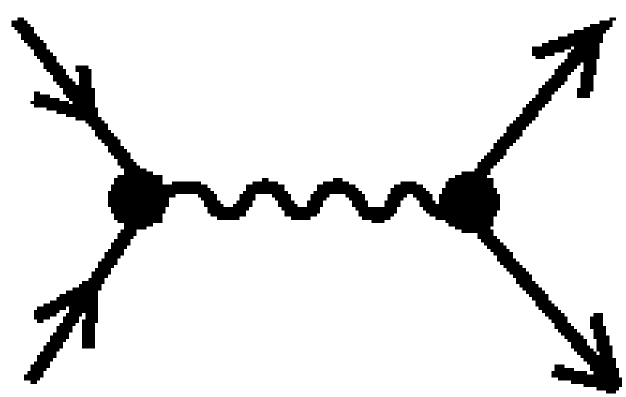
Graphical Bracket and Jones Polynomial for Knots and Links in Thickened Surfaces

by Louis H. Kauffman

Arxiv: 0712.2546
and
arXiv:0810.3858
(Arrow Poly with Heather Dye)

$$<< K >> = \Sigma_S < K|S > d^{||S||-1}[S]$$



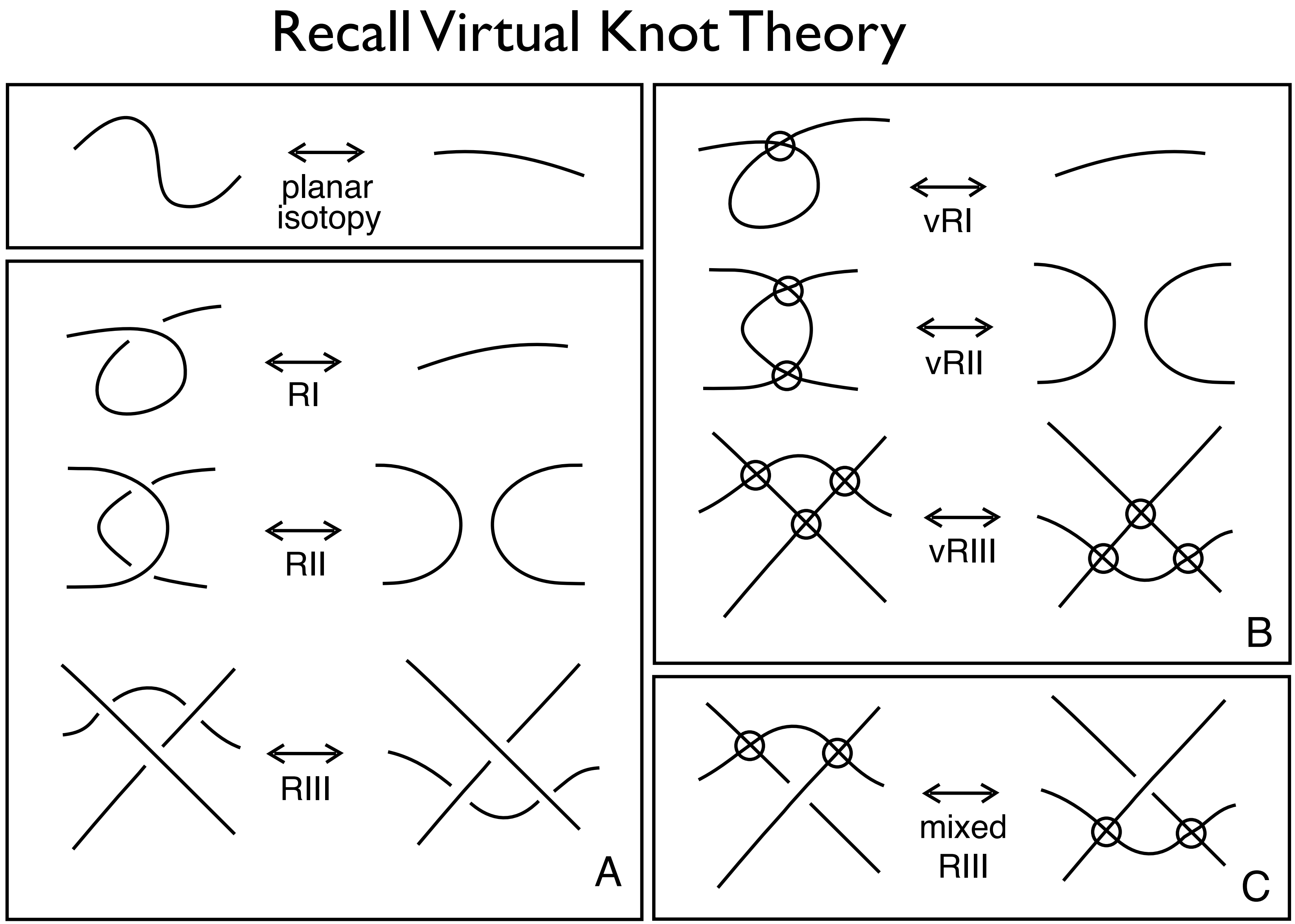
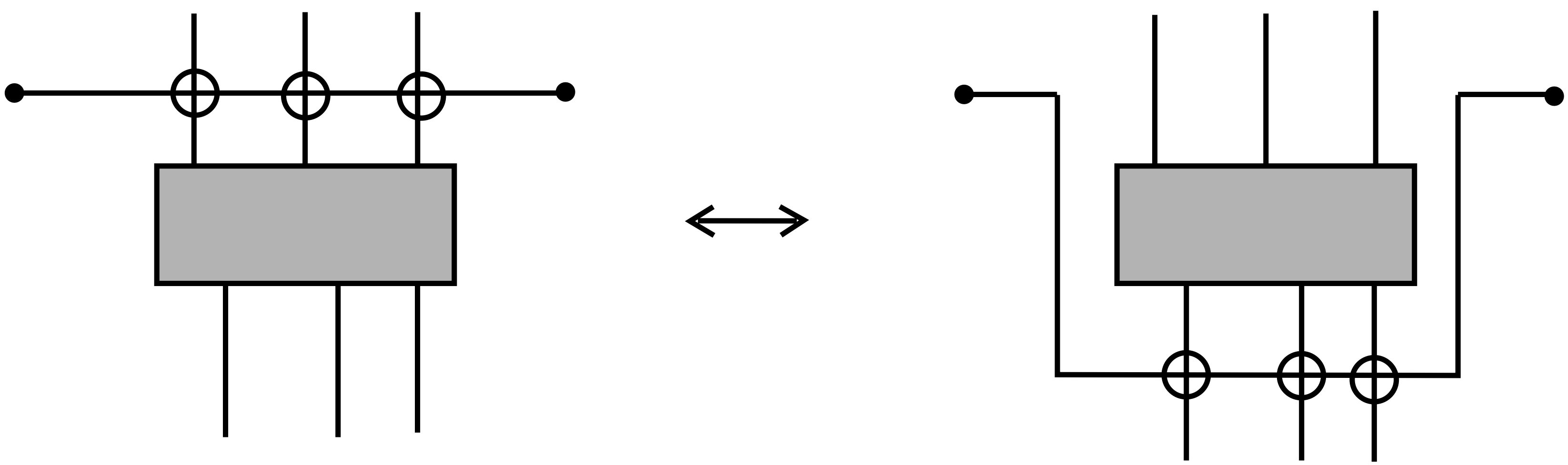


Figure 1. Moves

Detour Move



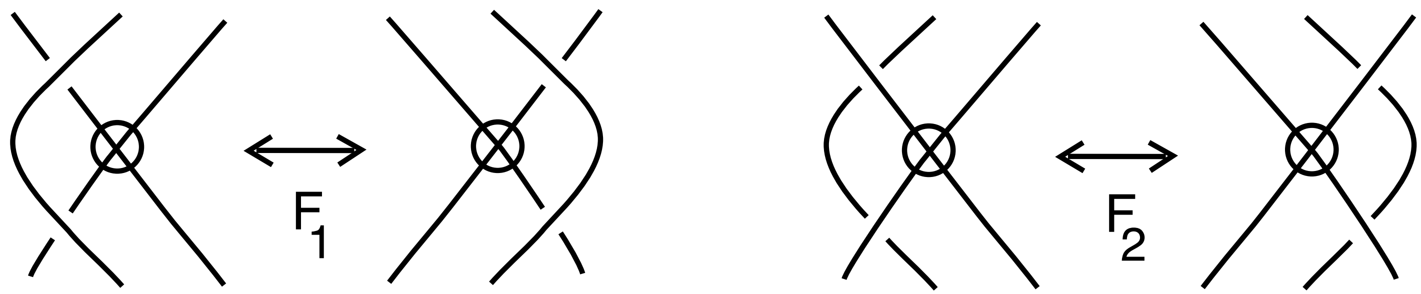


Figure 3. Forbidden Moves

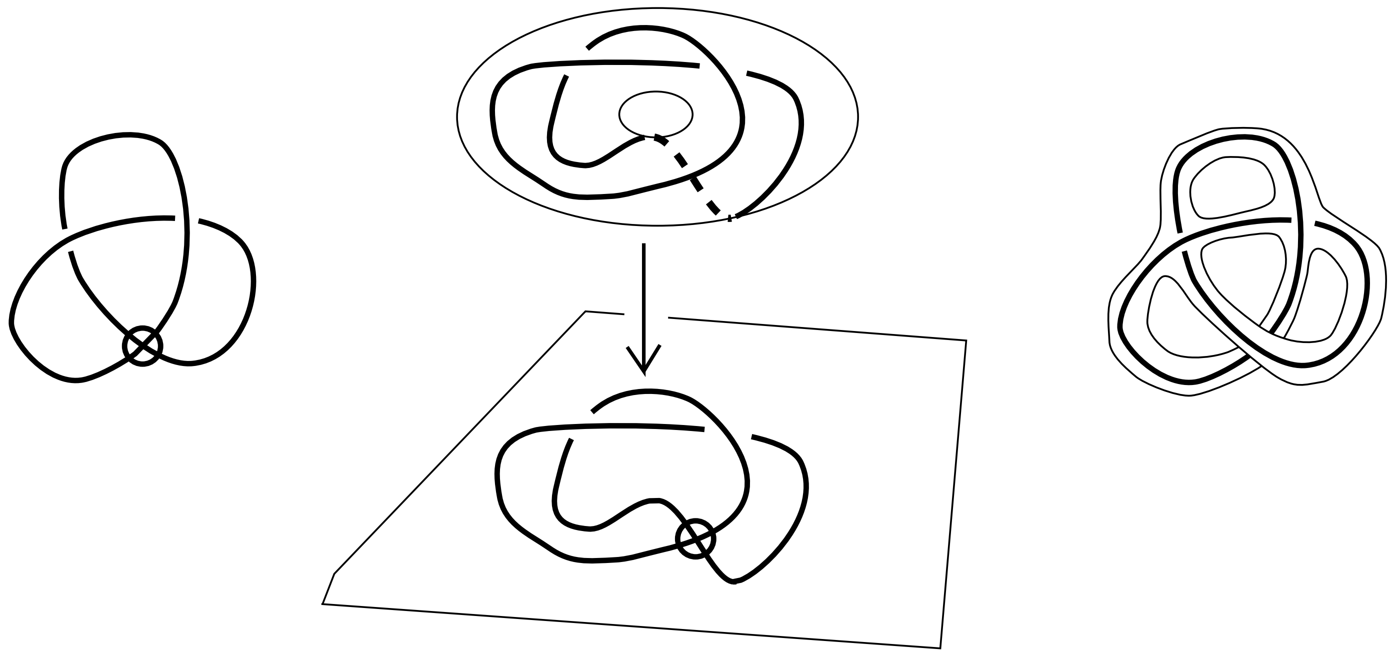
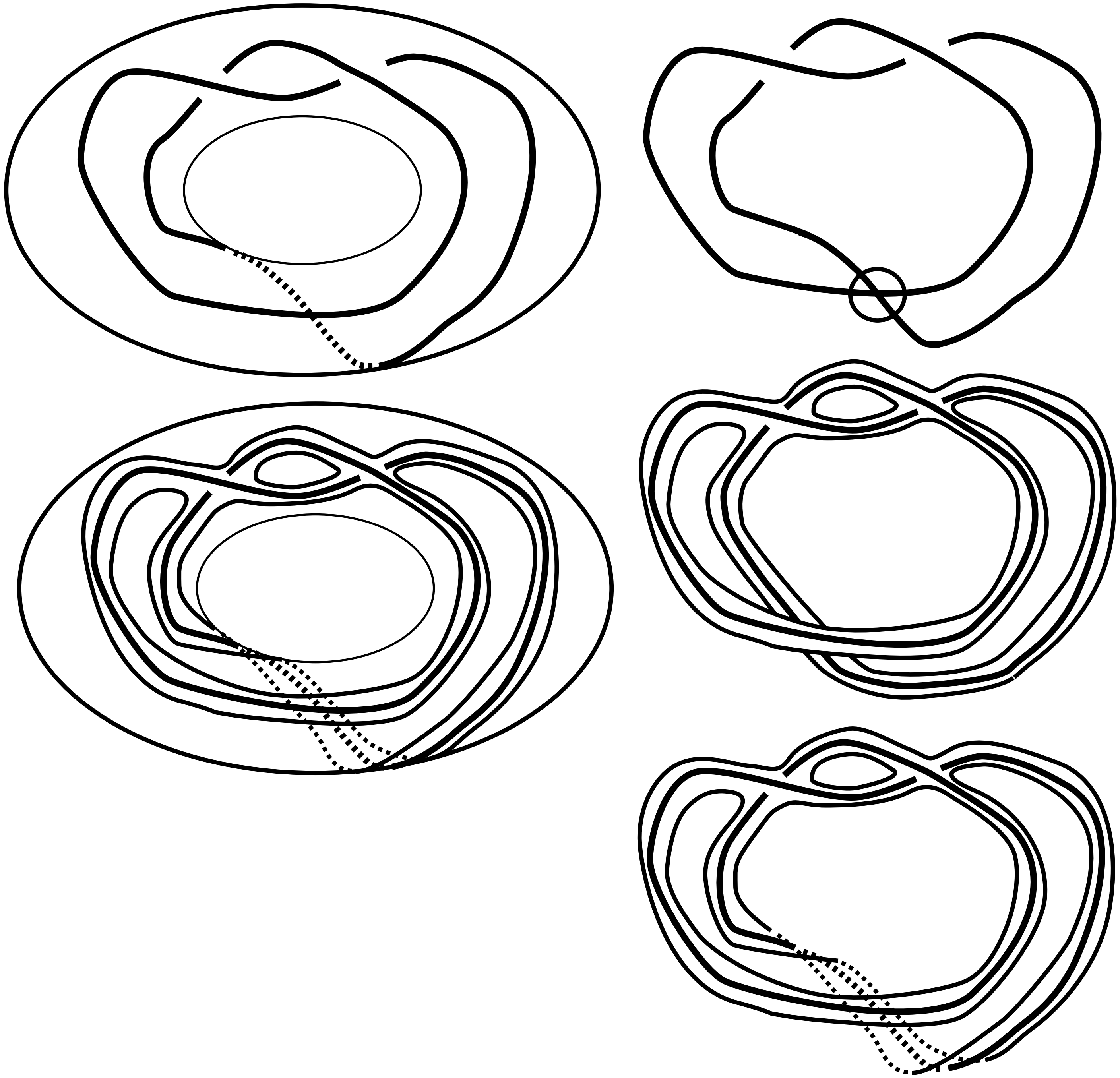
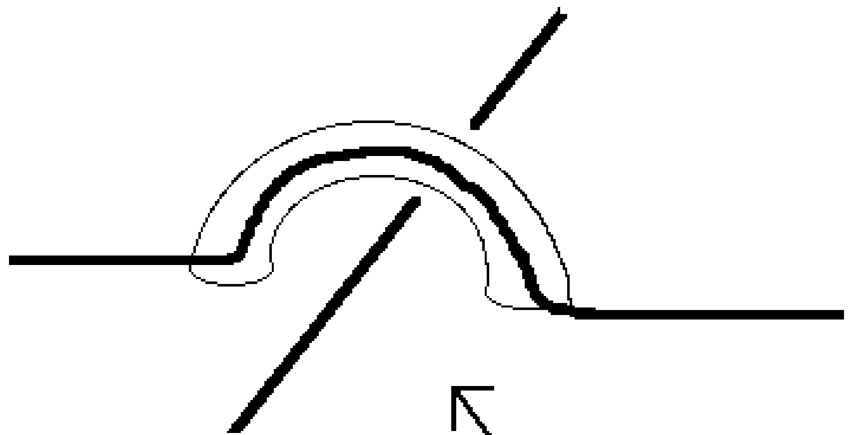
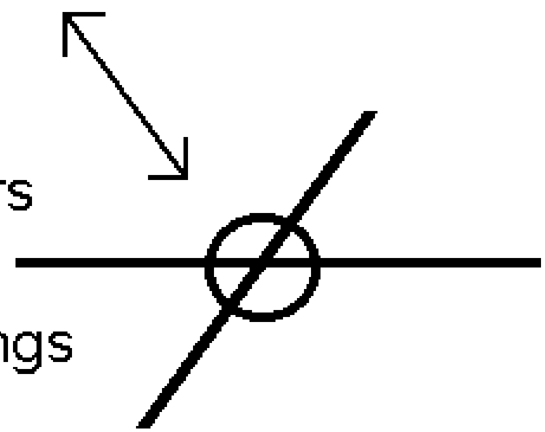


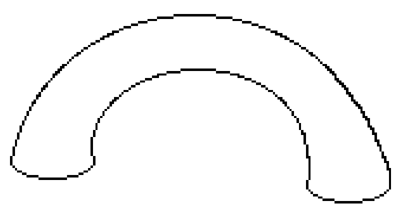
Figure 4. Surfaces and Virtuals





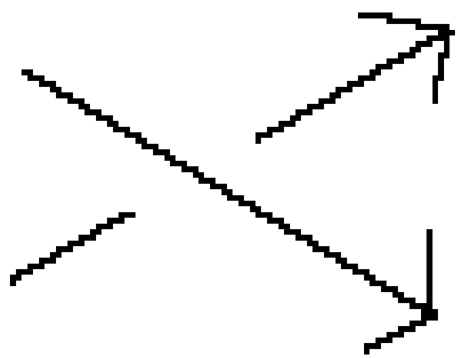


handle detours
for
virtual crossings

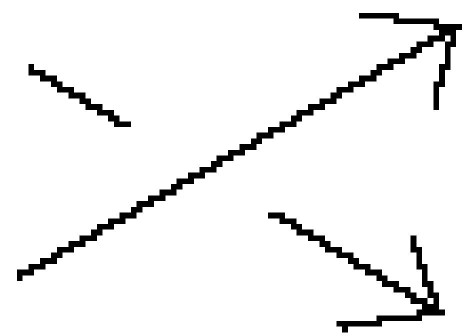


an empty
handle

Our Sign Convention

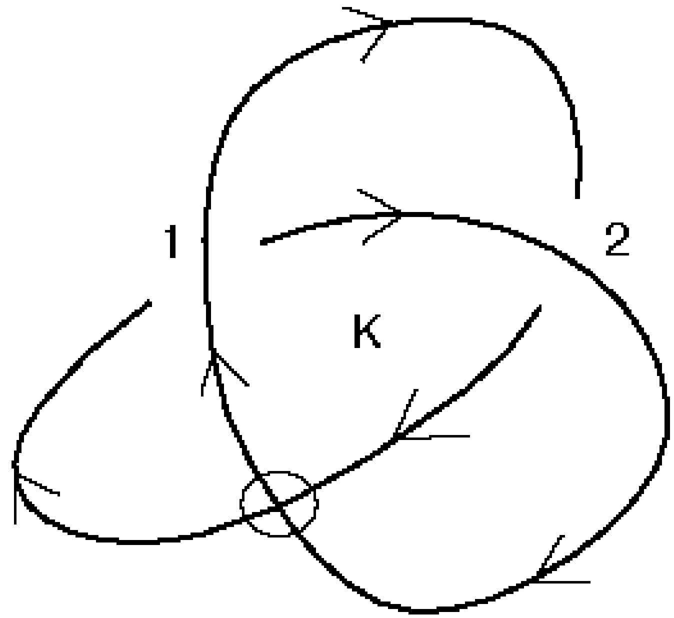


sign = +1



sign = -1

A Simple Invariant of Virtuals -- The Odd Writhe



Bare Gauss Code
1212

Crossings 1 and 2 are odd.

A crossing is odd if it flanks an odd number of symbols in the Gauss code.

The odd writhe of K, J(K).

J(K) = Sum of signs of the odd crossings of K.

Here J(K) = -2.

Facts: J(K) is an invariant of virtual isotopy.
J(K) = 0 is K is classical.
J(Mirror Image of K) = -J(K).

Hence this example is not classical and is not isotopic to its mirror image.

Long Flats Embed in Long Virtuals via the Ascending Map.

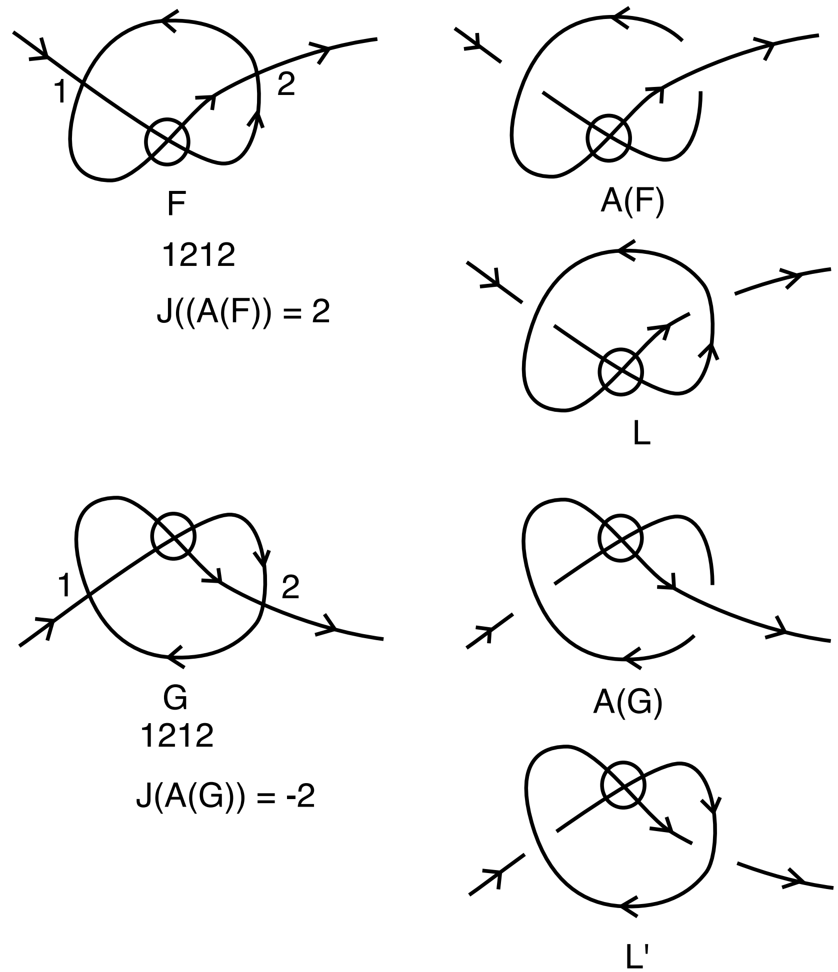
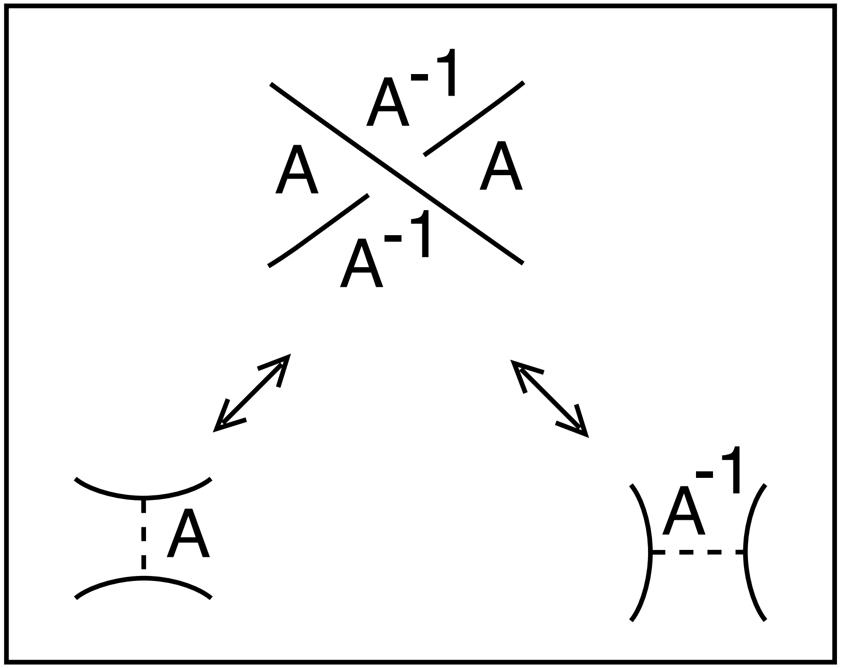
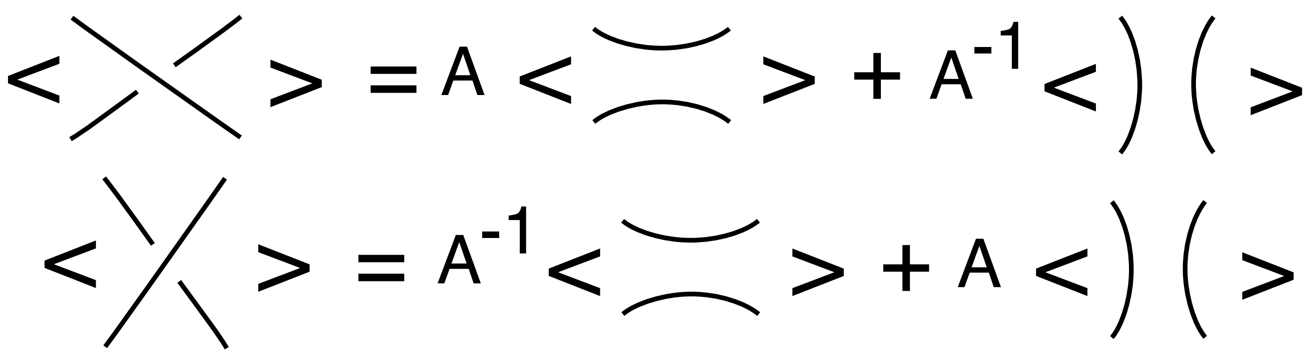


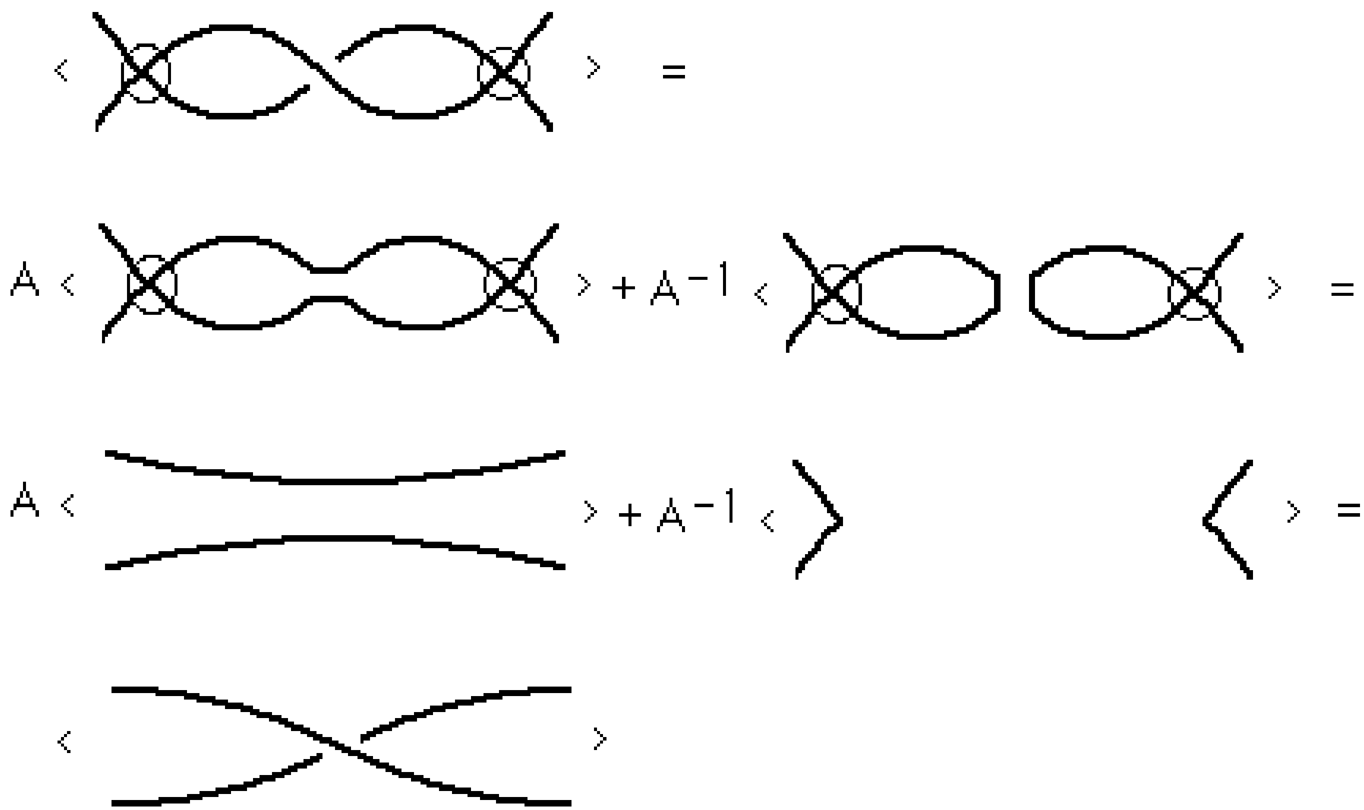
Figure 5. Ascending Map

The Bracket Polynomial Model for the Jones Polynomial Extends to Virtual Links.





Bracket Polynomial is Unchanged when smoothing flanking virtuals.



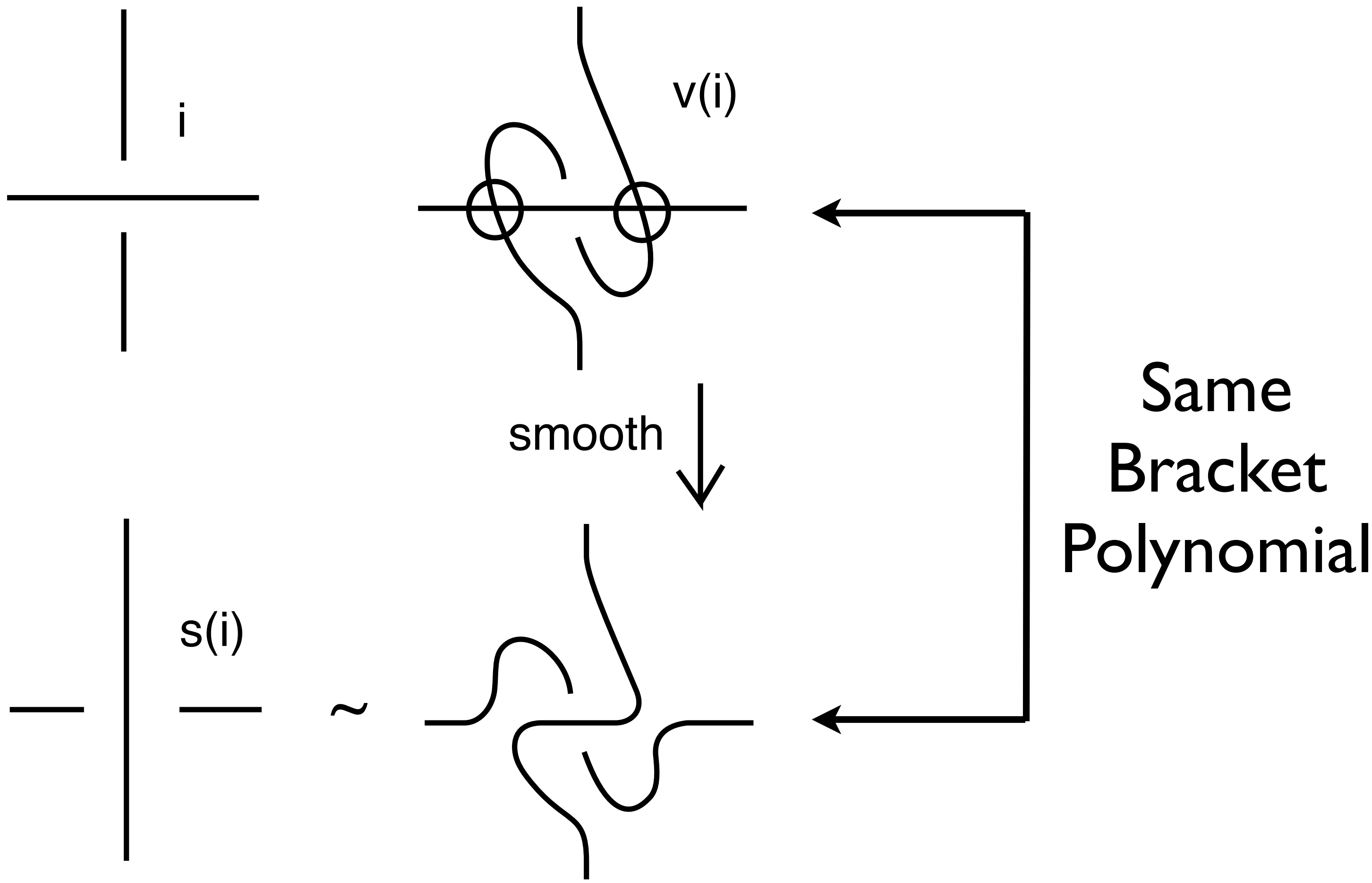


Figure 7. Switch and Virtualize

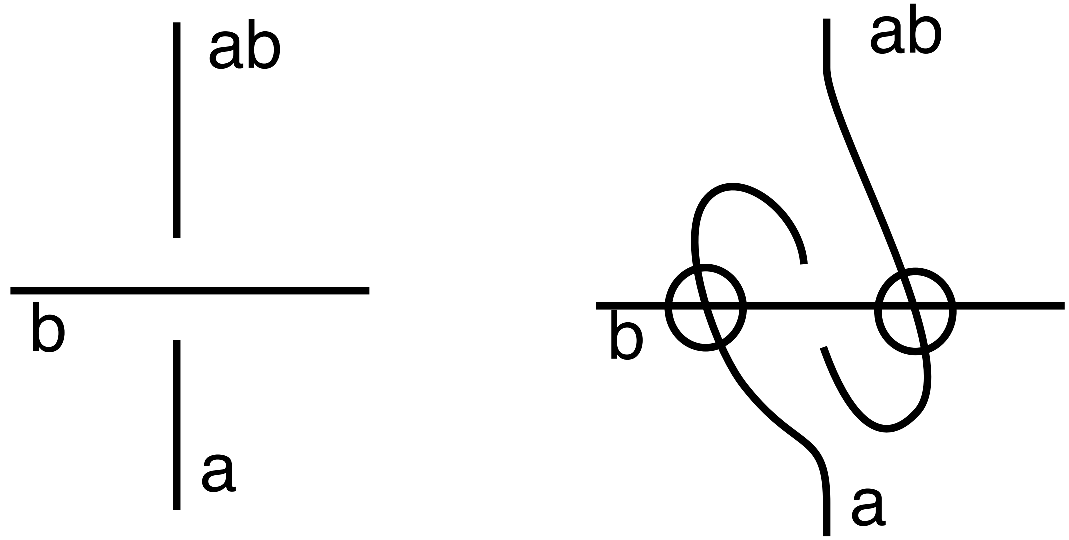
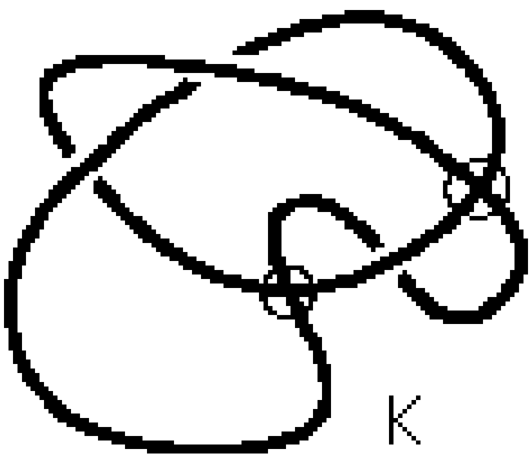


Figure 8. IQ(Virt)

<Virt(K)> = <Switch(K)>
and

IQ(Virt(K)) = IQ(K).

Conclusion: There exist infinitely many non-trivial Virt(K) with unit Jones polynomial.



K

U

A Well-Known Culprit

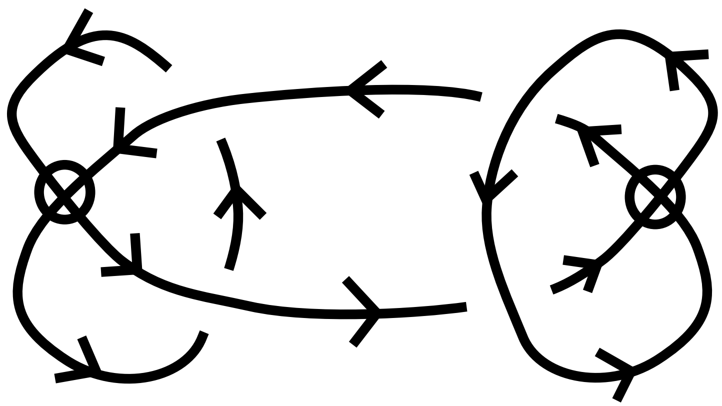
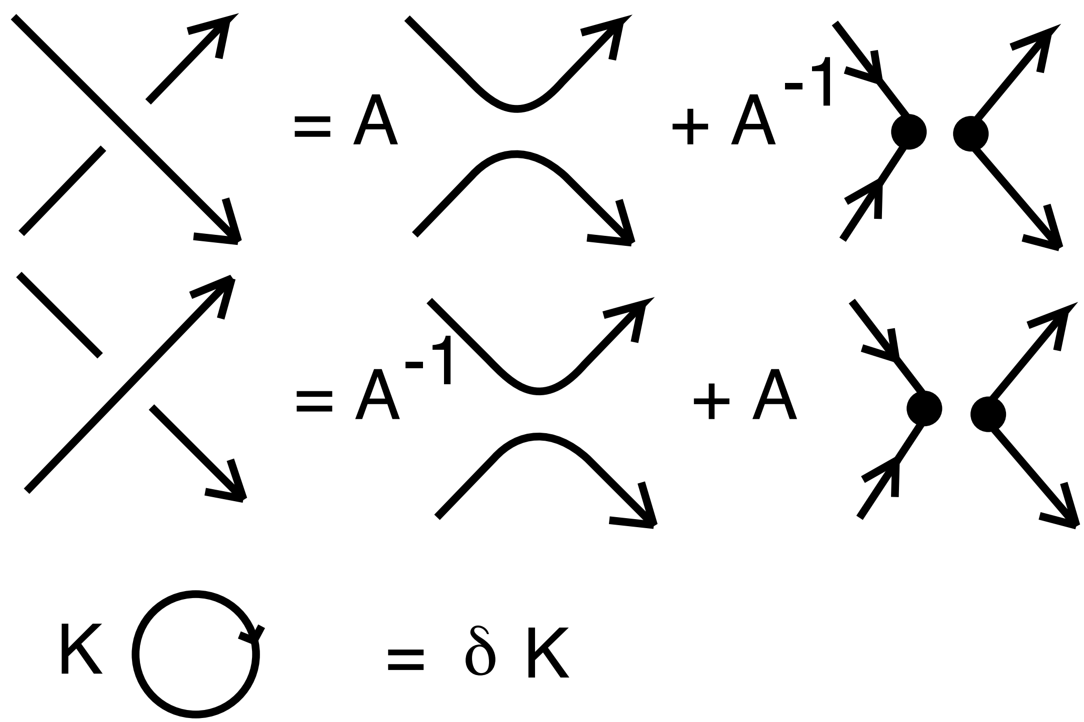


Figure 9. Kishino Diagram

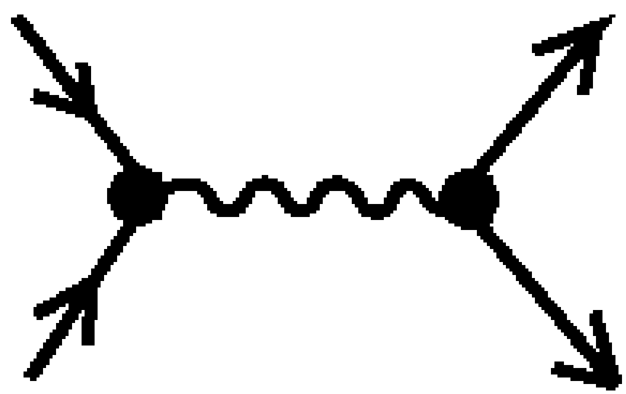
Oriented Bracket State Sum



$$\delta = - A^{2} - A^{-2}$$

Figure 10. Oriented Bracket Expansion

Our Approach:
Retain the reverse oriented vertex if possible.
Think of the reverse oriented vertex as
endowed with a spring that holds the ends together.
Reduce states to graphs.
Determine reduction rules from the
Reidemeister moves.



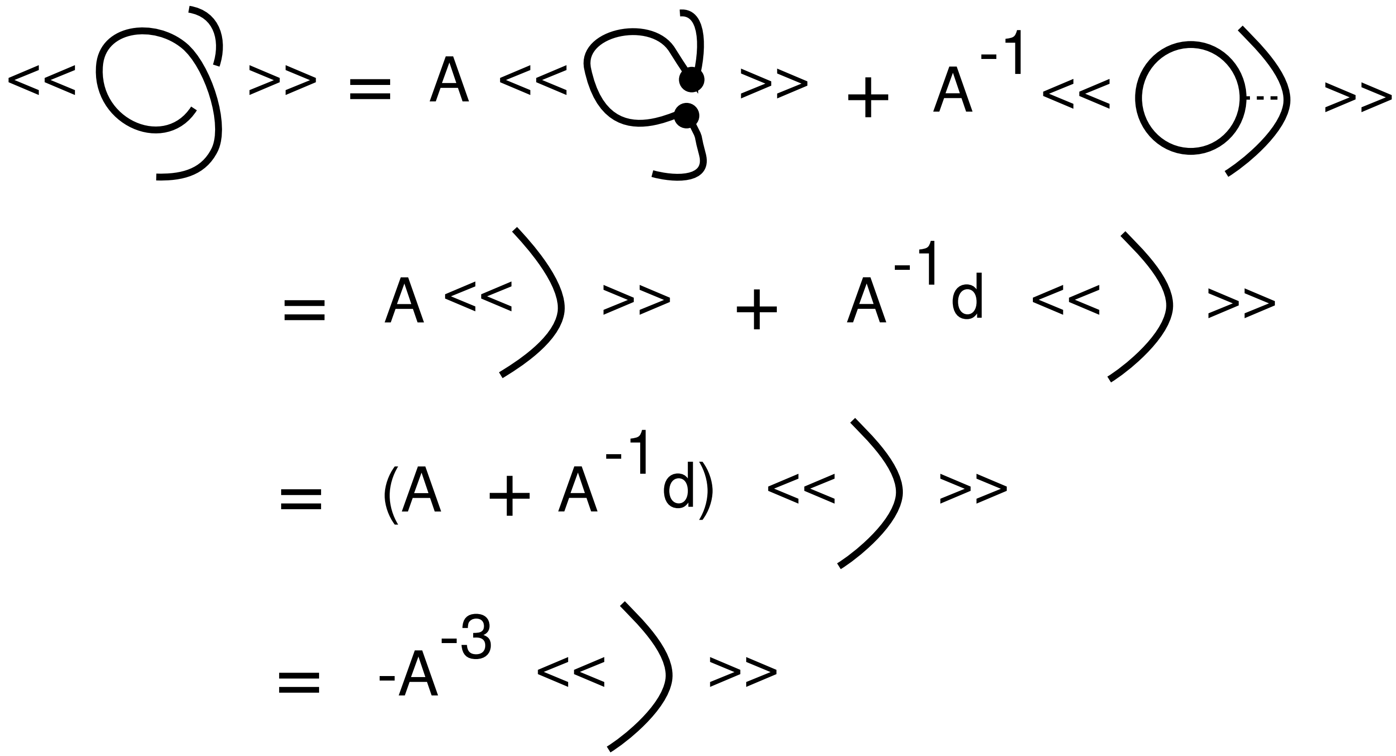


Figure 13: **The Type One Move**

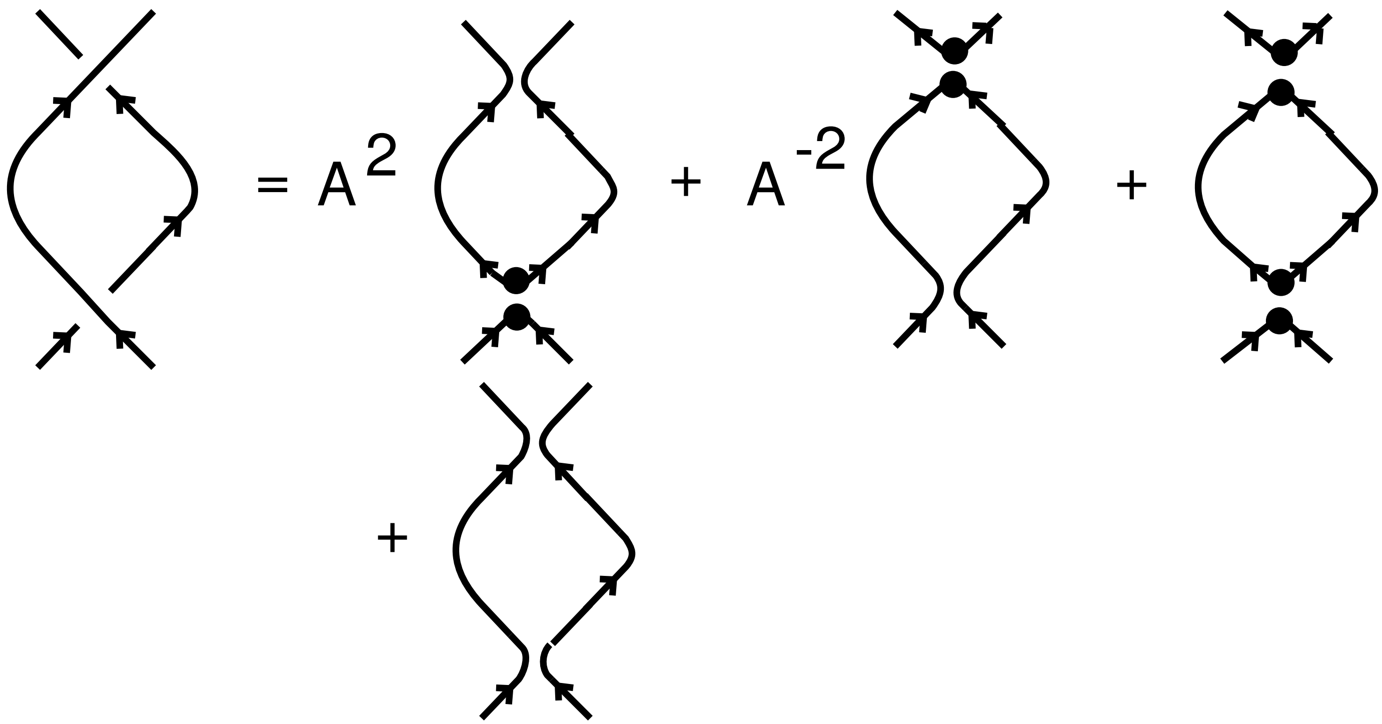


Figure 18: **Oriented Second Reidemeister Move**

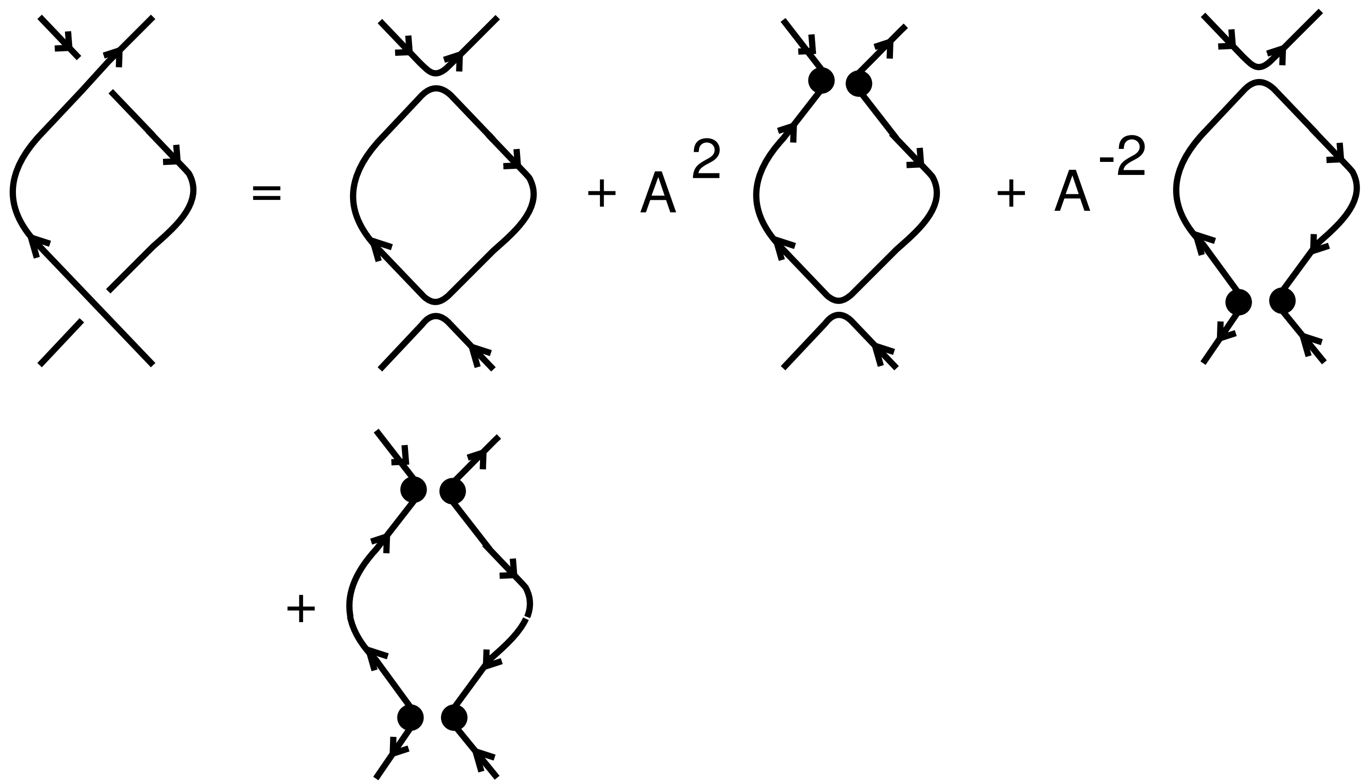
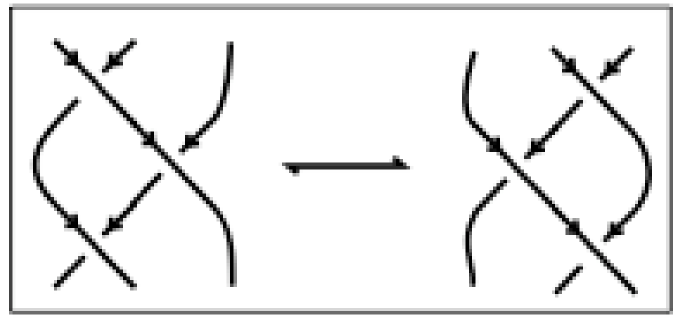
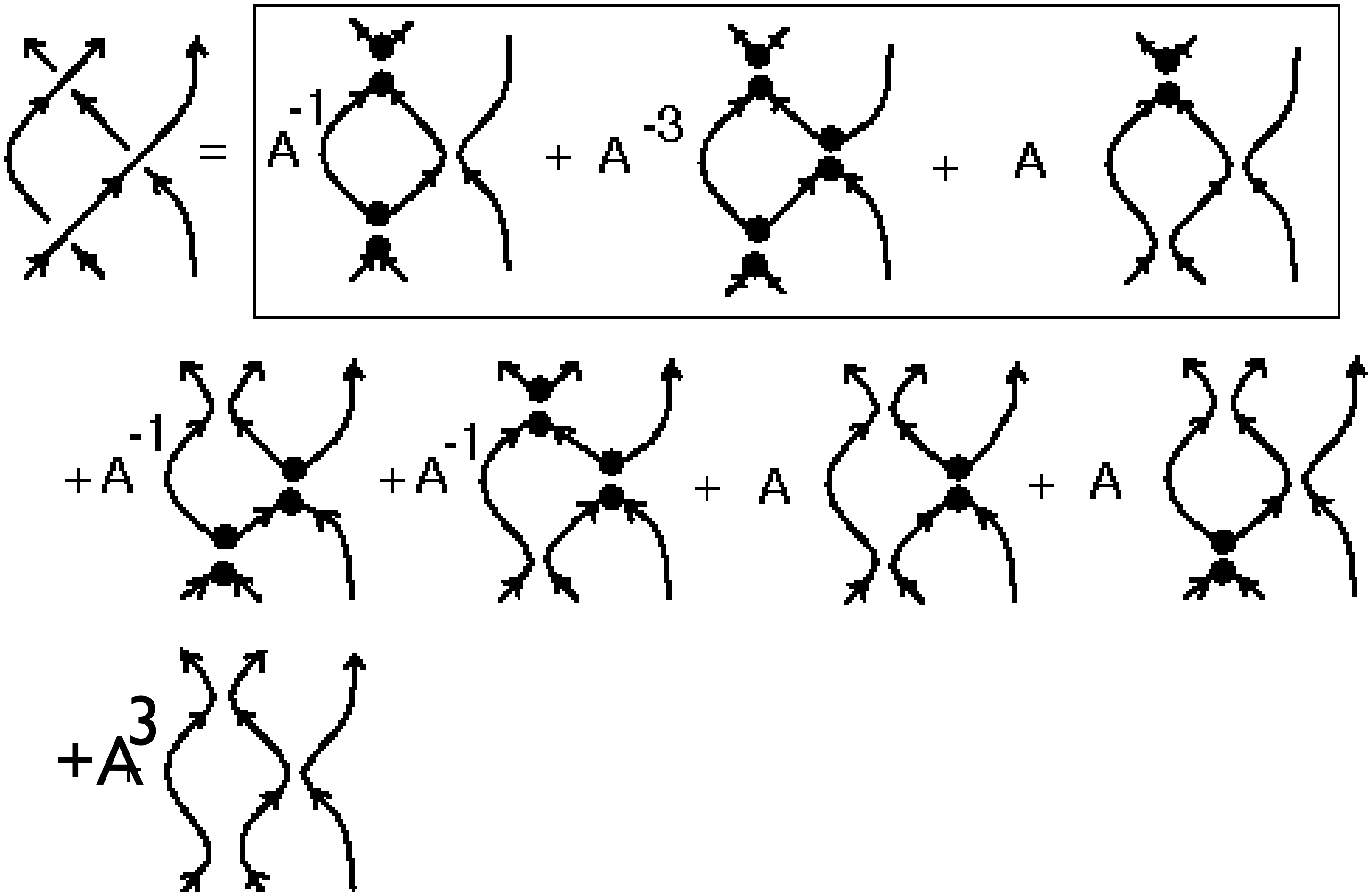


Figure 19: **Reverse Oriented Second Reidemeister Move**

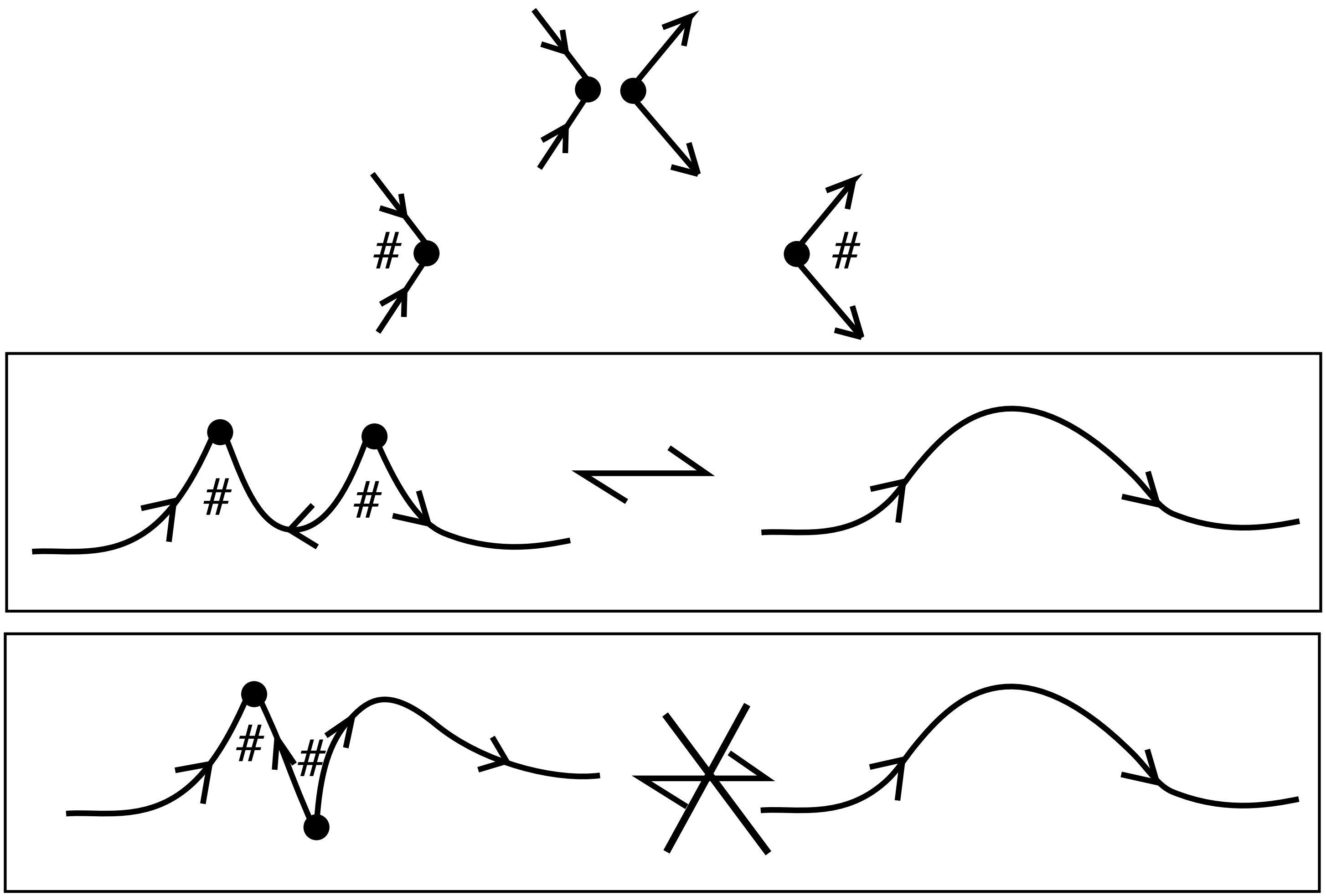
Third Reidemeister Move



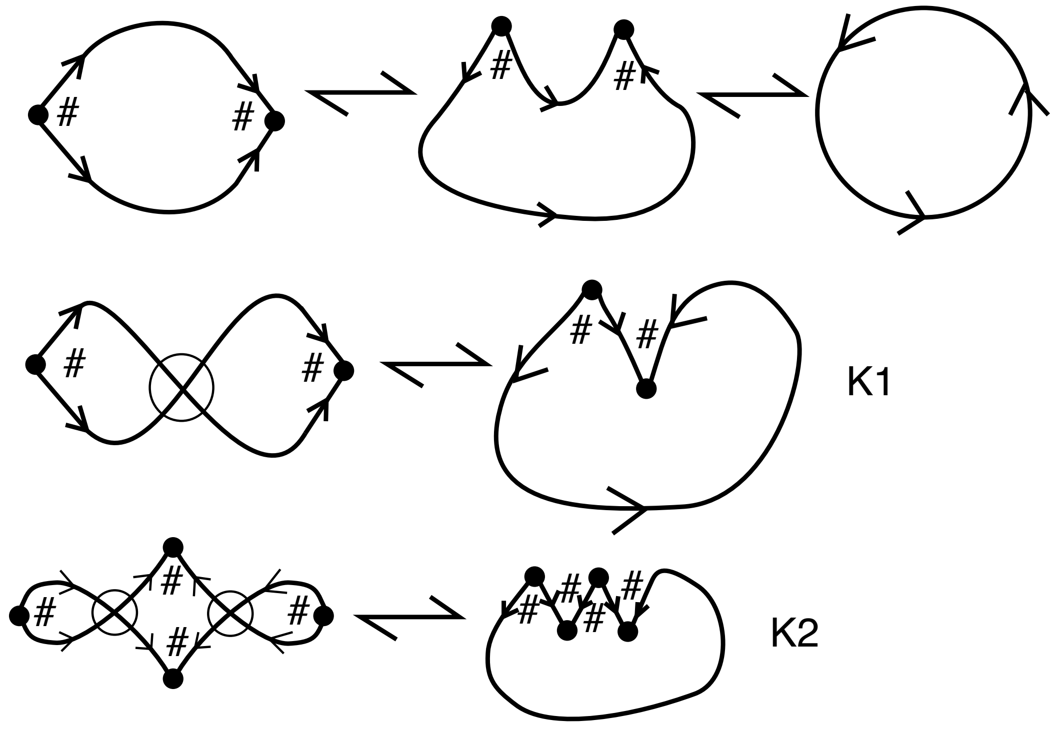


THE ARROW POLYNOMIAL

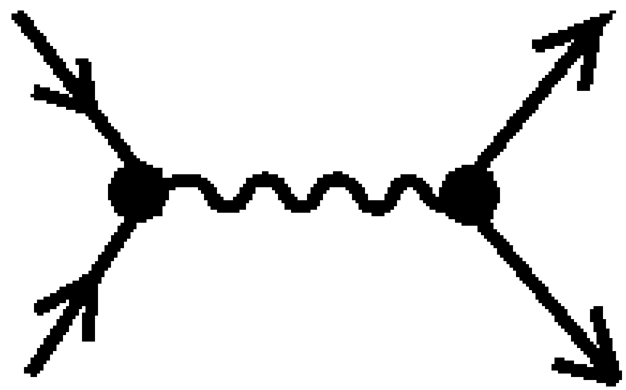
All paired vertices are allowed to come apart.

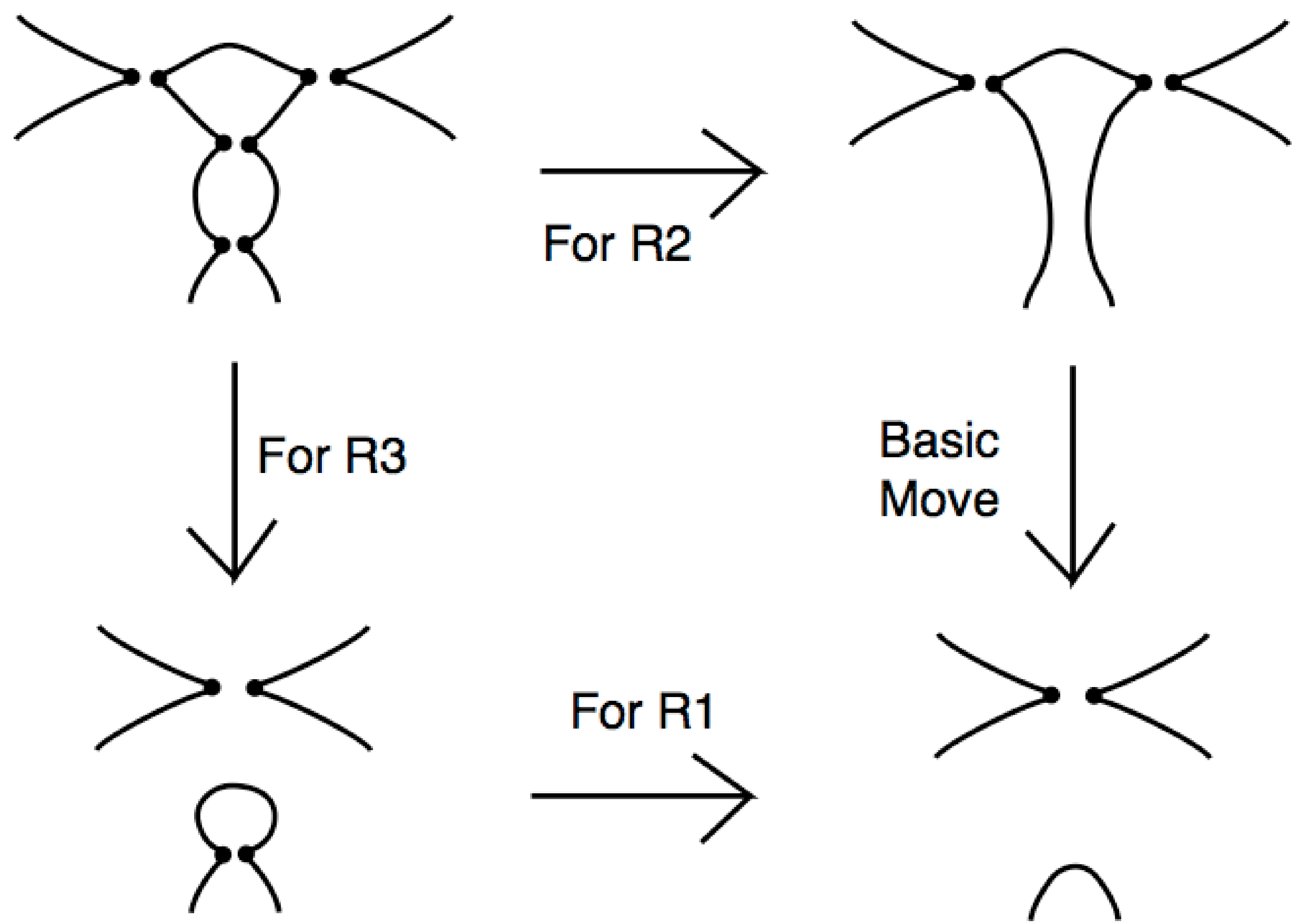


In the arrow polynomial the paired vertices at a disoriented crossing come apart and the reduction relations simplify. The end graphs are disjoint unions of simplified circle graphs. Each reduced circle graph becomes a new polynomial variable.



Returning to Extended Bracket




For R2
For R3
Basic
Move
For R1

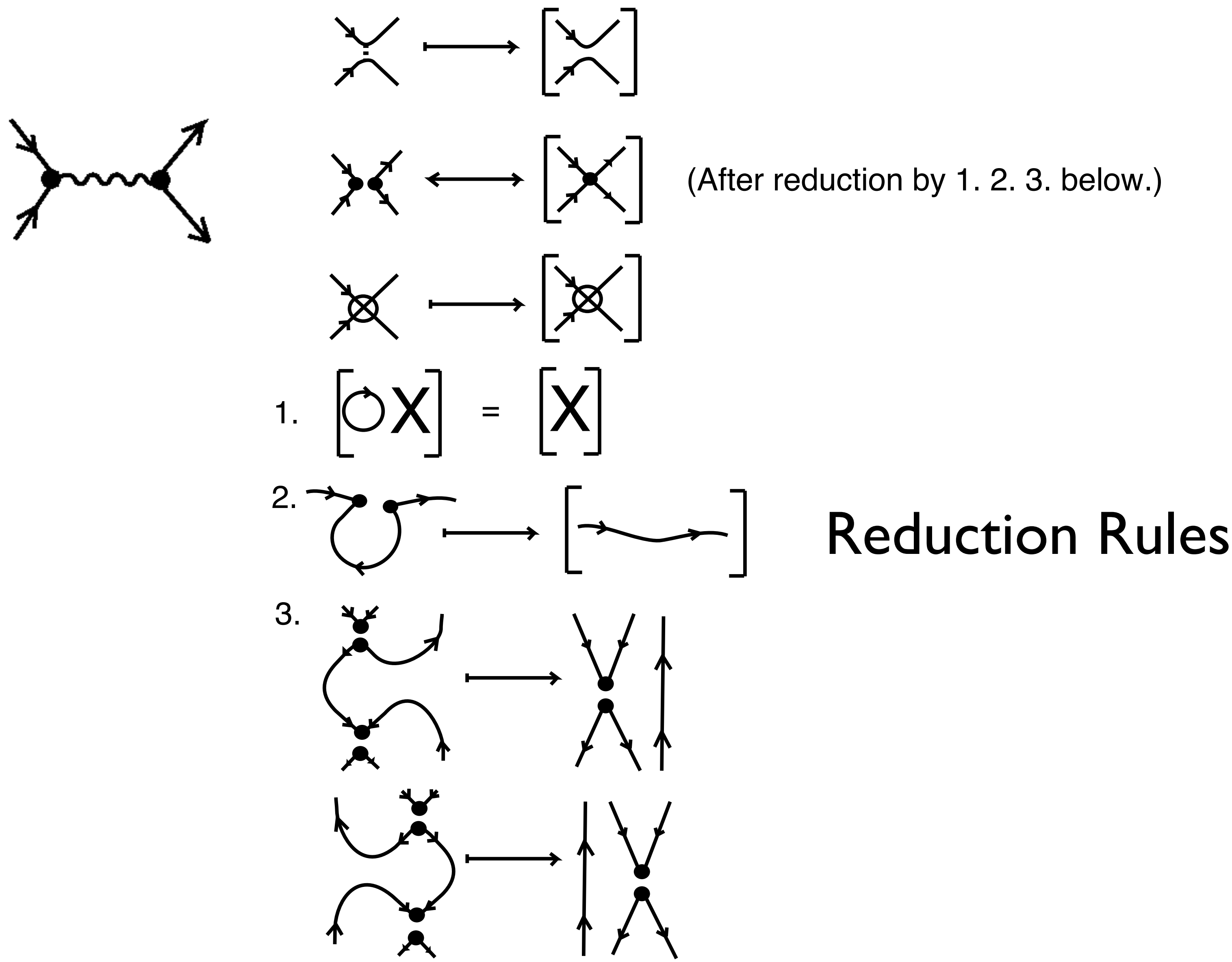
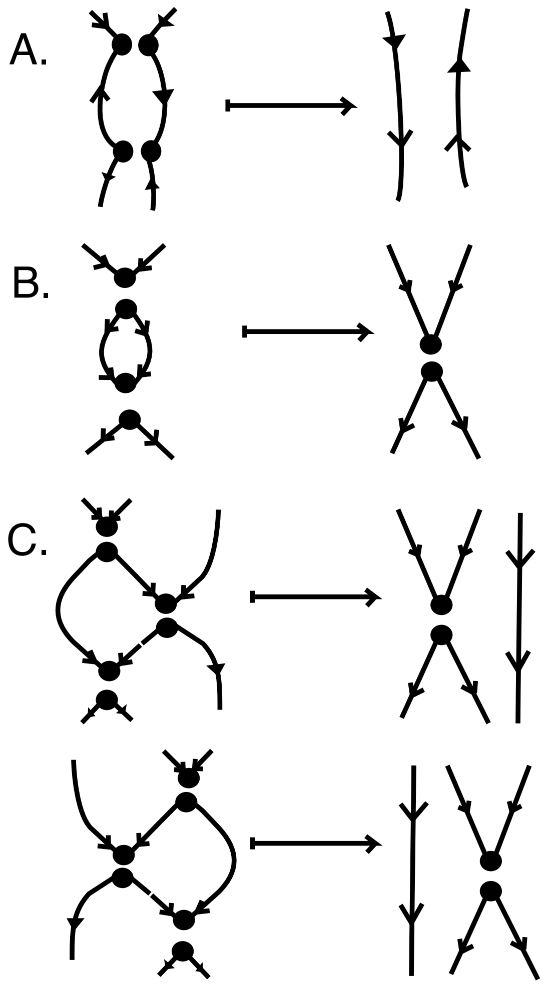


Figure 11. Basic Replacements



These replacements are consequences of the reduction rules.

C takes precendence over rule [3].

Figure 14. Special Replacements

Key Example

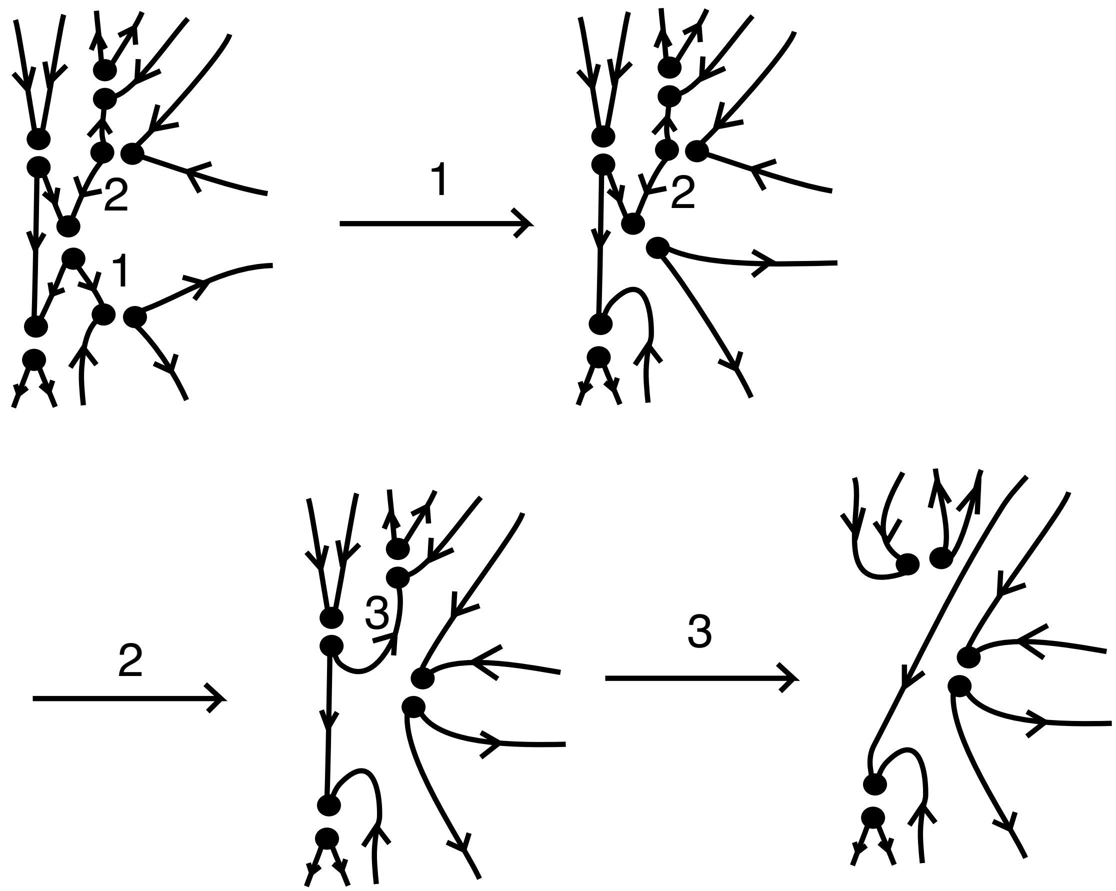


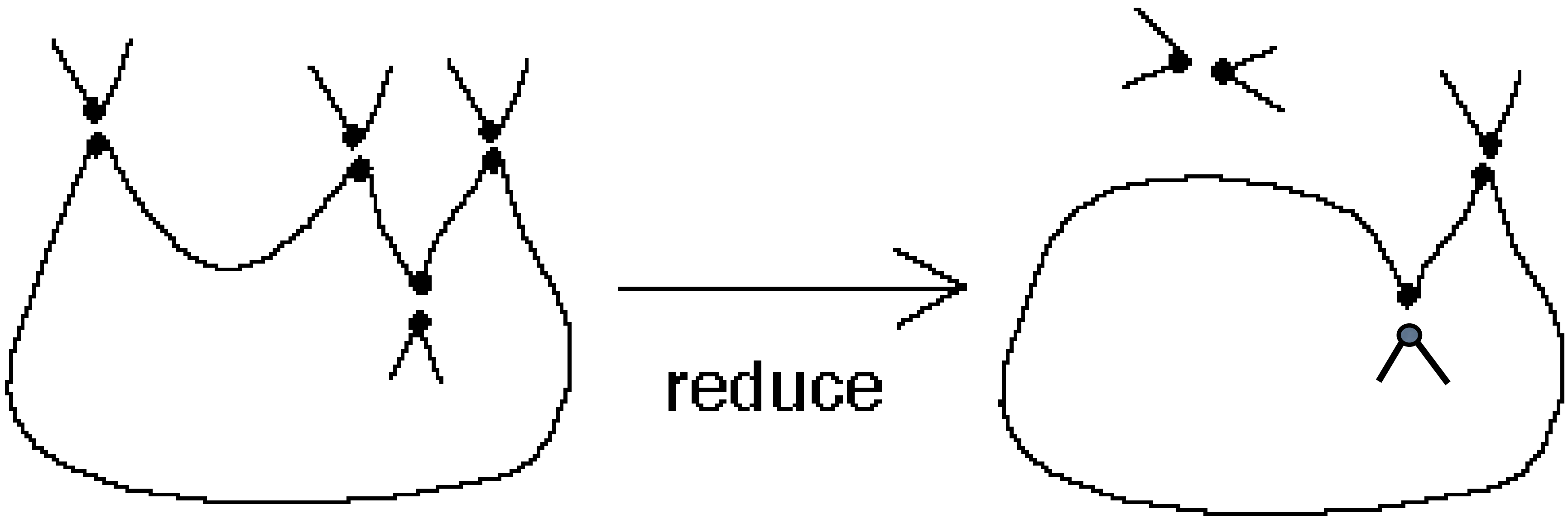
Figure 17: **Special Replacement C Requires a Precedence Rule**

If S^ is a state obtained from S by making one of these replacements, then S^ and S have the same unique graphical reduction. The summation

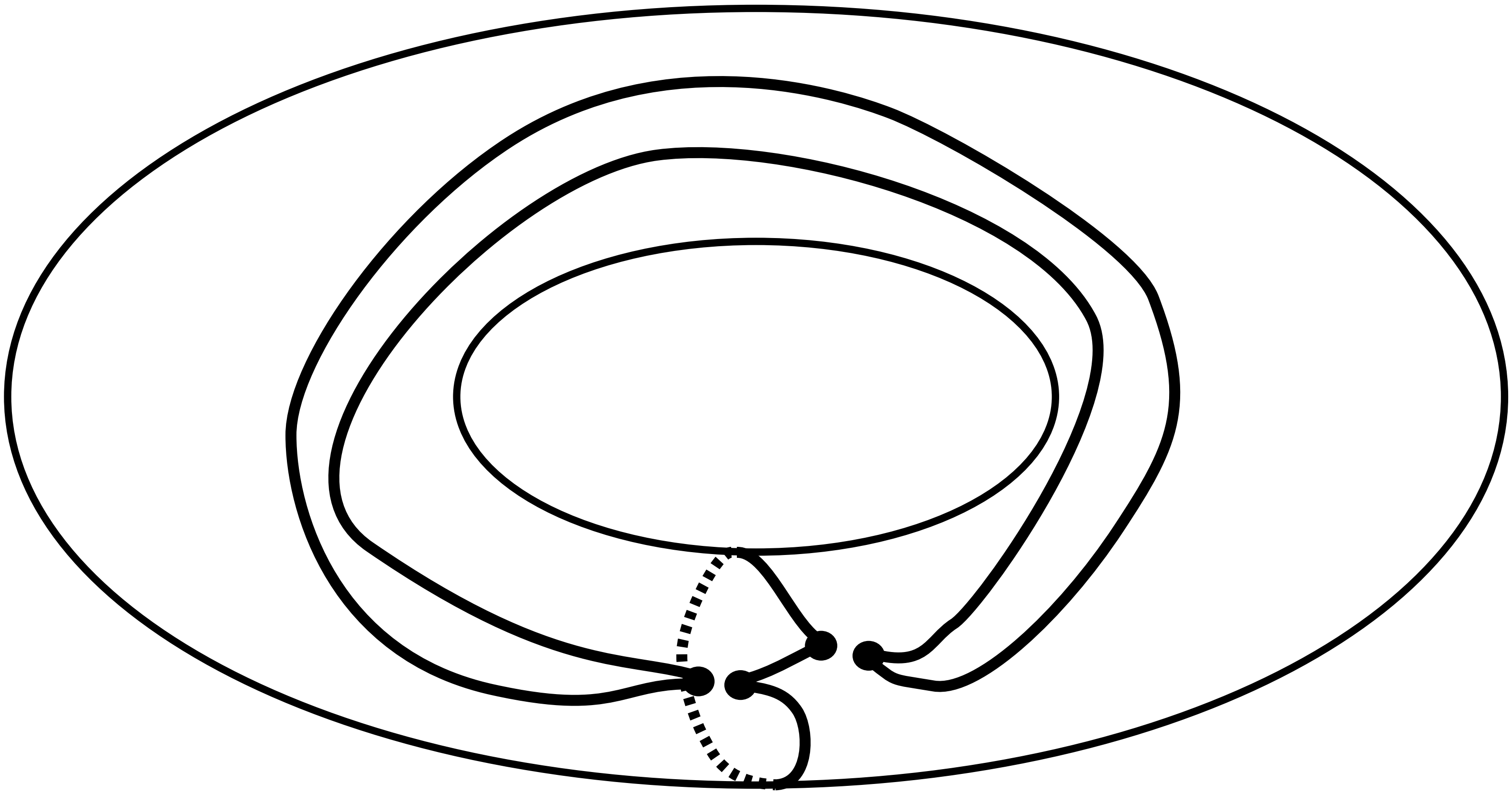
$$<< K >>= \Sigma_S < K|S > d^{||S||-1}[S]$$

where [S] denotes the reduced graph corresponding to the state S, is a regular isotopy invariant of virtual knots and links.

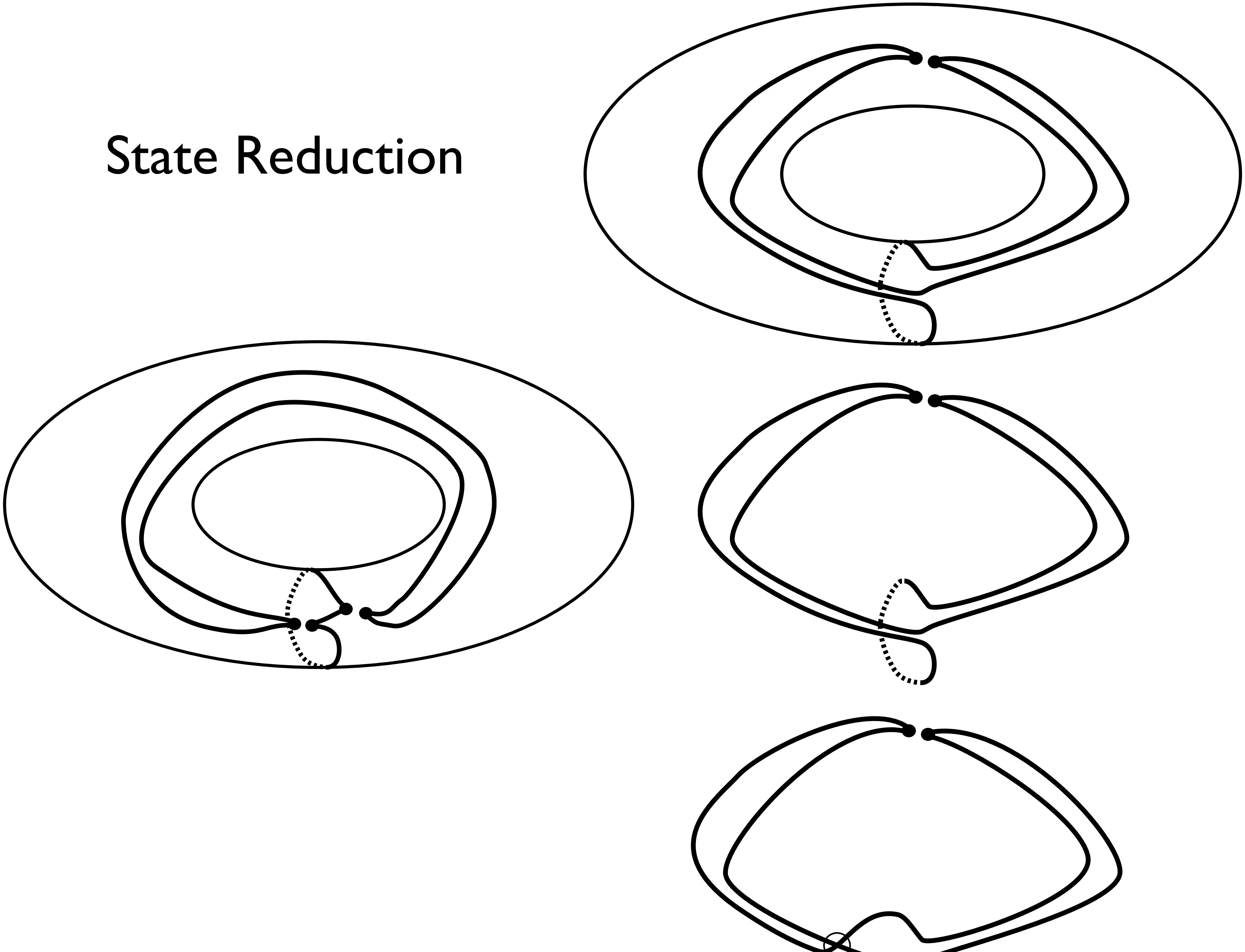
Reduced States with zigzags cannot be embedded in the plane.



Zig-zags survive in higher genus.



State Reduction



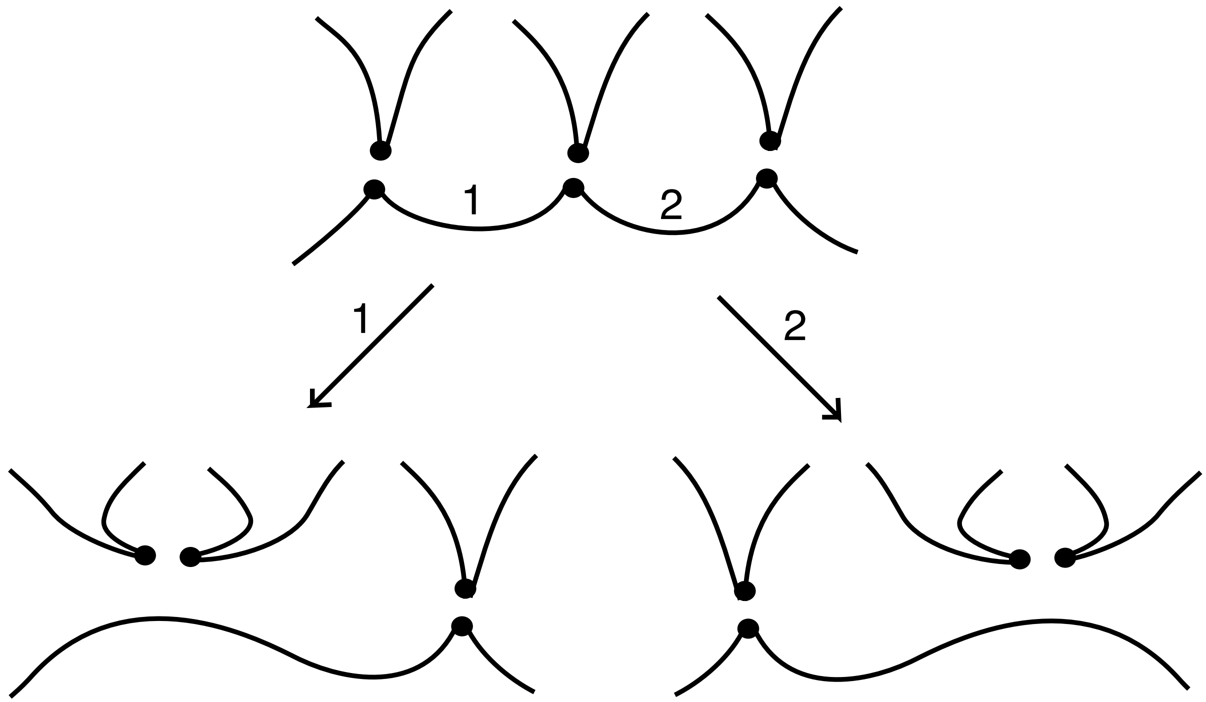


Figure 12. Multiplicity

Special Replacements Avoid Multiplicity

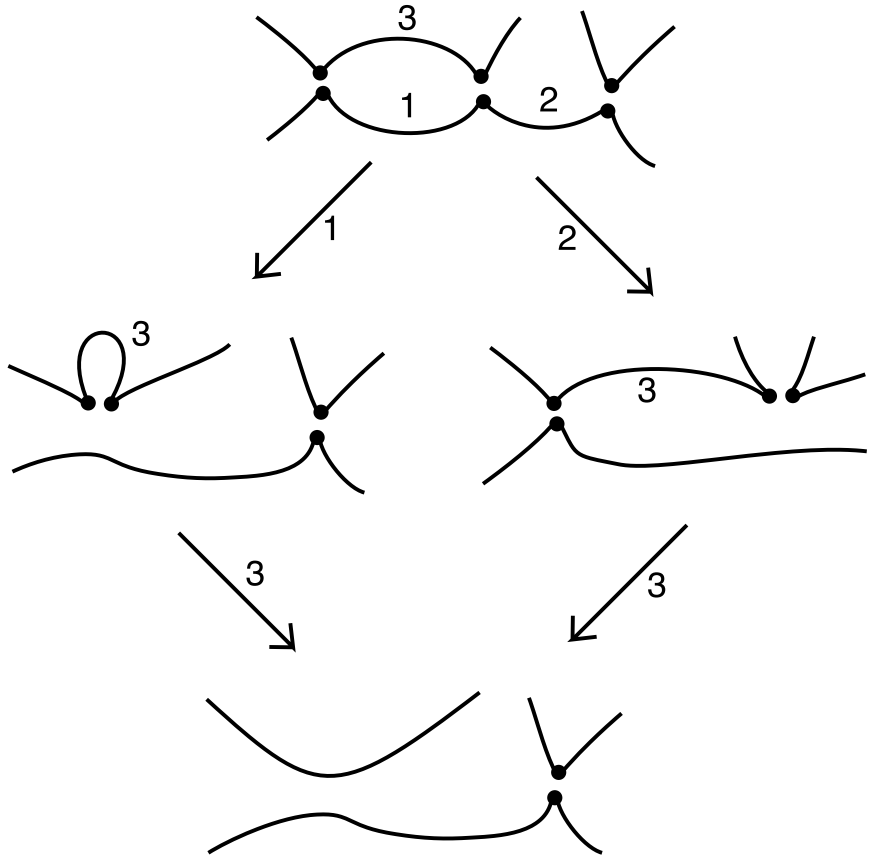


Figure 15. Well-definedness of Special Replacement A

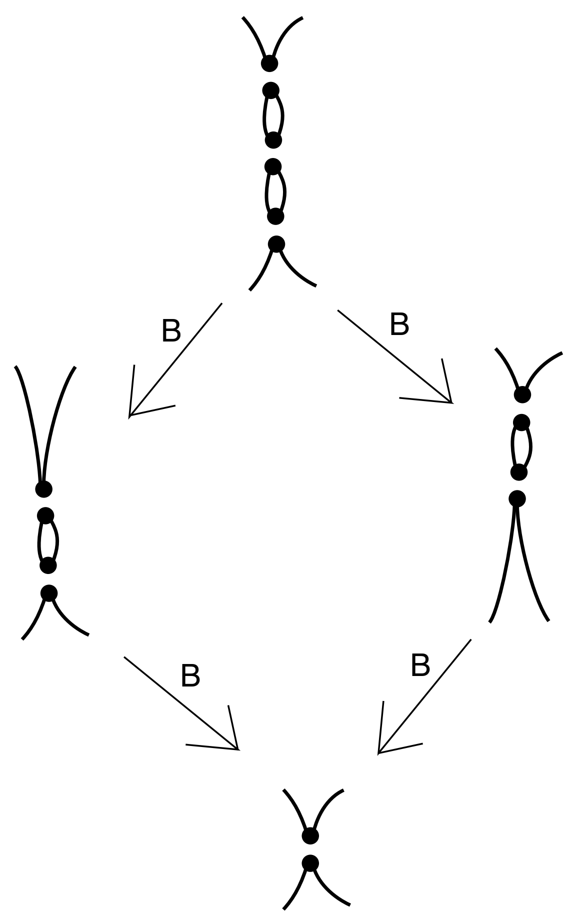


Figure 18: **Uniqueness of Special Replacement** B

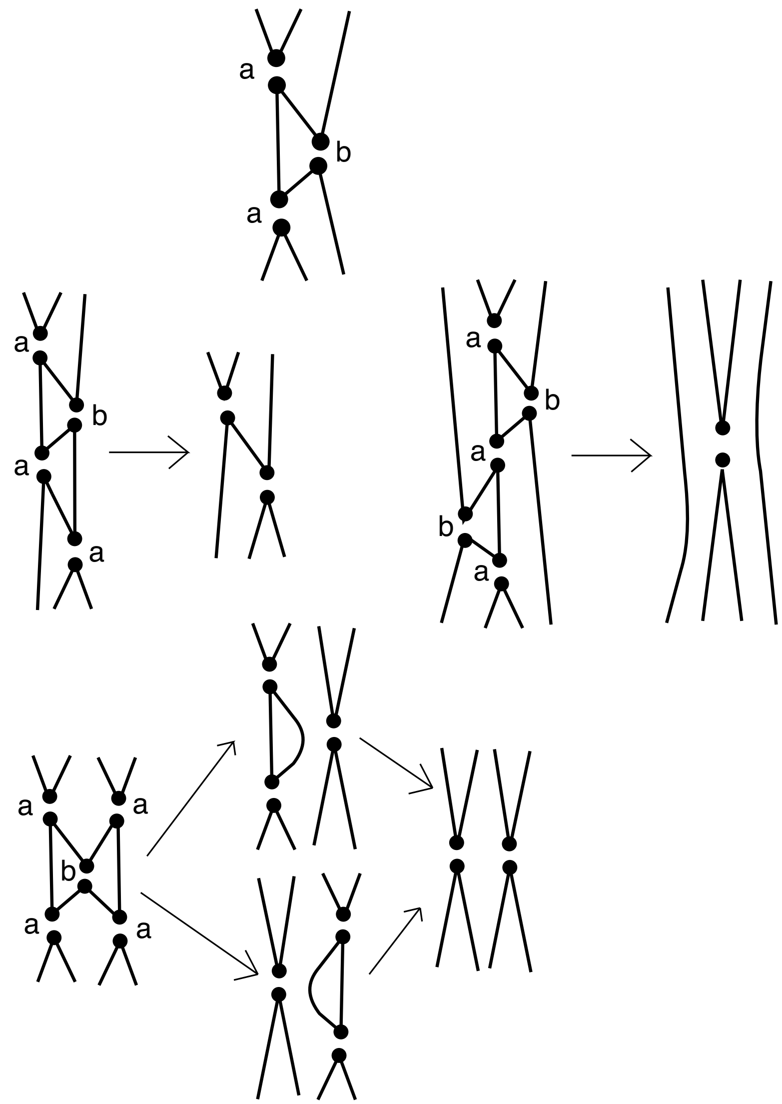


Figure 22: **Networks of C - Moves**

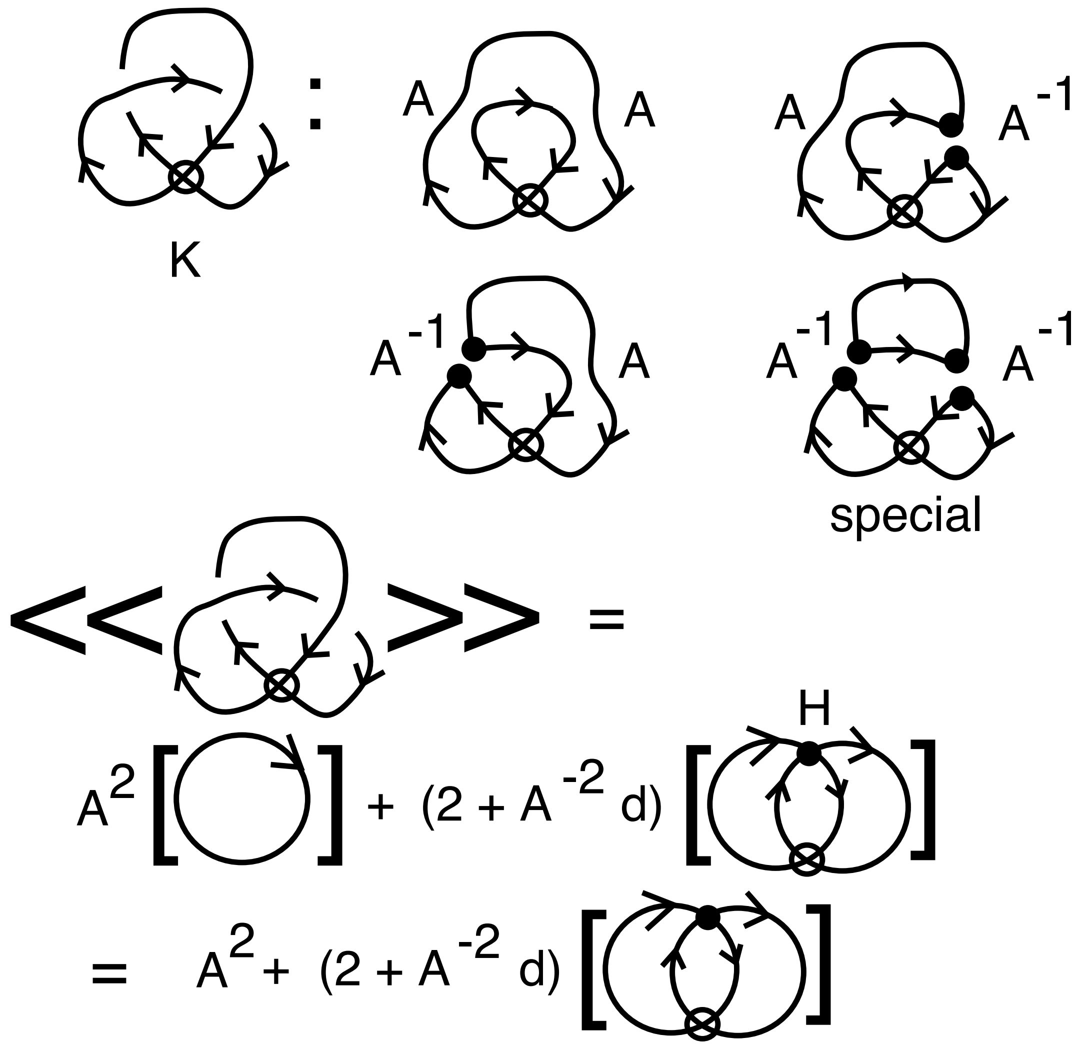


Figure 22. **Example1**

In this example <<L>> detects the non-triviality of a long virtual whose closure is unknotted.

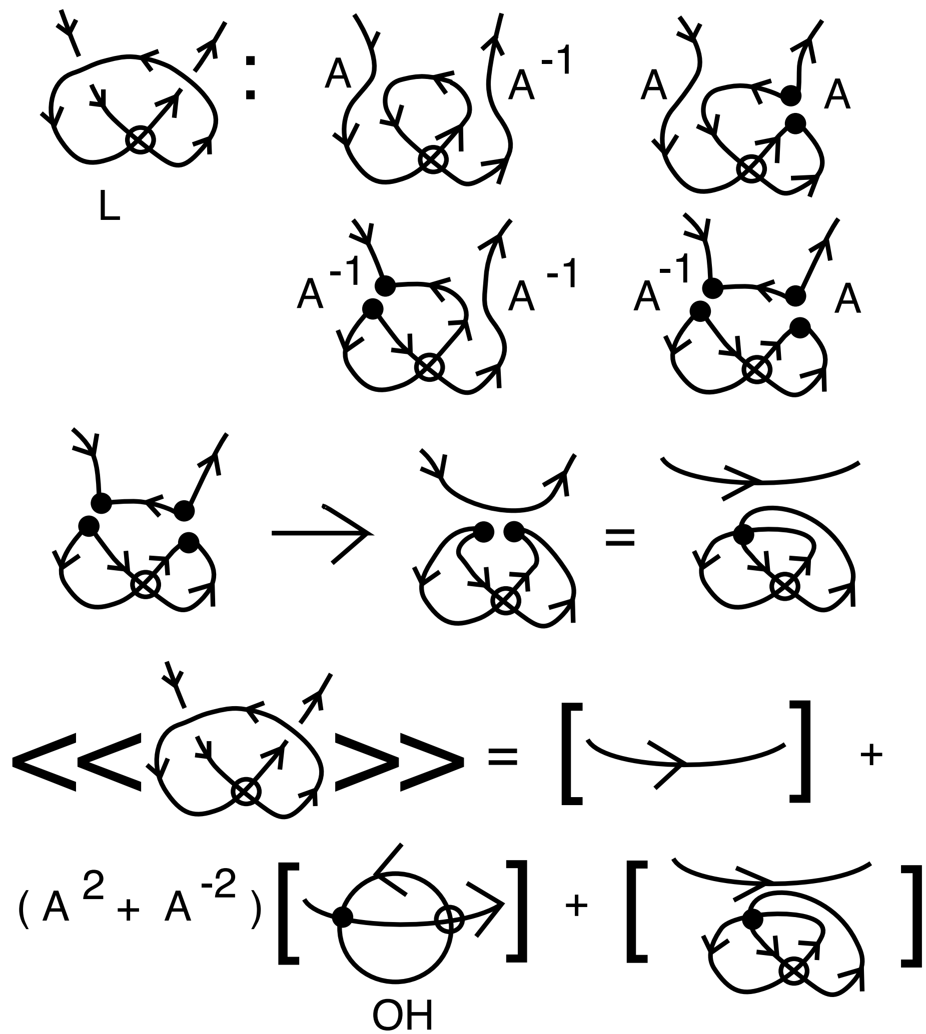


Figure 23. Example2

The Trivial Closure

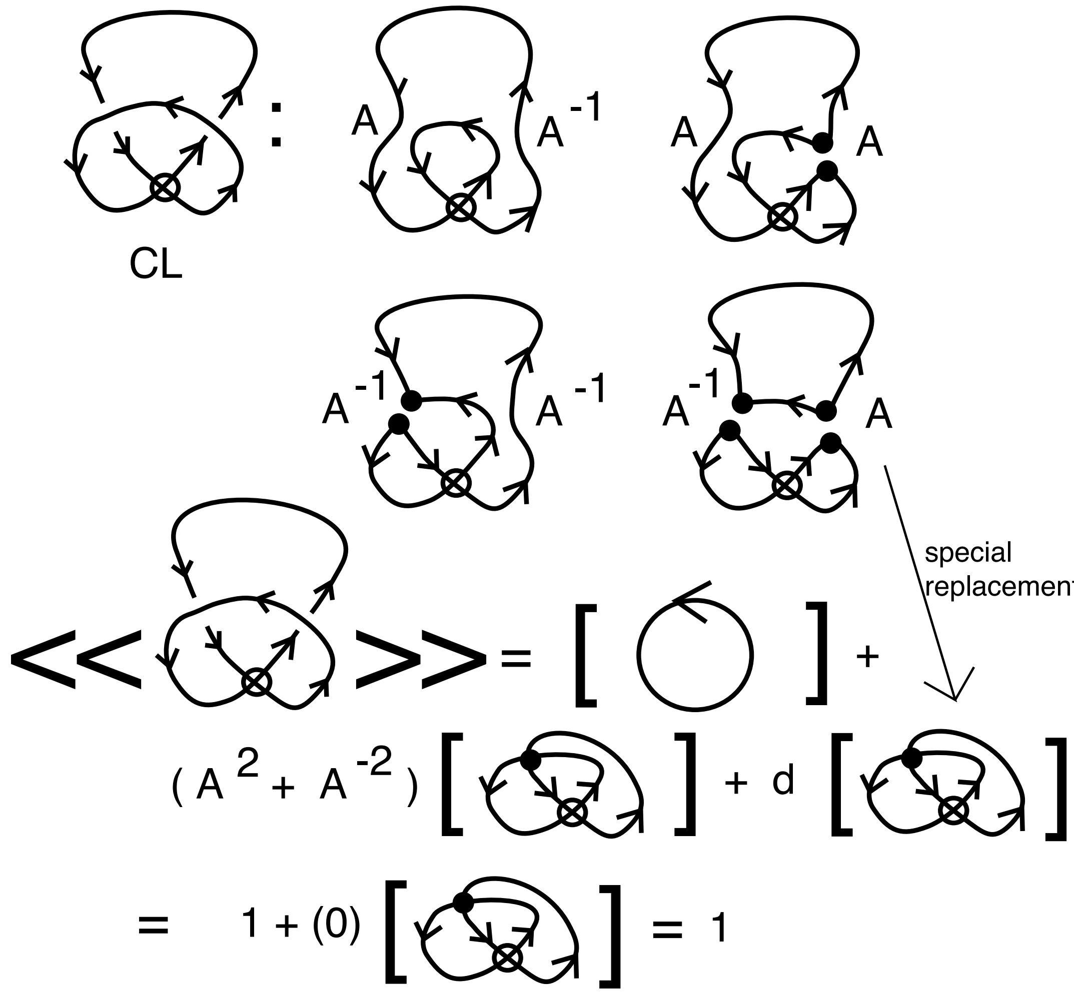


Figure 24. Example2.1

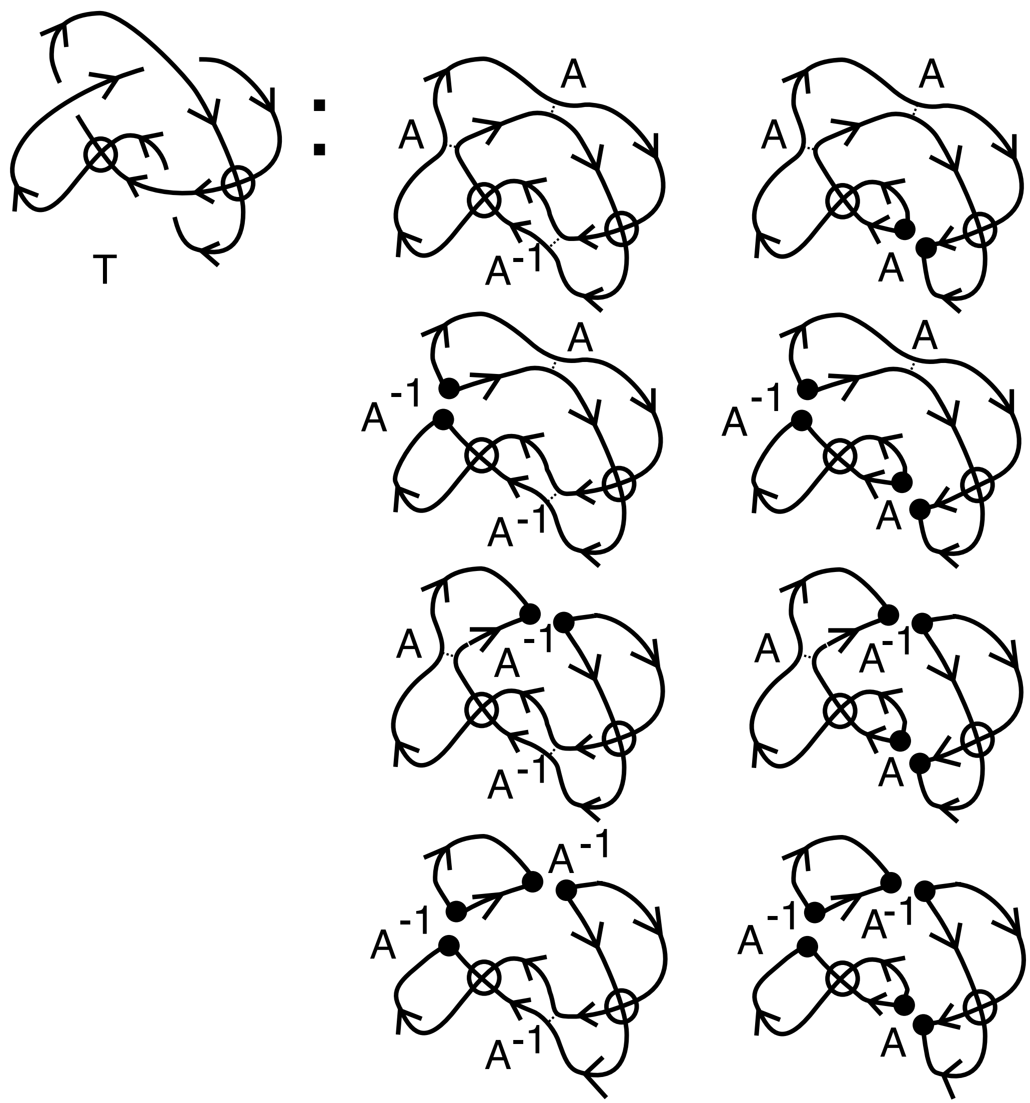


Figure 27. Virtualized Trefoil States

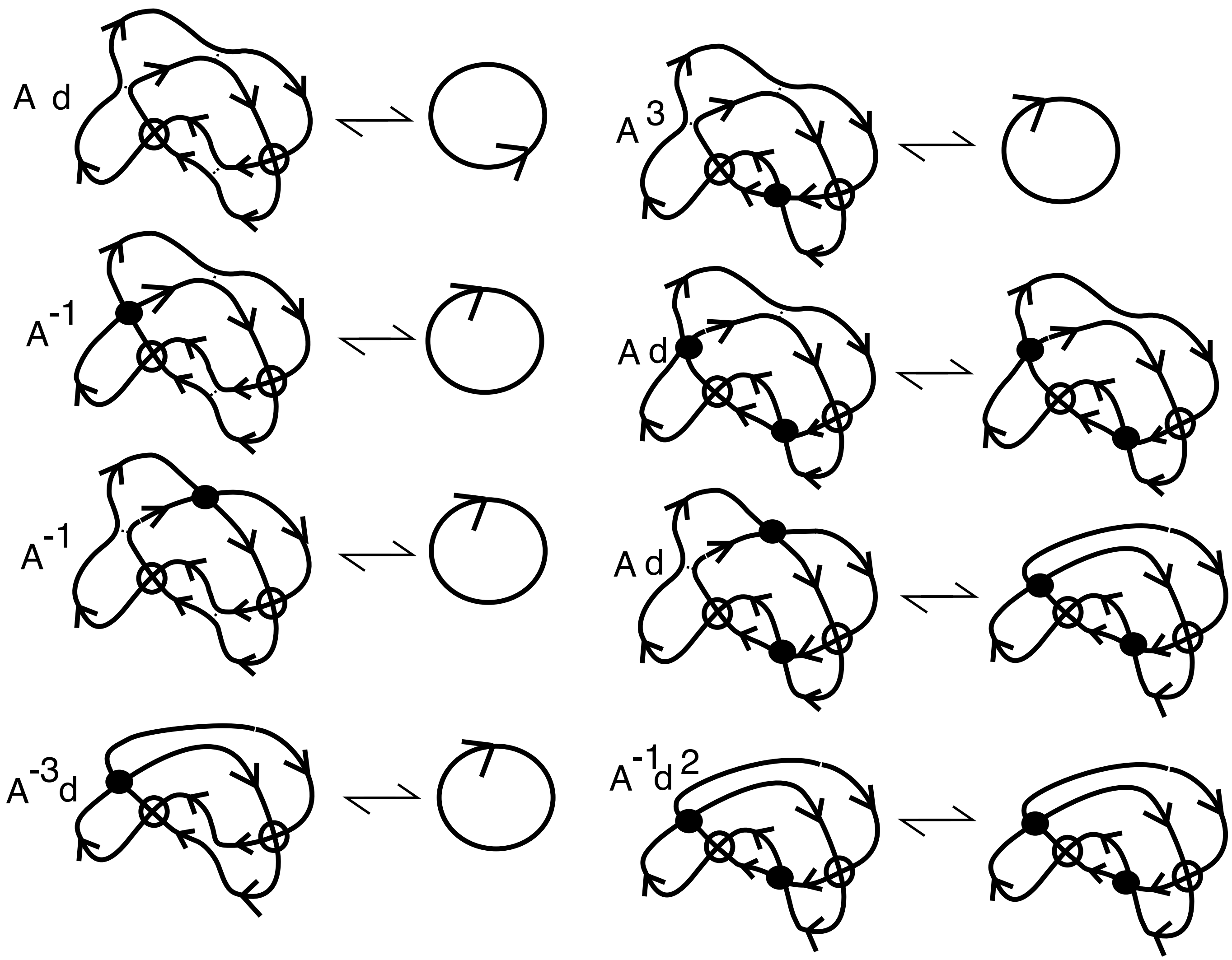
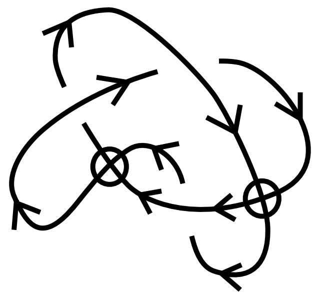
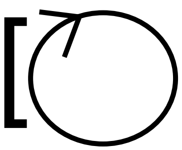


Figure 28. Flattened Virtualized Trefoil States

Virtualized Trefoil is Non-Classical with Virtual Crossing Number Two.

<<  >> =

$(A^{3} + Ad + 2A^{-1} + A^{-3}d)$ [] +

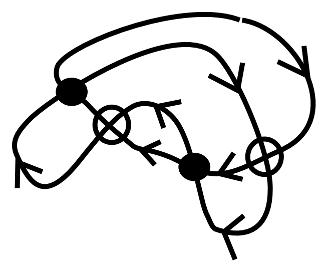
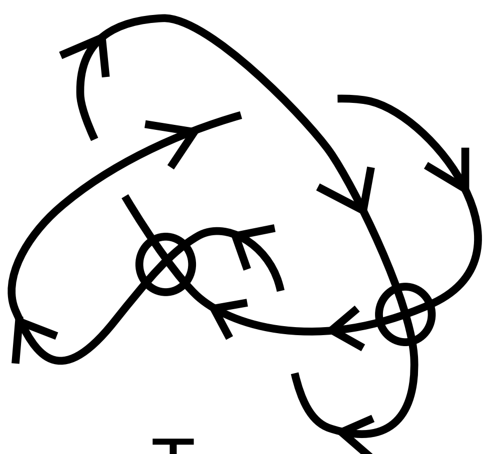
$(Ad + A^{-1}d^{2} + Ad)$ []

Figure 29. Extended Bracket for the Virtualized Trefoil

Let #<<K>> the maximal number of necessary virtual crossings among all the virtual graphs that appear in <<K>>.

THEOREM. The virtual crossing number of K is bounded below by #<<K>>.

Conclusion: The virtualized trefoil (previous slide) had virtual crossing number two.



T

Nota Bene. T lives on a torus.

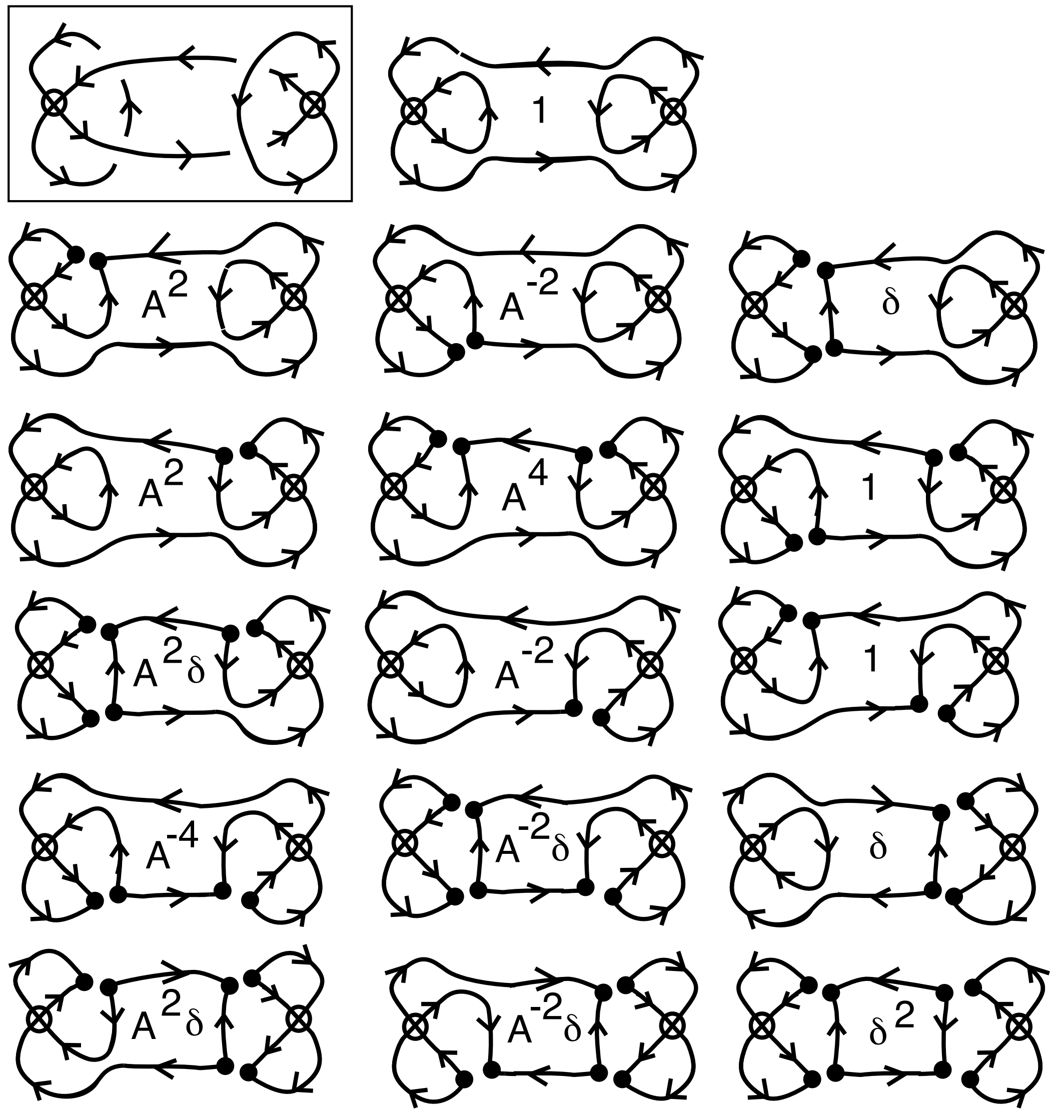
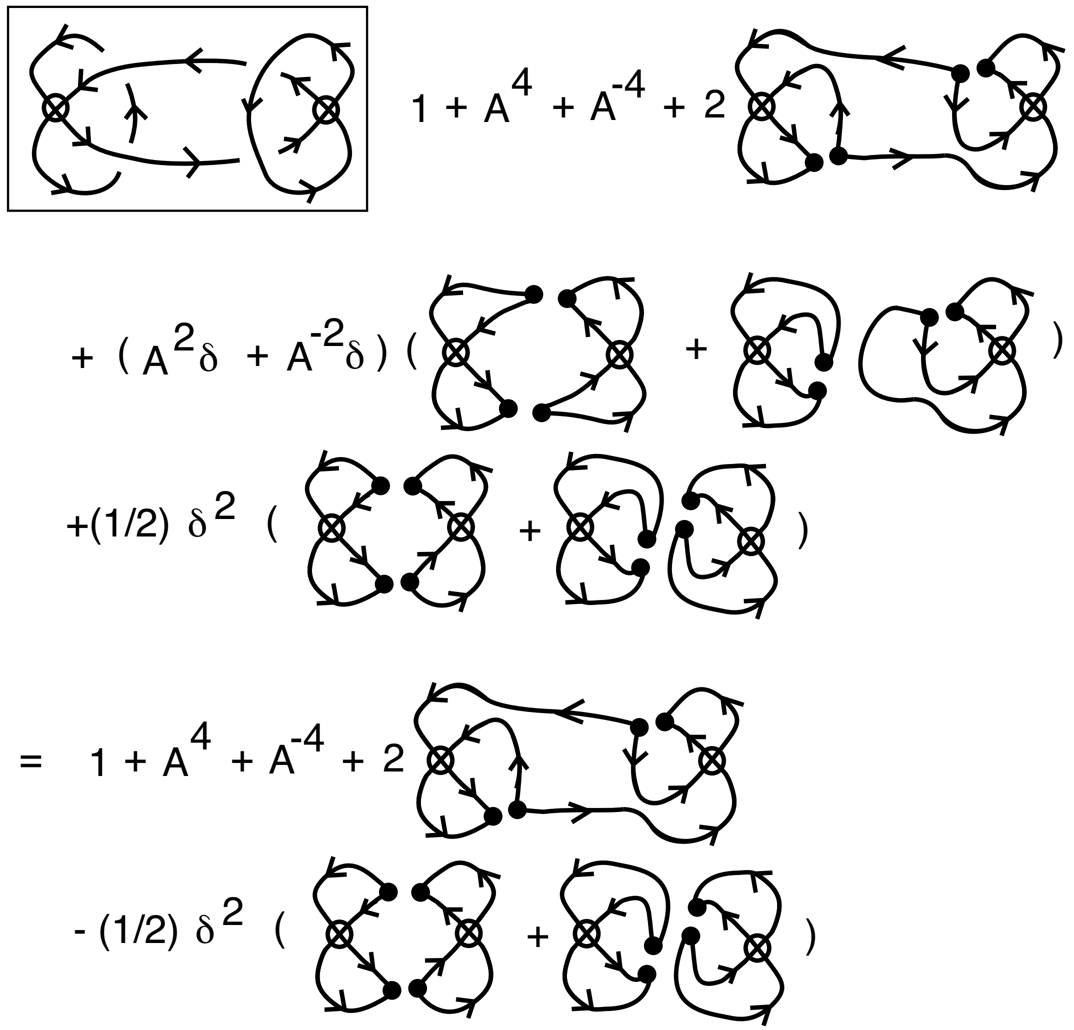
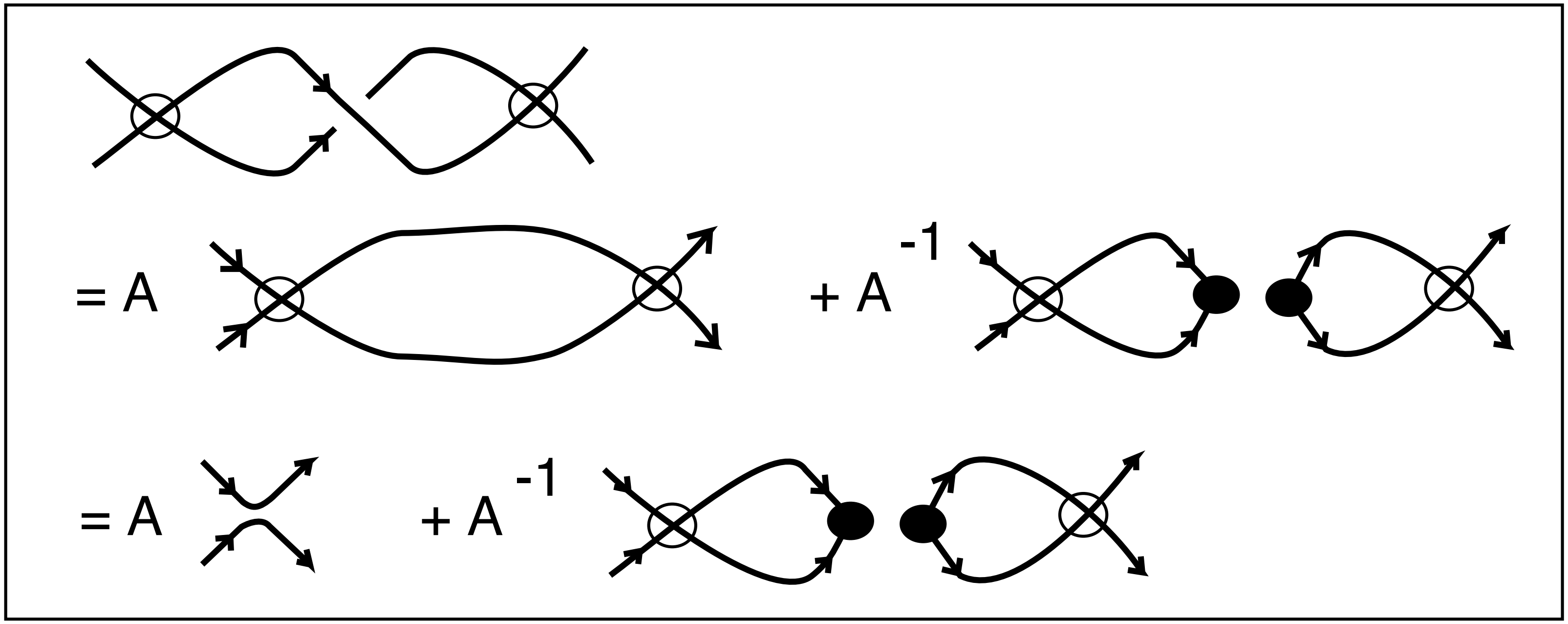


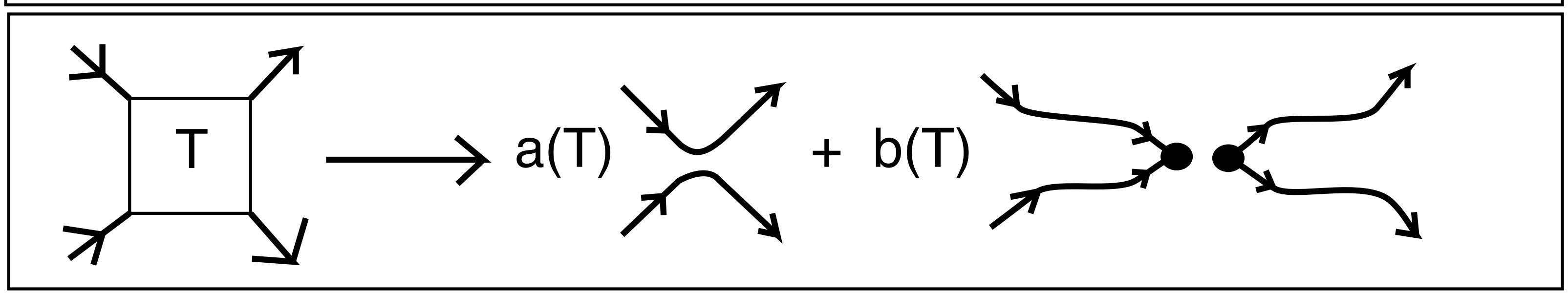
Figure 31. Kishino Diagram States



Extended Bracket for Kishino Diagram

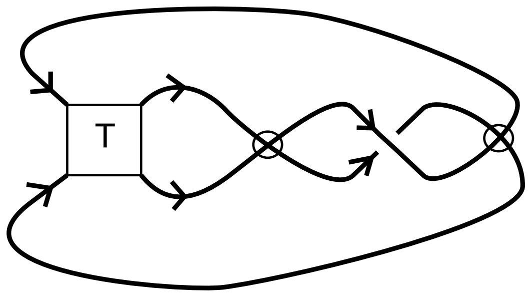
Expanding a Virtualized Crossing



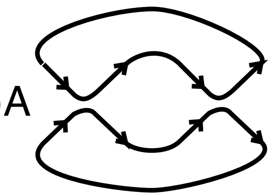


Expanding a Classical Tangle

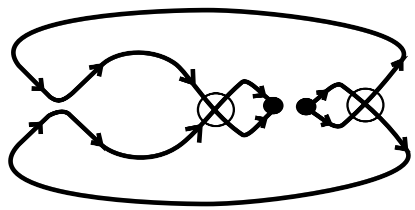
Detecting Non-Classicality of Single Virtualizations



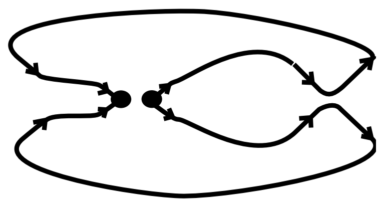
$\longrightarrow$ $a(T)A$



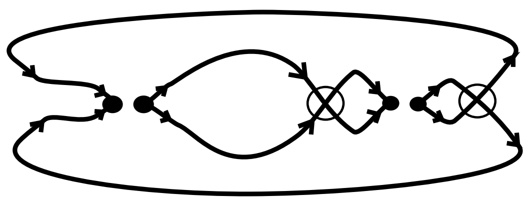
$+\ a(T)A^{-1}$



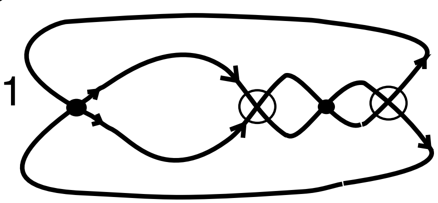
$+b(T)A$



$+b(T)A^{-1}$

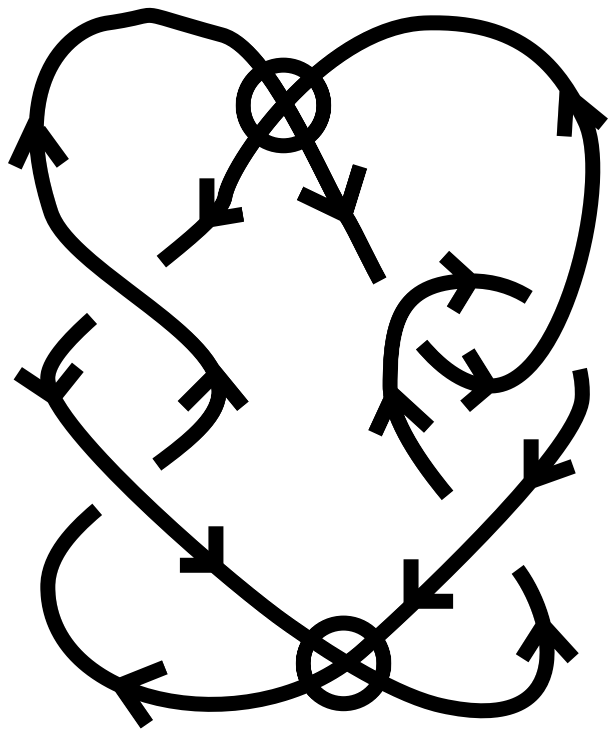


$= \ a(T)Ad + a(T)A^{-1} + b(T)A \ + b(T)A^{-1}$



Nobody's Perfect

A Culprit (discovered by Slavik Jablan)

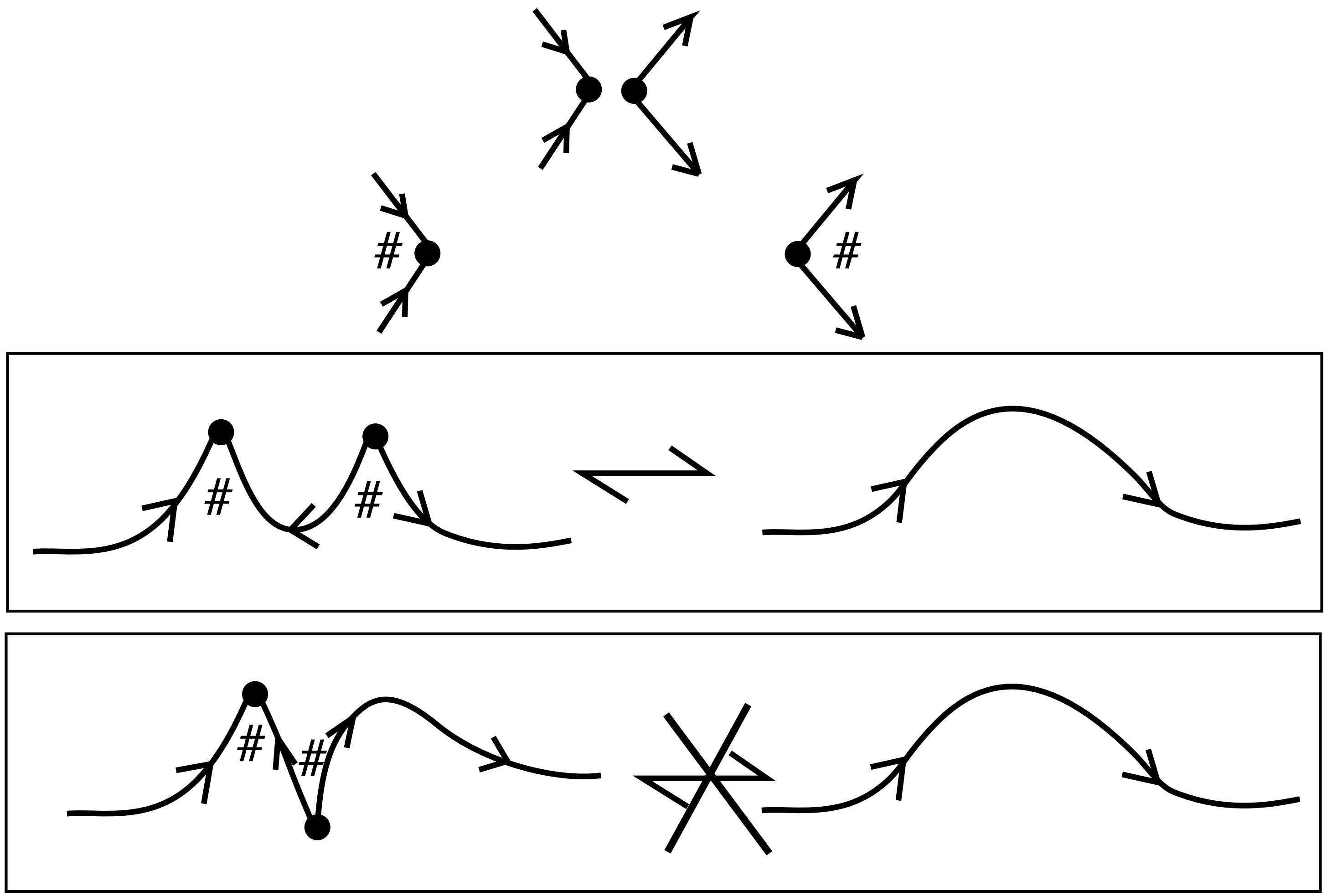


This virtual knot is undetectable by the extended bracket.

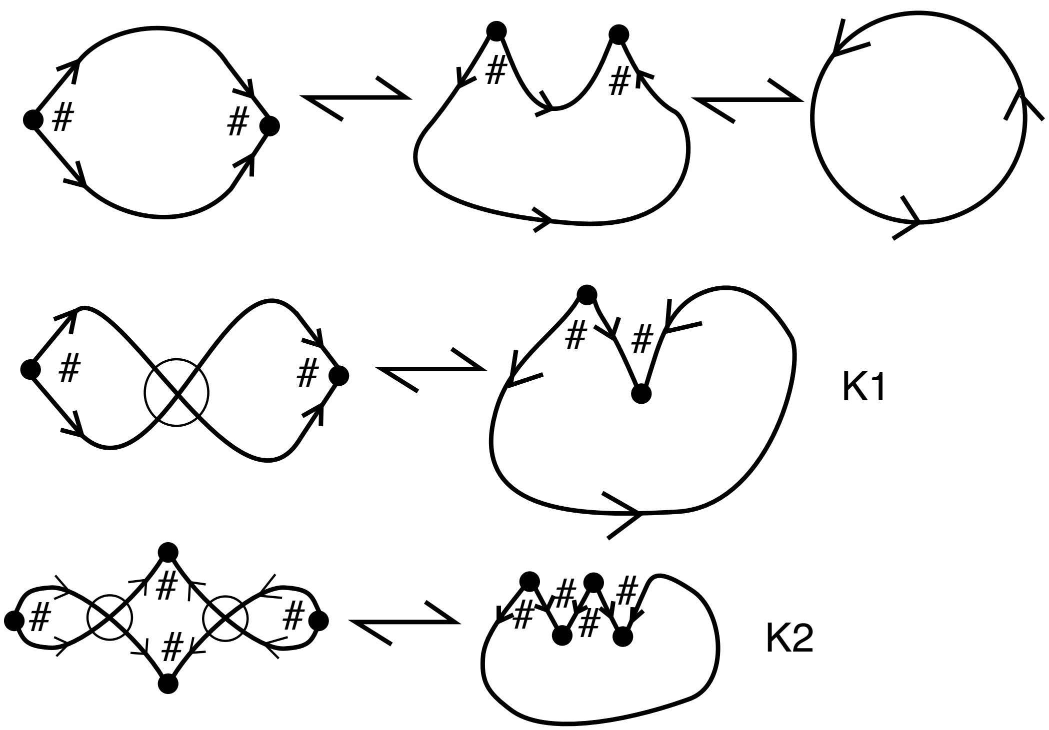
It is not classical as is shown by a look at its Alexander module.

THE ARROW POLYNOMIAL

All paired vertices are allowed to come apart.



In the arrow polynomial the paired vertices at a disoriented crossing come apart and the reduction relations simplify.
The end graphs are disjoint unions of simplified circle graphs. Each reduced circle graph becomes a new polynomial variable.



The arrow polynomial A[K] is
presented here as a natural simplication of
the extended bracket <<K>>.

In joint work with Heather Dye, we
found the very same invariant
by a different set of motivations
related to the work by
Miyazawa and Kamada.

HD and LK show that the maximum monomial
degree of the variables Kn
with deg(Kn) = n
gives a
a lower bound on the crossing number of
the knot.

We let A[K] denote the arrow polynonmial.

A[K] = <<K>> (replacing each graph by the corresponding product of Kn's)

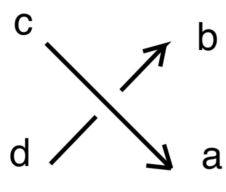
Setting all Kn = 1 gives the old bracket.

<K> = B[K] (1 = K1 = K2 = K3 = ...)

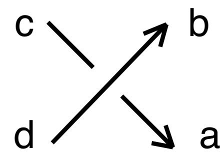
Setting A = 1 gives a polynomial invariant of flat virtuals.

F[K] = B[K](A =1)

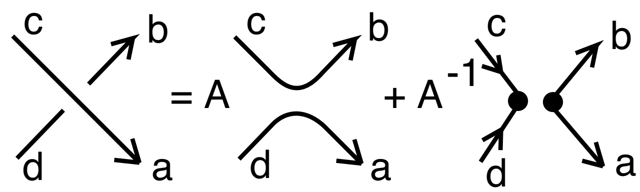
Coding A[K]

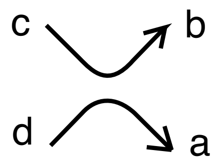


X[a,b,c,d]



Y[a,b,c,d]

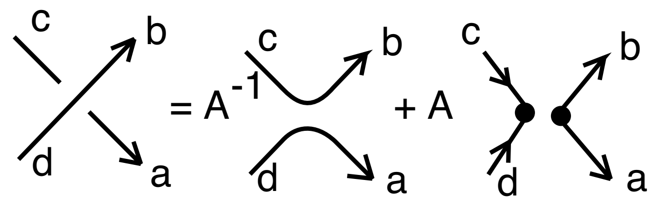


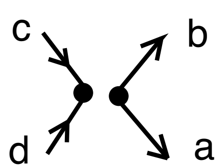


del[c,b] del[d,a]

X[a,b,c,d] = A del[c,b] del[d,a] + (1/A) led[a,b] led[c,d]

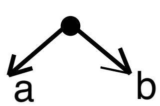
del[a,b] = del[b,a]
del[a.b] del[b,c] = del[a,c]



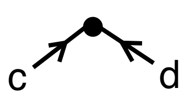


led[c,d] led[a,b]

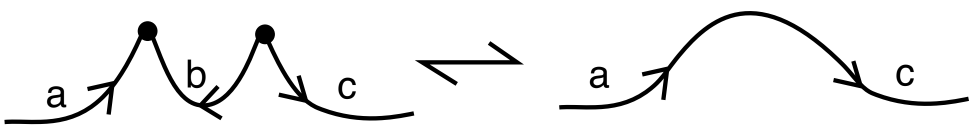
Y[a,b,c,d] = (1/A) del[c,b] del[d,a] + A led[a,b] led[c,d]



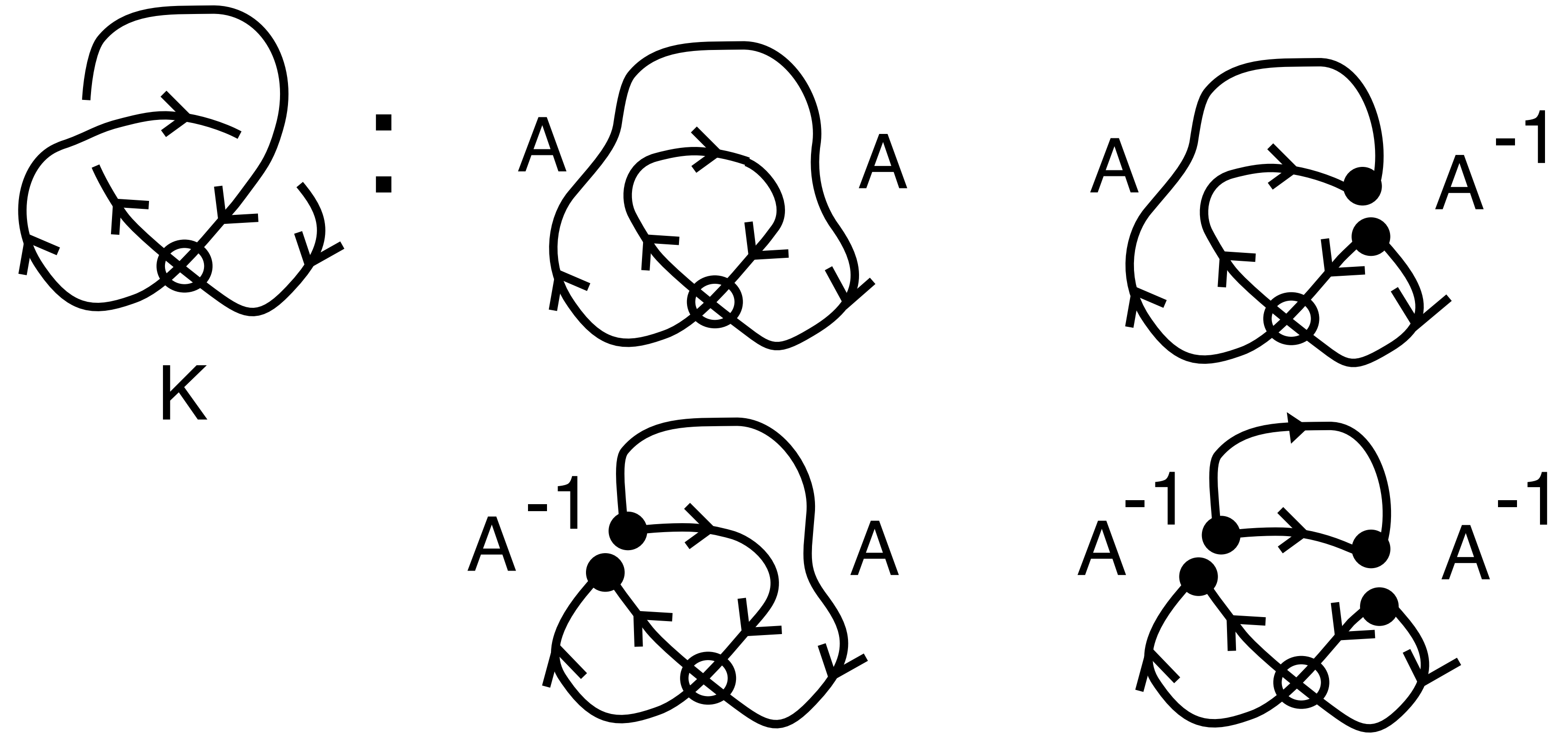
led[a,b]



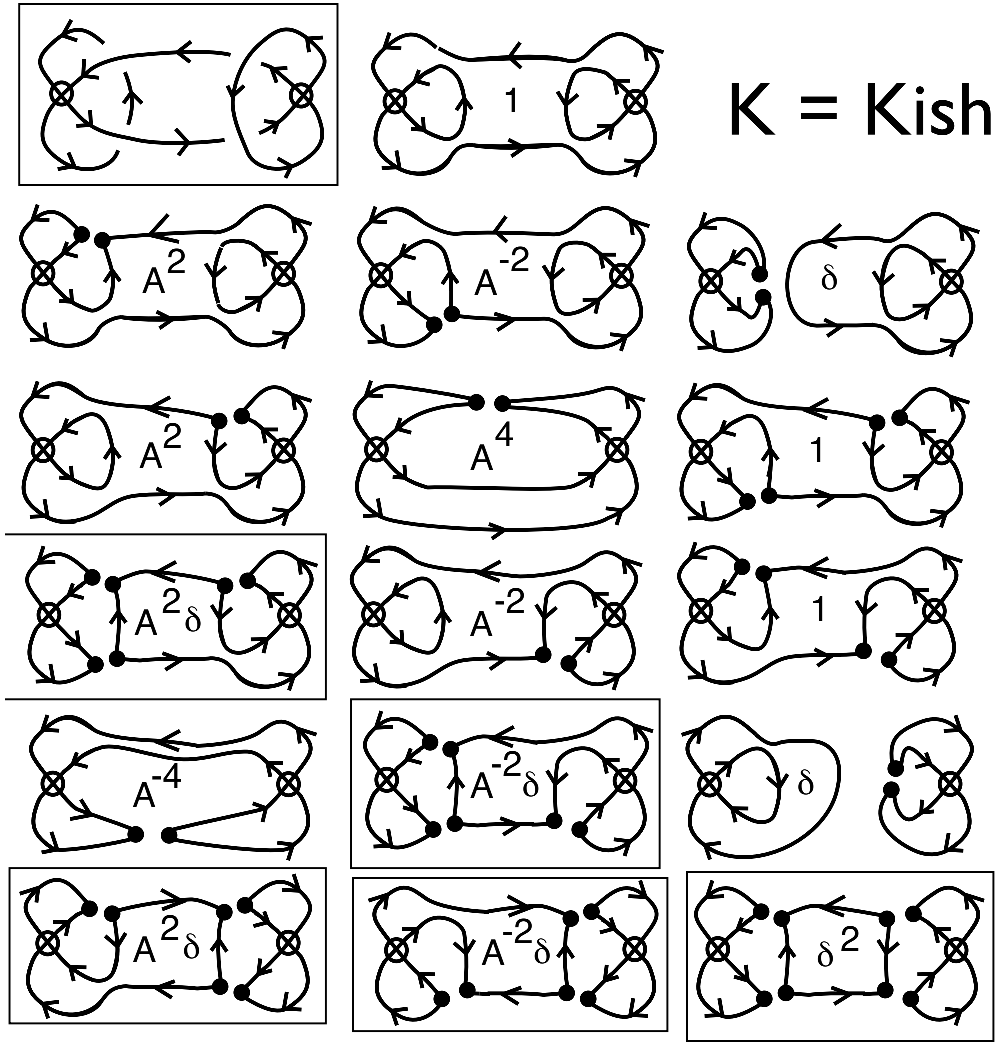
led[c,d]



led[a,b] led[b,c] = del[a,c]



$$A[K] = A^{2} + (1 - A^{(-4)})K1$$



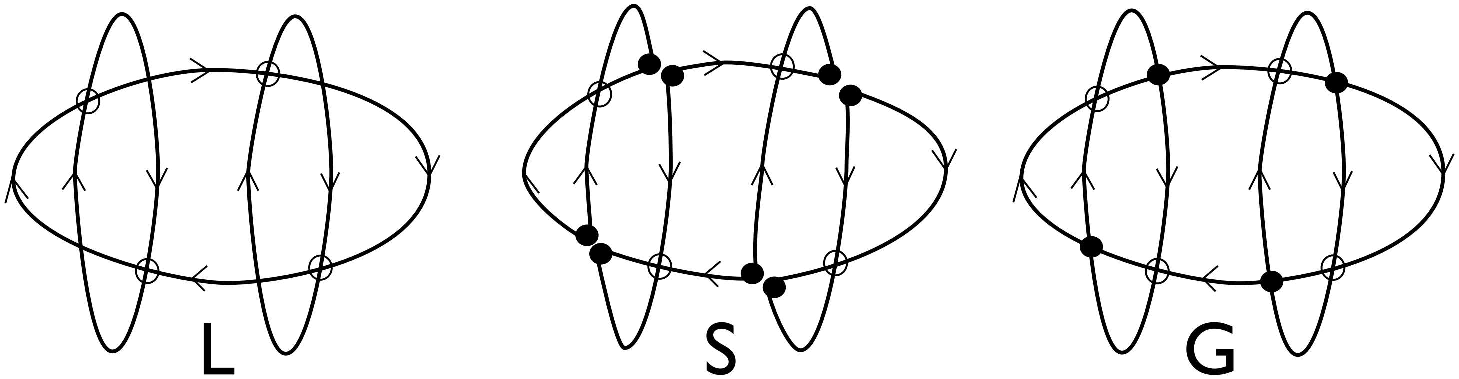
K = Kishino Diagram

$$\mathcal{A}[K] = 1 + A^4 + A^{-4} - d^2 K_1^2 + 2K_2$$

$$F[K] = 3 + 2K_2 - 4K_1^2.$$

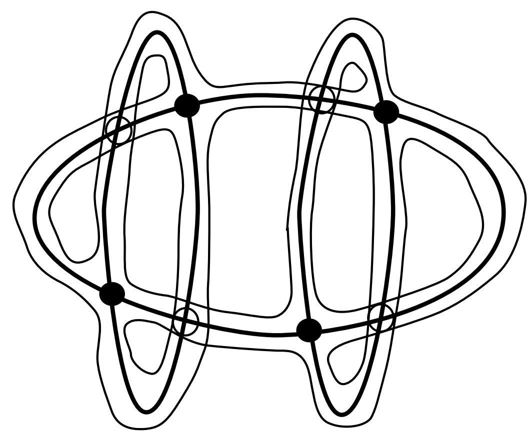
Using the Extended Bracket to
Determine Virtual Genus.

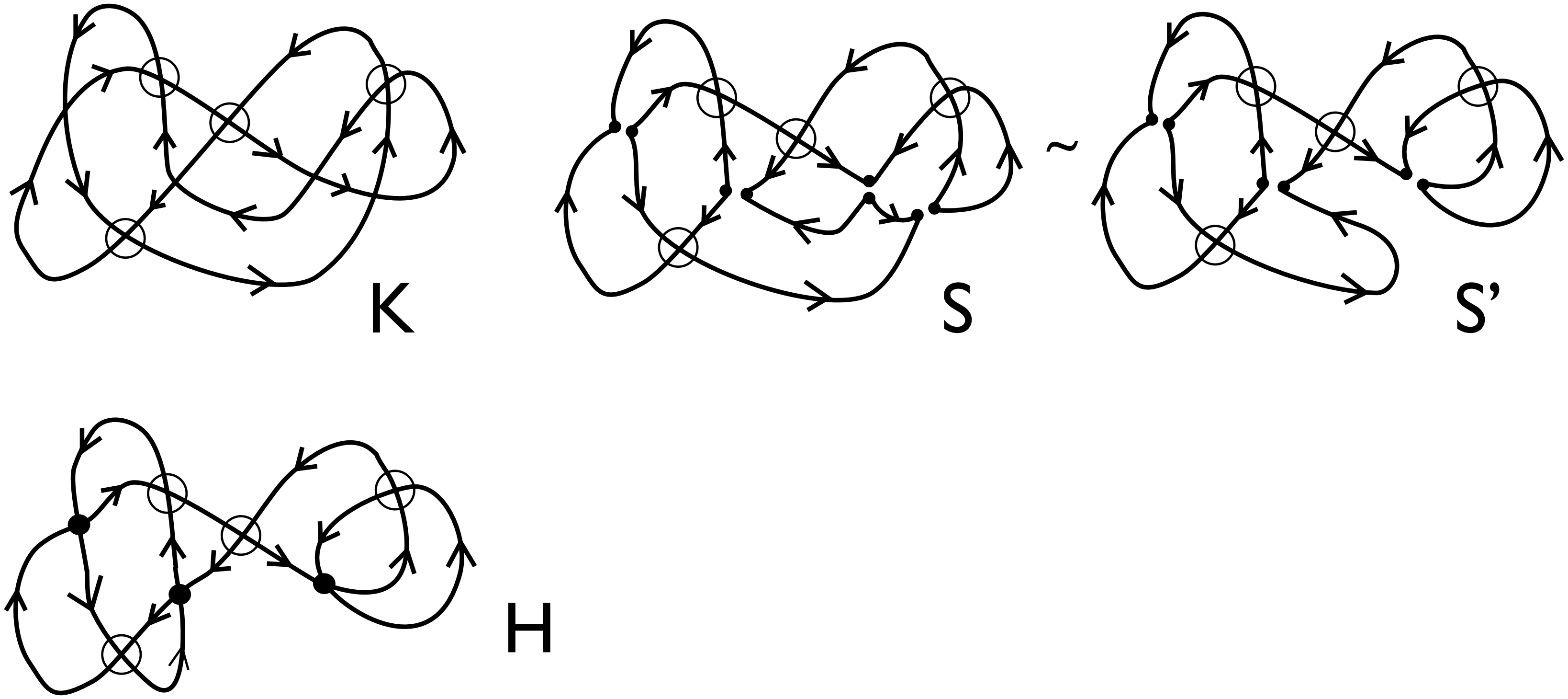
The virtual genus is the least genus orientable surface on which the virtual knot (or flat virtual knot) can be represented.



L is a flat virtual link whose virtual genus is 2.
We prove this by using the arrow polynomial to show that the state S survives and thus the graph G survives in the extended bracket.
One then sees that G is a virtual graph of genus 2.

This example shows how extended bracket has more information than arrow poly.



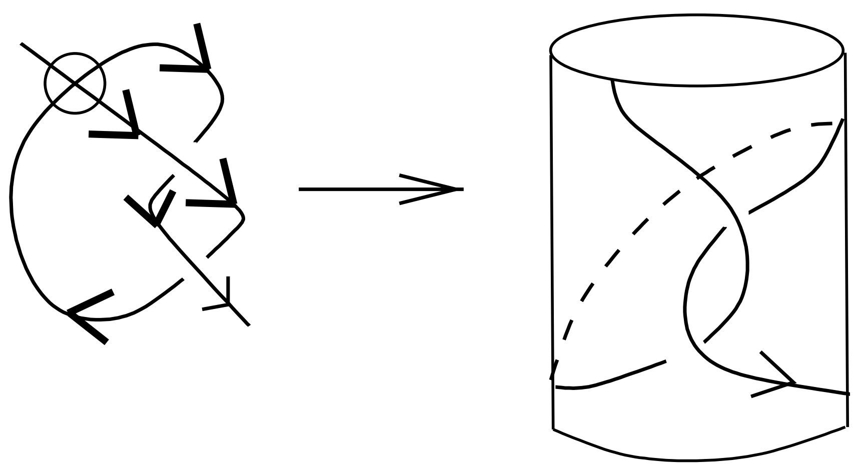


Here we have a similar story for the flat virtual knot K. The state S reduces to S'. And S' gives the surviving graph H. H has genus 2. And the graph of K itself has genus 2. This proves that K is a virtual flat knot of virtual genus 2.

The Arrow Polynomial for Surface Embeddings

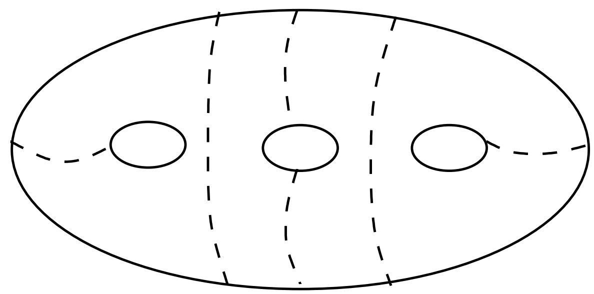
Lemma 4.1. *Let C be a curve in a state of the generalized arrow polynomial applied to a link in a surface. If C has non-zero arrow number then C is an essential curve in the surface.*

Proposition 4.2. *For any $i \geq 1$, there exists a virtual knot (and a virtual link), L, with minimal genus 1 such that some summand of $\langle L \rangle_A$ contains the variable K_i.*



Theorem 4.3. *Let S be an oriented, closed, 2-dimensional surface with genus $g \geq 1$. If $g = 1$, then S contains at most 1 nonintersecting, essential curve and if $g > 1$, then S contains at most $3g - 3$ non-intersecting, essential curves.*

Theorem 4.4. *If S is an oriented, closed, 2-dimensional surface that contains $3g - 3$ non-intersecting, essential curves with $g \geq 2$ then the genus of S is at least g.*

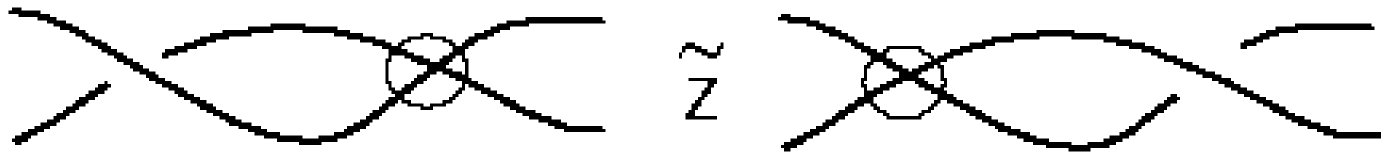


Theorem 4.5. *Let L be a virtual link diagram with arrow polynomial $\langle L \rangle_A$. Suppose that $\langle L \rangle_A$ contains a summand with the monomial $K_{i_1} K_{i_2} \cdots K_{i_n}$ where $i_j \neq i_k$ for all i, k in the set $\{1, 2, \ldots, n\}$. Then n determines a lower bound on the genus g of the minimal genus surface in which L embedds. That is, if $n \geq 1$, then the minimum genus is 1 or greater and if $n \geq 3g - 3$ then the minimum genus is g or higher.*

Proof. The proof of the this theorem is based on Theorem 4.3. Let L be a virtual link diagram with minimal genus one. Suppose that the arrow polynomial contains a summand with the monomial $K_i K_j$ with $i \neq j$. The summand corresponds to a state of expansion of L in a torus that contains two non-intersecting, essential curves with non-zero arrow number. As a result, these curves cobound an annulus and either share at least one crossing or both curves share a crossing with a curve that bounds a disk in some state obtained from expanding the link L. Smoothing the shared crossings results in a curve that bounds a disk and has non-zero arrow number (either $|i-j|$ or $|i+j|$) resulting in a contradiction. Hence, the minimum genus of L can not be one.

Suppose that L is a virtual link diagram and that $\langle L \rangle_A$ contains a summand with the factor $K_{i_1} K_{i_2} \cdots K_{i_{3g-3}}$. Hence, the corresponding state of the skein expansion contains $3g-3$ non-intersecting, essential curves in any surface representation of L. If any of these curves cobound an annulus in the surface, then some state in the expansion of L contains a curve that bounds a disk and has non-zero arrow number, a contradiction. Hence, none of the $3g-3$ curves cobound an annulus and as a result, the minimum genus of a surface containing L is at least g. $\square$

Z - Equivalence

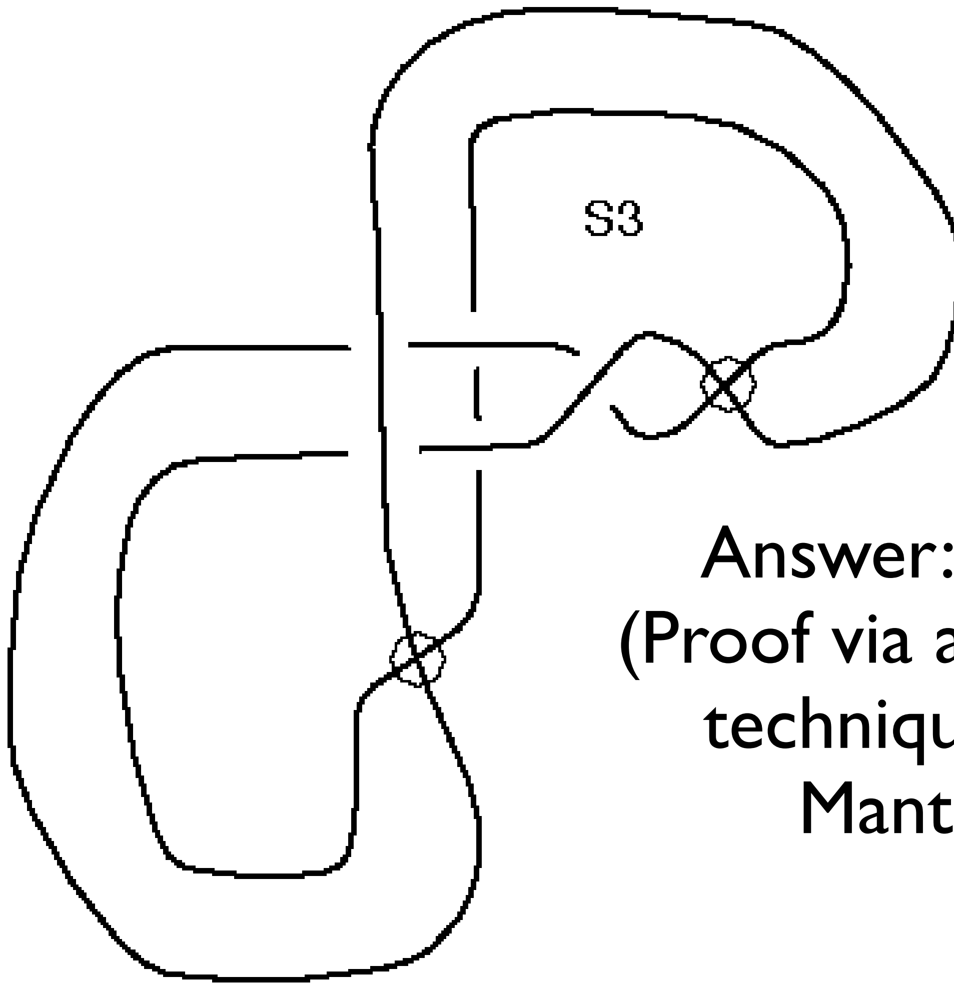


Z - Equivalent Links have the
same Jones polynomial

Kauffman, Fenn, Manturov conjectured that virtual knots of unit Jones polynomial are Z-equivalent to classical knots.

Here are some recent examples to ponder.

The Knot S3 (work with Slavik Jablan) has unit Jones polynomial. Is it Z-equivalent to a classical knot?

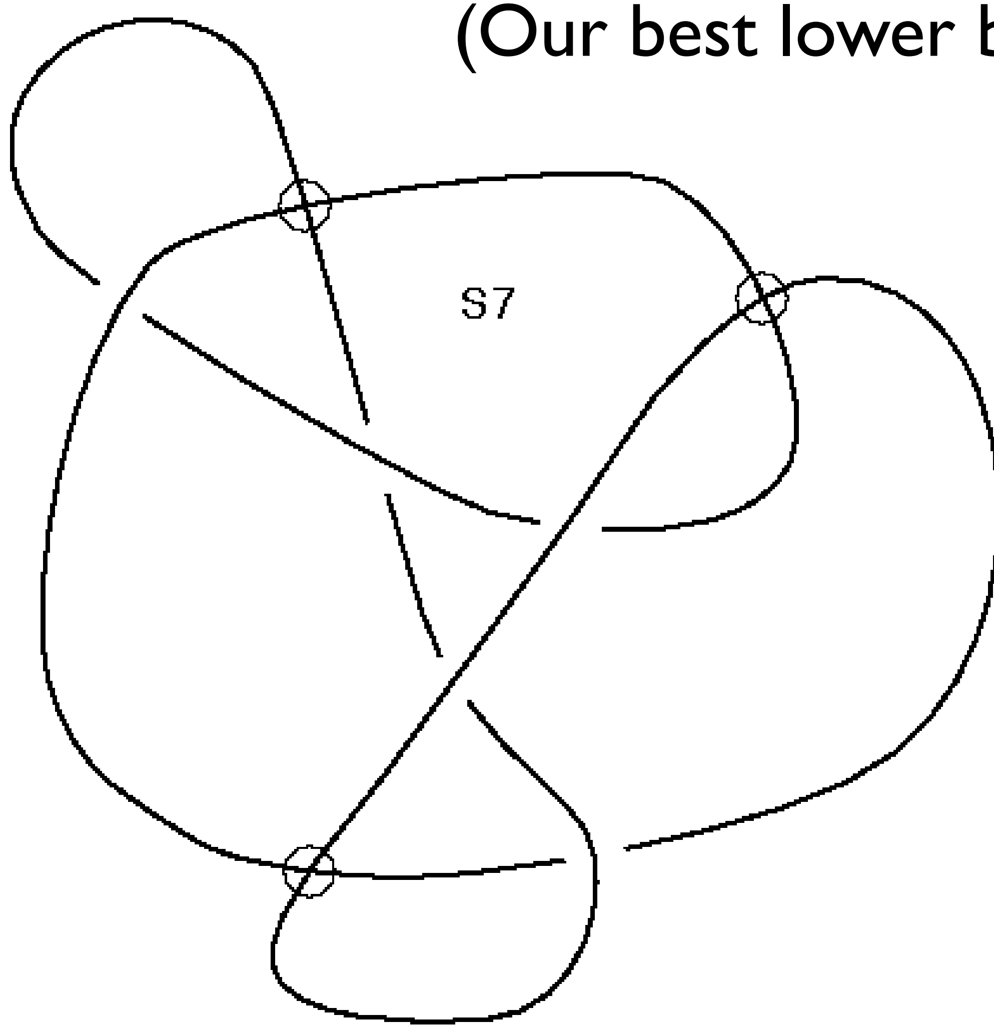


Answer: It is not!
(Proof via a new parity technique due to Manturov.)

A[S3] = -2K1^2 + K2 + A^4 (1 - 2K1^2 + K2)

The knot S7 has unit Jones polynomial. Is it Z-equivalent to a classical knot? Does it have crossing number 3?

(Our best lower bound is 2.)



A[S7] = -(A^(-1) + A^3)K1^2 + (A^(-1))K2

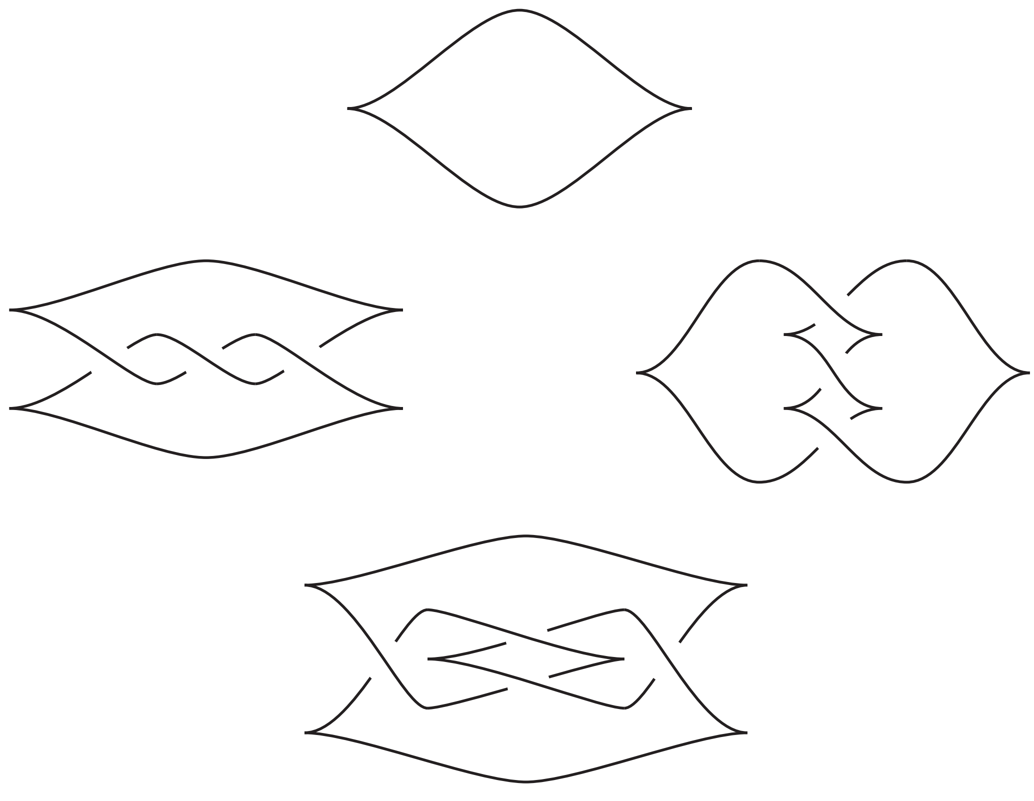
Legendrian Knots

$$x'(t)y(t) = z'(t)$$

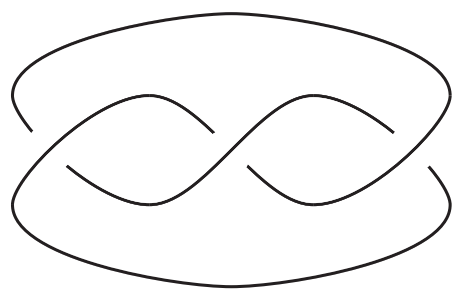
no tangents parallel to z-axis

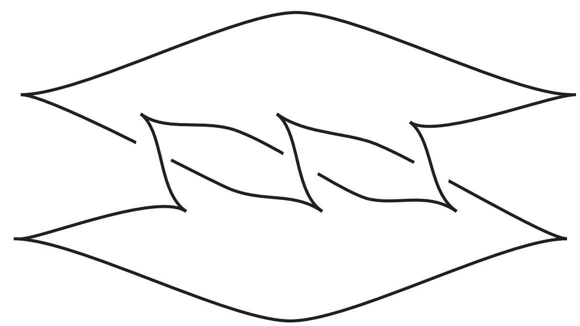
project into x-z plane

finite number of points with tangent parallel to y axis



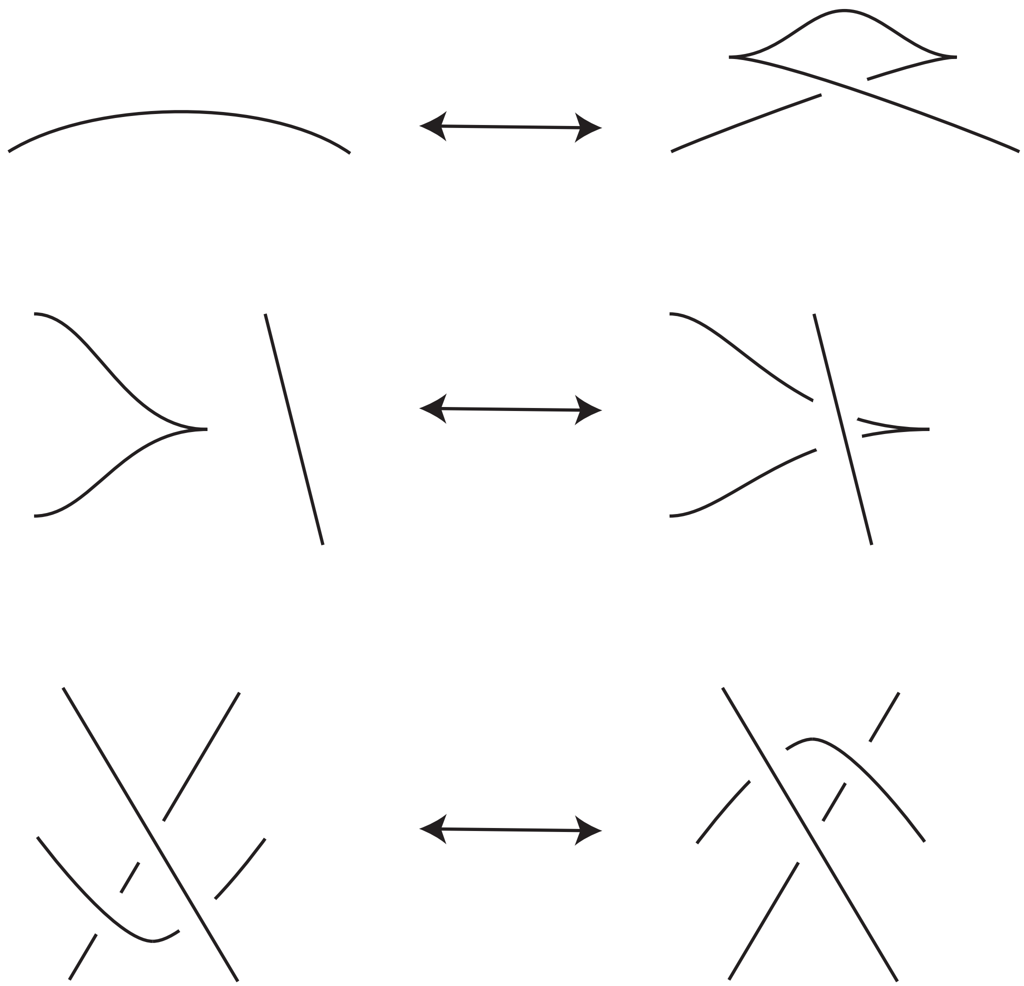
no vertical tangencies.
only non-smooth points are generalized cusps.
at each crossing the slope of th overcrossing is smaller (more negative) than the slope of the undercrossing.





Converting a knot diagram (left) into a Legendrian front (right).

See survey article by John Entnyre.



Legendrian Reidemeister Moves

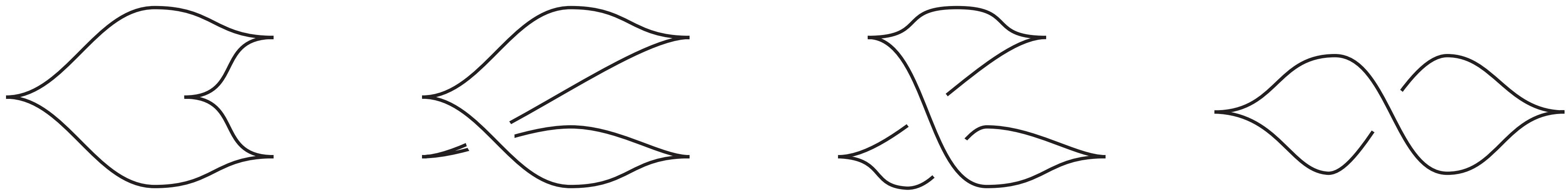


FIGURE 9. Various fronts of the same Legendrian unknot.

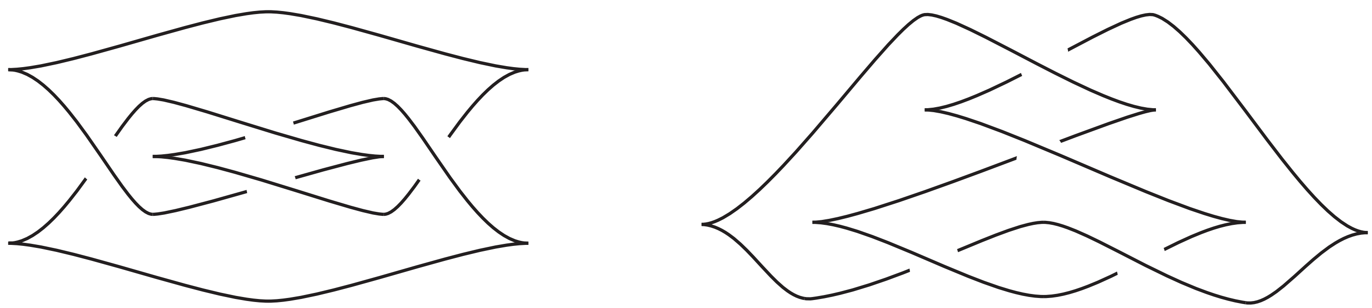


FIGURE 10. Two fronts of the same Legendrian figure eight knot.

Work In Progress:
The Arrow Polynomial generalizes to an invariant of Legendrian knots.

Stay tuned for more developments.

Many Questions

1. Find better bounds on virtual crossing numbers.
2. Understand virtual graph classes.
3. Relative strength of <<K>> and A[K].
4. Categorify these invariants (work with Heather Dye and Vassily Manturov. see recent paper on arxiv.)
5. Relationship of these invariants with with the virtual Temperley Lieb algebra.
6. Second order generalizations to invariants of knots in surfaces and to long flats.
7. Deeper oriented structure in other state sums?
8. Legendrian knots.