On a relation between algebraic and geometric properties of braids

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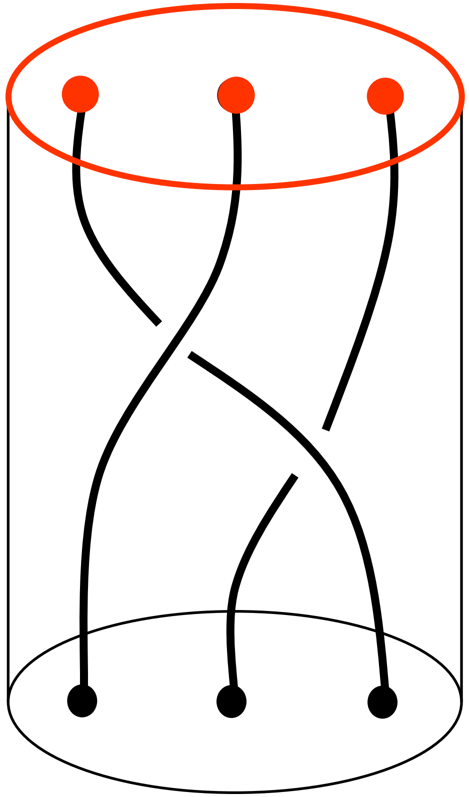
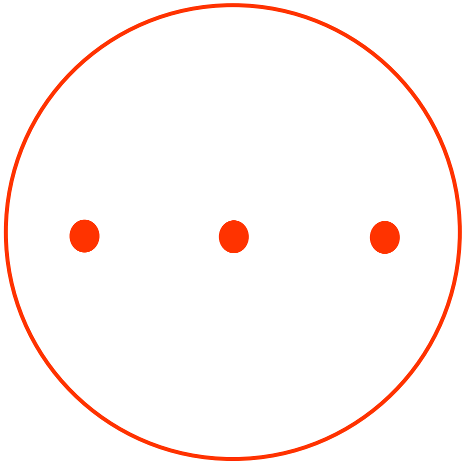
Introduction



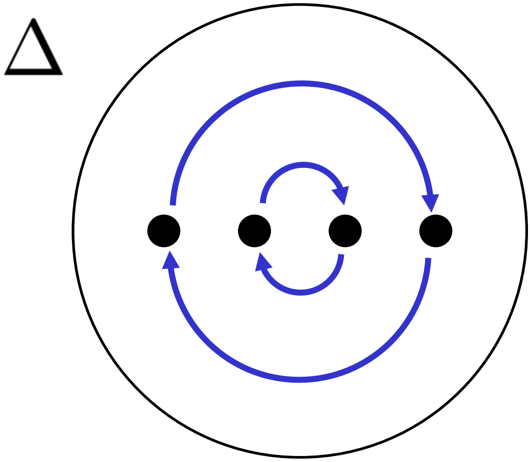


Braids form a group: B_n

The geometric side

Braids can also be seen as automorphisms of the punctured disc.

Center of B_n : $\langle \Delta^2 \rangle$



Modulo this center,
braids can be seen as automorphisms of the **punctured sphere**.

Hence one can apply **Nielsen-Thurston** theory to braids.

The geometric side

Geometric classification of braids:

Periodic braids: Finite order elements in $B_n / \langle \Delta^2 \rangle$

That is, roots of Δ^m.

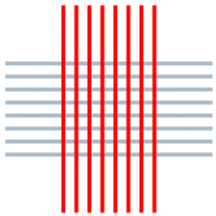
Rotations

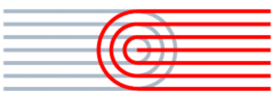
The geometric side

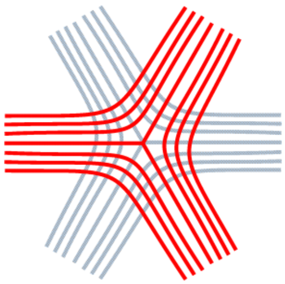
Geometric classification of braids:

Periodic braids: Rotations

Pseudo-Anosov braids: Preserve two transverse measured foliations…







…scaling the measure of $\mathcal{F}^u$ by $\lambda > 1$

and the measure of $\mathcal{F}^s$ by λ^{-1}

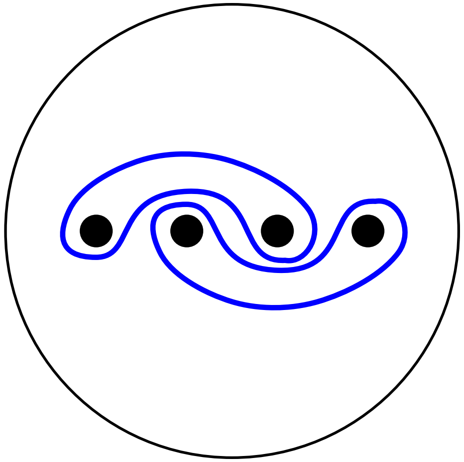
The geometric side

Geometric classification of braids:

Periodic braids: Rotations

Pseudo-Anosov braids: Preserve two transverse measured foliations…

Reducible braids: Preserve a family of disjoint simple curves



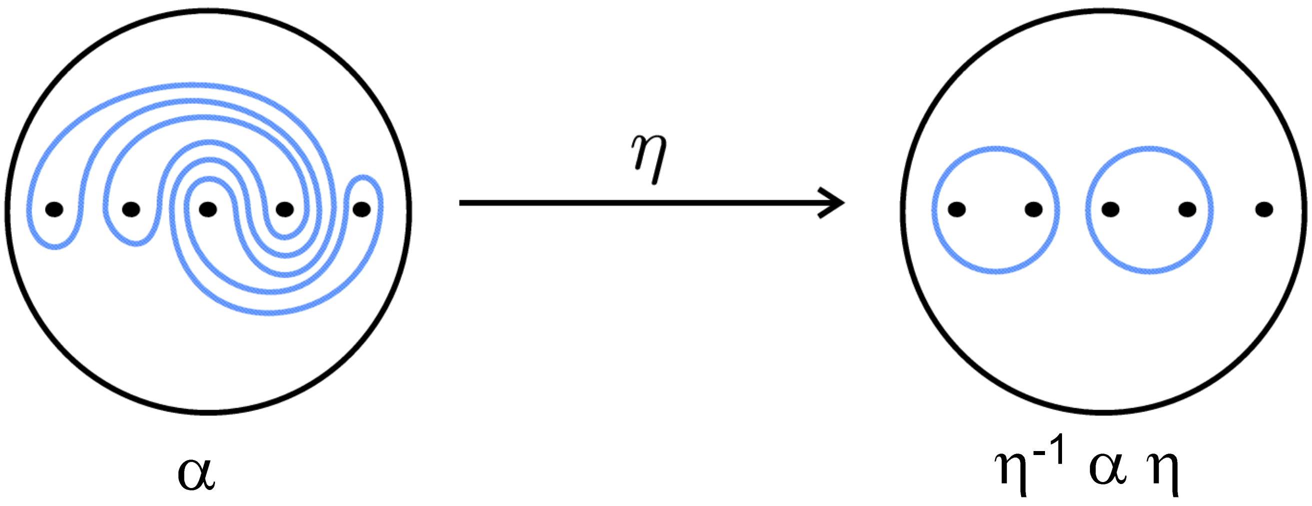
Reduction system

The geometric side

Every braid has a **canonical reduction system**. (Birman-Lubotzky-McCarthy)

In general, the canonical reduction system can be quite complicated.

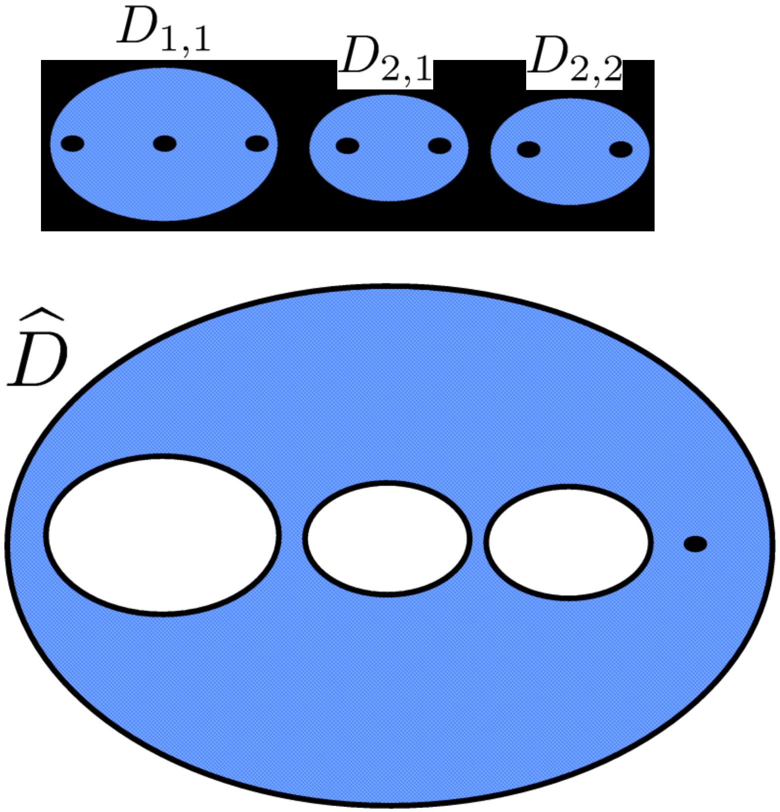
But it can always be simplified by an automorphism



This corresponds to a conjugation

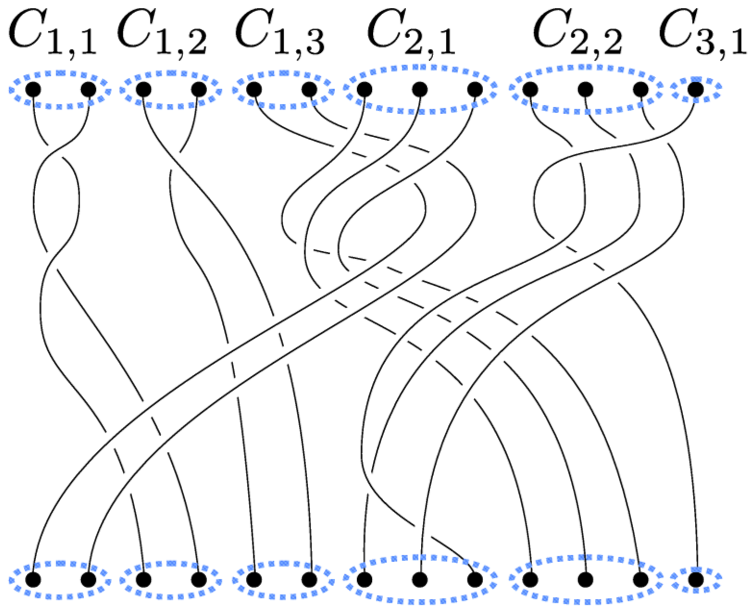
The geometric side

We can then decompose the disc D along the reduction curves:



The geometric side

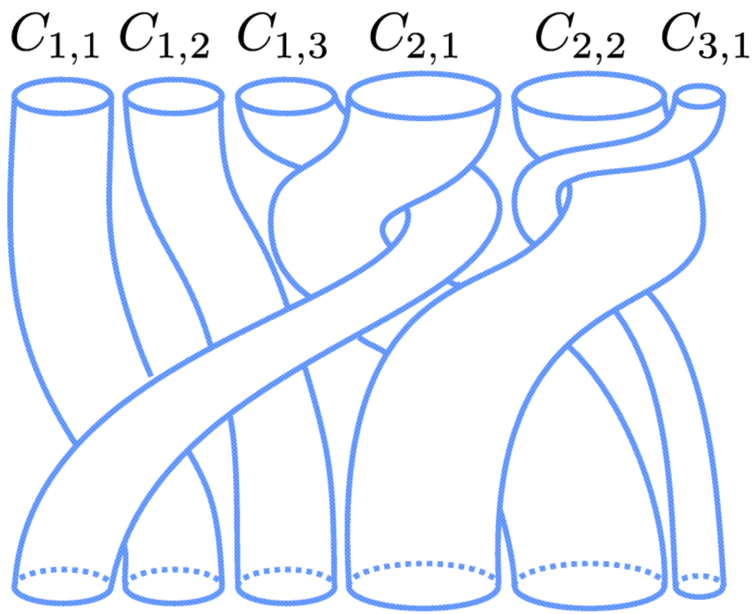
Example:



Every reducible braid can be decomposed into simpler braids

The geometric side

Example:



Every reducible braid can be decomposed into simpler braids

The geometric side

Theorem: (Thurston) (applied to braids)

Every braid is either **periodic**, or **pseudo-Anosov**, or **reducible**.

Theorem: (Thurston) (applied to braids)

The canonical reduction system decomposes a braid into braids which are either

periodic or **pseudo-Anosov**

The geometric side

Thurston decomposition allows to show **theoretical** results.

First solving the **periodic** and **pseudo-Anosov** case…

… and then deducing the **reducible** case from the above ones.

Examples:

(GM, 2003) The n-th root of a braid is unique up to conjugacy.

(GM-Wiest, 2004) Structure of the centralizer of a braid.

The geometric side

In **practice**, knowing the canonical reduction system **improves algorithms**.

Example: (Birman-Gebhardt-GM, 2003)
A project to solve the conjugacy problem in braid groups.

But… how can we compute the canonical reduction system?

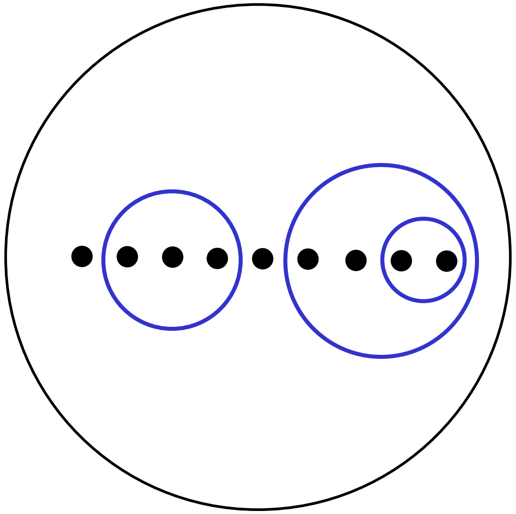
Remark: There is a well-known algorithm by Bestvina-Handel, using train tracks.

We will use a different approach.

The geometric side

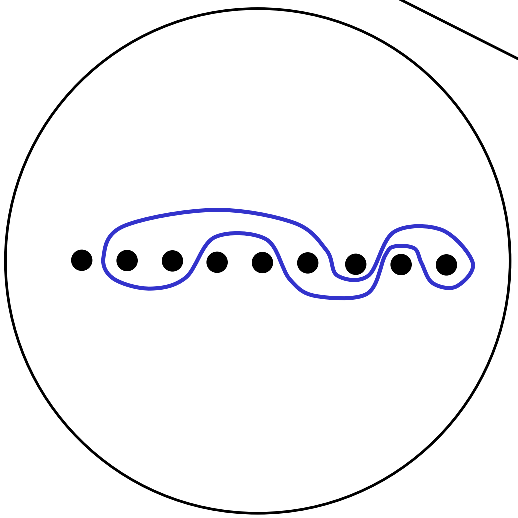
If the reduction curves are **round**, they are very easy to detect.

Benardete-Gutiérrez-Nitecki, 1991



The geometric side

If the reduction curves are **almost round**, they are very easy to detect...
… if the interior braid is trivial.

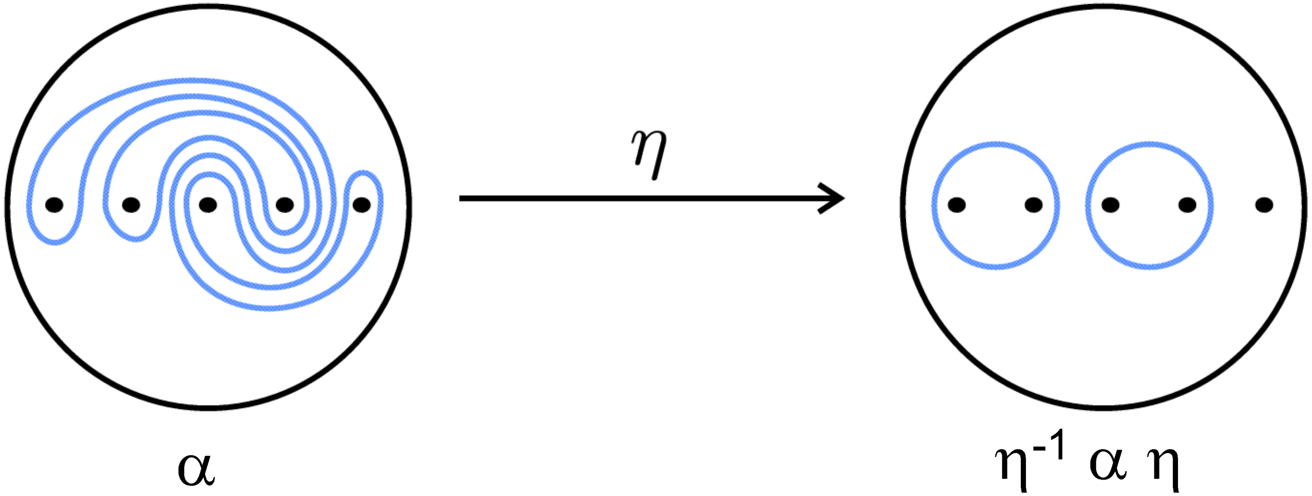


One max. and one min.
in the horizontal direction

Theorem: (GM-Wiest, 2009) There is a polynomial algorithm which decides whether a braid admits an almost-round reduction curve, with trivial interior braid.

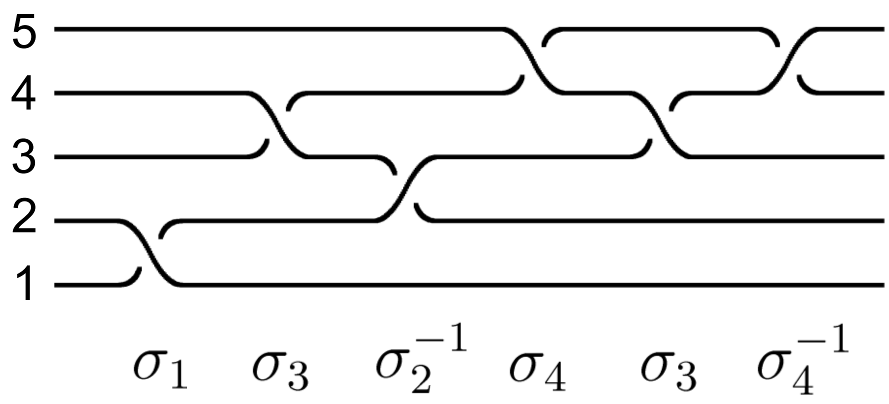
But…what happens if the reduction curves are very tangled?

Recall that they can be simplified by a conjugation.



But… what conjugation?

The algebraic side



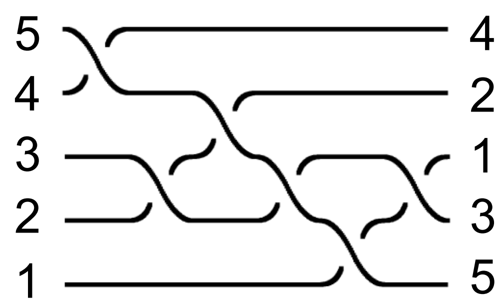
E. Artin (1925)

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leq i \leq n-2) \end{array} \right\rangle$$

The algebraic side

Positive elements: Braids in which every crossing is positive

Simple elements: Positive elements in which every pair of strands cross at most once.



Simple elements of B_n $\xleftrightarrow{\text{Bij.}}$ Permutations of Σ_n

Δ = Biggest simple element = *Half twist*

The algebraic side

Garside (1969), Deligne (1972), Adyan (1984), Elrifai-Morton (1988), Thurston (1992).

Left normal form: $x = \Delta^p x_1 \cdots x_r$

Simple elements ($x_1 \cdots x_r$)

Every product $x_i \, x_{i+1}$ must be **left-weighted**.

Biggest possible simple element in any such writing of this product.

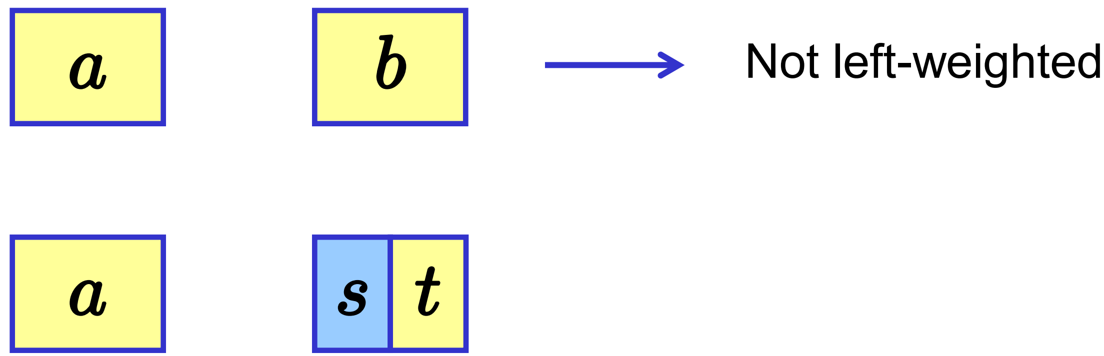
The algebraic side

In general: Given simple elements a, b,

a b $\longrightarrow$ Not left-weighted

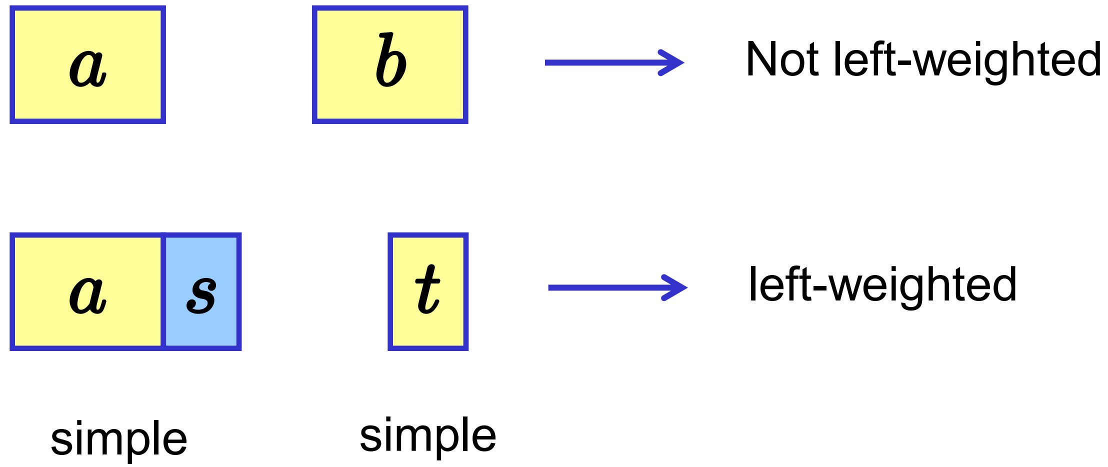
The algebraic side

In general: Given simple elements a, b,



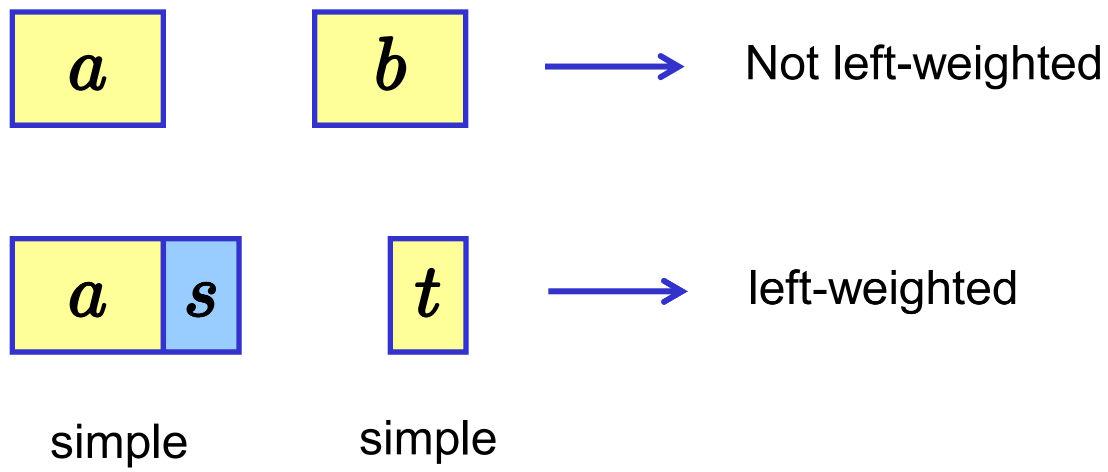
The algebraic side

In general: Given simple elements $a, \; b,$



The algebraic side

In general: Given simple elements $a, \; b,$



We call this procedure a **local sliding** applied to $a\, b$.

Remark: Possibly $as = \Delta$, or $t = 1$.

The algebraic side

Computation of a left normal form, given a product of simple elements:

$$x = s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9\ s_{10}$$

Apply all possible local slidings,
until all consecutive factors are left weighted

$$x = \Delta\ \Delta\ \Delta\ x_1\ x_2\ x_3\ x_4\ x_5\ 1\ 1$$

Left normal form:

$$x = \Delta^3 x_1\ x_2\ x_3\ x_4\ x_5.$$

Maximal power of Δ.

Minimal number of factors.
(canonical length)

Conjugacy problem

Garside (1969)

ElRifai-Morton (1988)

Birman-Ko-Lee (1998)

Franco-GM (2003)

Gebhardt (2005)

Birman-Gebhardt-GM (2008)

Gebhardt-GM (2009)

Charney: (1992) Artin-Tits groups of spherical type are biautomatic.

Conjugacy problem

Given an element x, compute the set of simplest conjugates of x.

(in some sense)

x and y are conjugate $\Leftrightarrow$ their corresponding sets coincide.

Conjugacy problem

$$x = \Delta^3 x_1 x_2 x_3 x_4 x_5.$$

Could x be simplified by a conjugation?

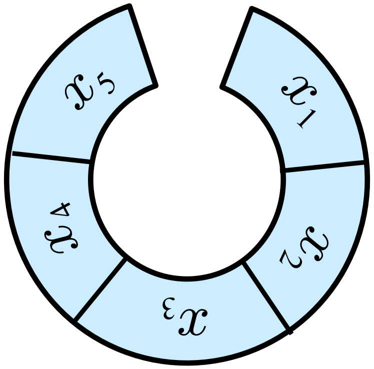
For simplicity, we will assume that there is no power of Δ :

$$x = x_1 x_2 x_3 x_4 x_5.$$

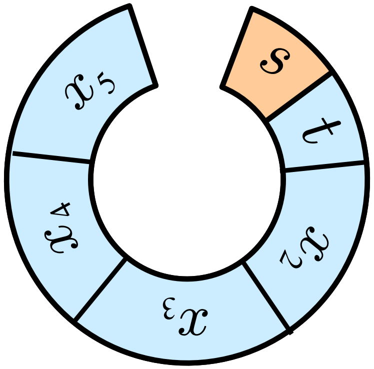
Consecutive factors are left-weighted . What about x_5 and x_1 ?

Up to conjugacy, we can consider that x_5 and x_1 are consecutive

Conjugacy problem

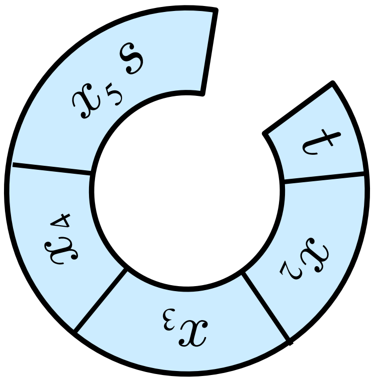


Cyclic sliding:



Conjugacy problem

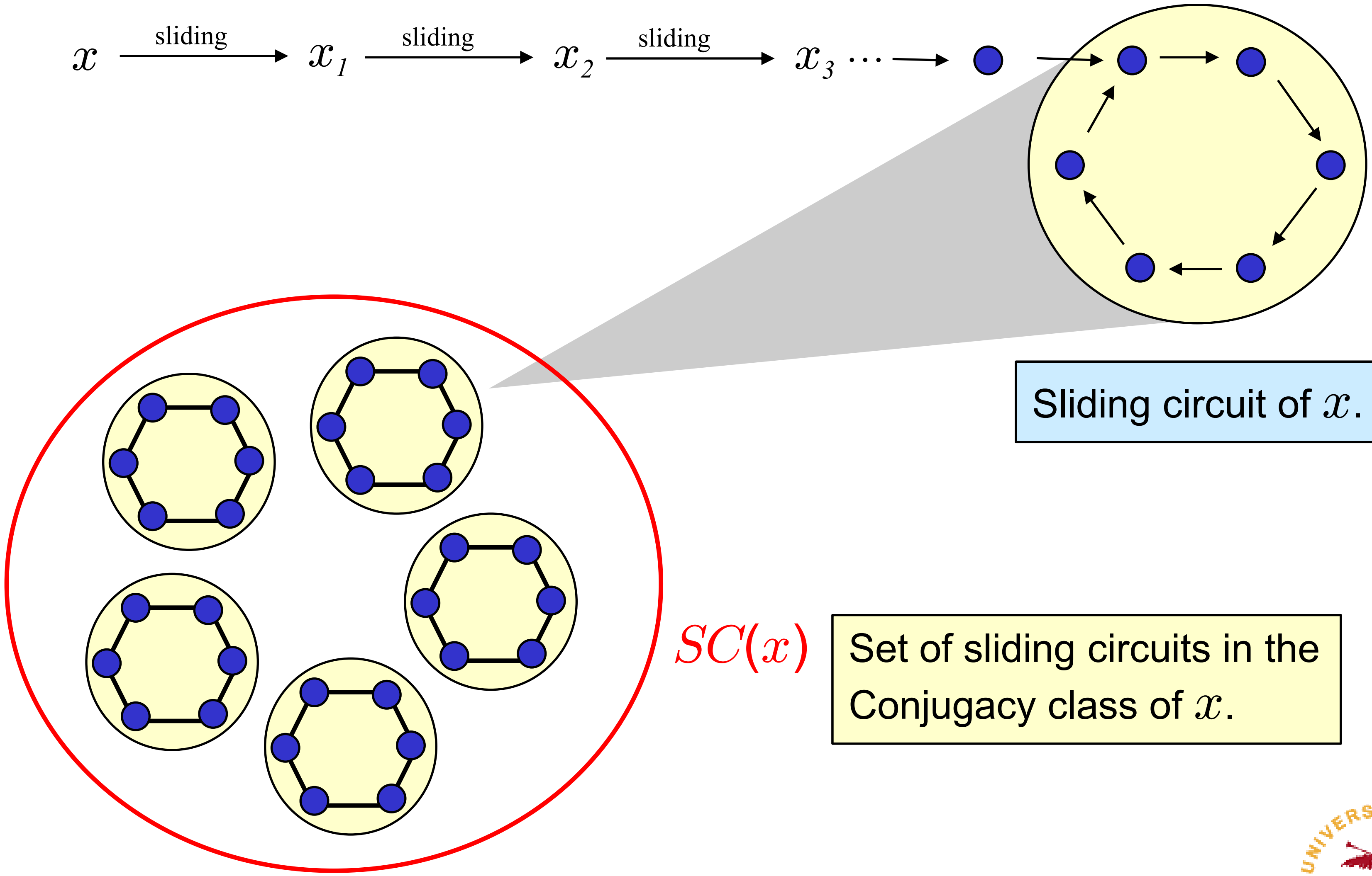
Cyclic sliding:



The resulting braid is not
In left normal form

We compute its left normal form, and it may become simpler

Iterate…



Sliding circuit of x.

Set of sliding circuits in the Conjugacy class of x.

Two braids x and y are conjugate $\Leftrightarrow$ $SC(x) = SC(y)$

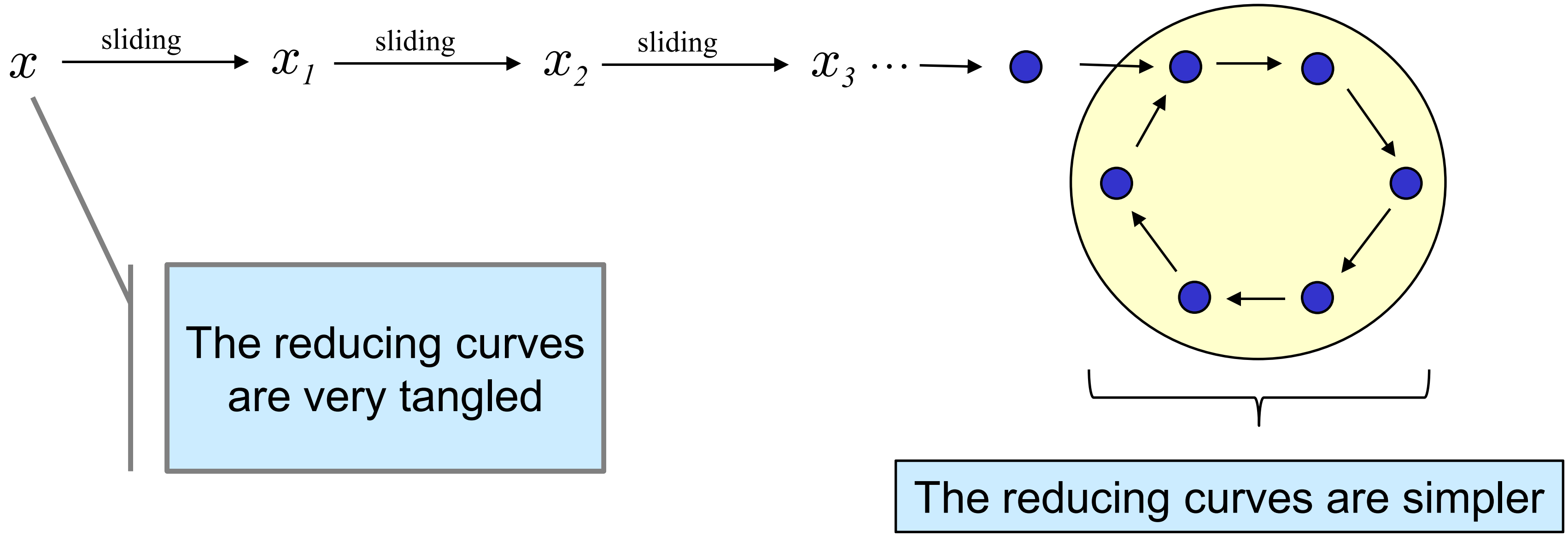
We solve the conjugacy problem by computing these sets.

Remark: Cyclic sliding simplifies braids algebraically…

…but also geometrically!

Simplifying reduction curves

Suppose that x is reducible.



In general, either round or almost-round!

Simplifying reduction curves

Theorem: (GM-Wiest, 2009) Suppose that $y \in B_n$ belongs to a sliding circuit and so do $y^2, y^3, \ldots, y^m$, where $m = (n(n-1)/2)^3$.
Then y admits a reduction curve which is either round or almost-round.

Ideas of the proof:

1) Reduction curves of a braid are preserved by powers.

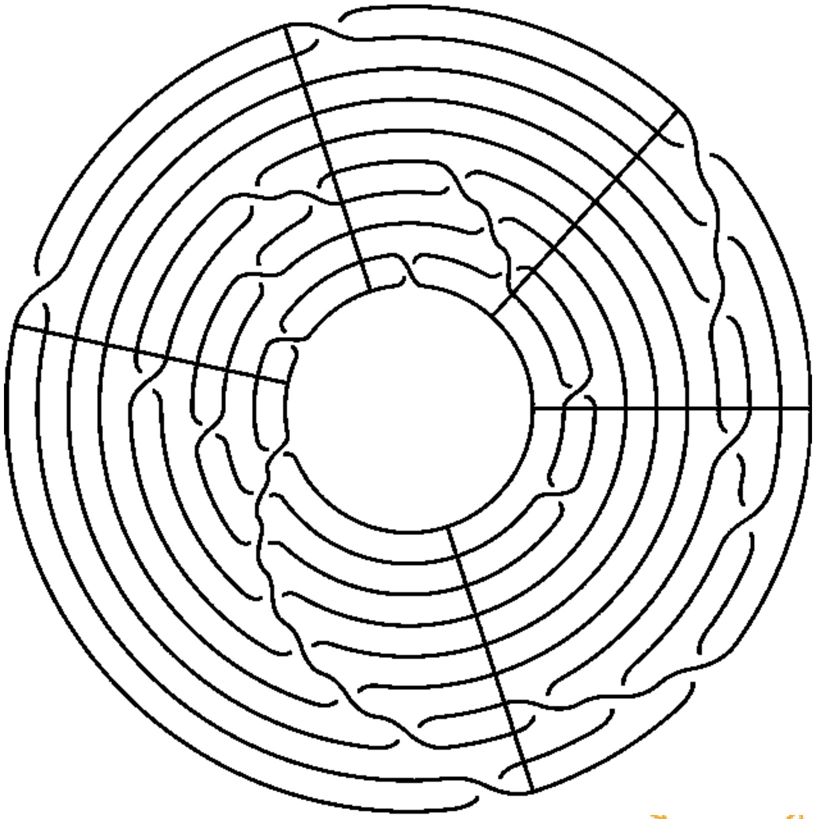
2) Some small power of y has the right property, provided it belongs to a sliding circuit.

Simplifying reduction curves

Suppose that y is **rigid**.

This means that it is in left normal form, even if considered *around a circle.*

Then, the braid inside an innermost tube must be either trivial or pseudo-Anosov.



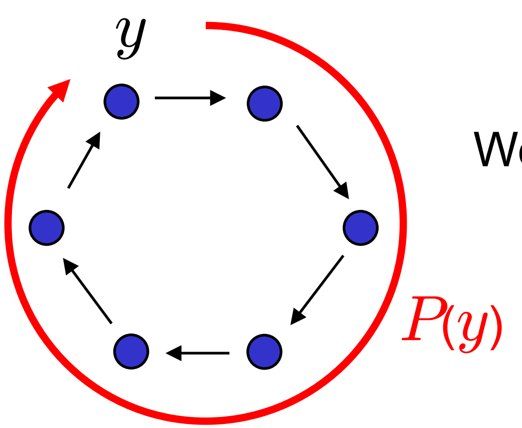
Simplifying reduction curves

Proposition: (GM-Wiest, 2009) If a rigid braid has a pseudo-Anosov interior braid, the corresponding reduction curve is round.

Proposition: (GM-Wiest, 2009) If a braid has a trivial interior braid, the corresponding reduction curve is either round or almost-round.

The rigid case is solved!

Simplifying reduction curves



We investigate the conjugating element along the circuit.

Up to replacing y by a small power, $P(y)$ is **rigid**.

The reduction curves of $P(y)$ are reduction curves of y.

We are done!?

Unless $P(y)$ has no reduction curves. But then it is trivial. $\Rightarrow$ y is rigid.

Conclusion

Cyclic sliding simplifies left normal forms, and also reduction curves.

This provides an algorithm to determine the geometric type of the braid, and to find the reducing curves.

This algorithm has polynomial complexity if this distance has a polynomial bound.