ON THE MILNOR FIBRATION OF MIXED FUNCTIONS

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Happy birthday, Anatoly !!

1. Introduction

Let $f(\mathbf{z})$ be a holomorphic function of n-variables $z_1, \dots, z_n$ such that $f(\mathbf{0}) = 0$.

J. Milnor proved $f/|f| : S_\varepsilon^{2n-1} \setminus K_\varepsilon \to S^1$ is a locally trivial fibration for any positive ε with $\varepsilon \le \varepsilon_0$ where $K_\varepsilon = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ ([12]).

Our situation: links coming from a pair of real-valued real analytic functions

$$V = \{g(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) = 0,\ K_\varepsilon = V \cap S_\varepsilon$$

$$f(\mathbf{x}, \mathbf{y}) := g(\mathbf{x}, \mathbf{y}) + i\,h(\mathbf{x}, \mathbf{y}) : \mathbb{R}^{2n} \to \mathbb{C}$$

When $f/|f| : S_\varepsilon^{2n-1} \setminus K_\varepsilon \to S^1$ to be a fibration?

The difficulty is that for an arbitrary choice of g, h, it is usually not a fibration. A breakthrough is given by the work of Ruas, Seade and Verjovsky [20]. After this work, many examples of pairs $\{g, h\}$ which give real Milnor fibrations have been investigated. However in most papers, certain restricted types of functions are mainly considered ([5, 6, 22, 19, 11, 18, 3]).

We consider a complex valued analytic function f expanded in a convergent power series of variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} = (\bar z_1, \dots, \bar z_n)$

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\, \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$$

where $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar z_1^{\mu_1} \cdots \bar z_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$) as usual. Here $\bar z_j$ is the complex conjugate of z_j. We call $f(\mathbf{z}, \bar{\mathbf{z}})$ a mixed analytic function (or a mixed polynomial, if $f(\mathbf{z}, \bar{\mathbf{z}})$ is a polynomial) of $z_1, \dots, z_n$. We are interested in the topology of the hypersurface $V = \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$, which we call a mixed hypersurface.

This approach is equivalent to the original one.

$\mathbf{z} = \mathbf{x} + i\,\mathbf{y}$ with $z_j = x_j + i\,y_j$ $j = 1, \dots, n$, $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$,

$$f(\mathbf{z}, \bar{\mathbf{z}}) \mapsto f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}) + i\,h(\mathbf{x}, \mathbf{y}), \quad g := \Re f,\ h := \Im f$$

Conversely, for a given real analytic variety $W = \{g(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) = 0\}$ which is defined by two real-valued analytic functions g, h, we can consider W as a mixed hypersurface by introducing a mixed function $f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ where

$$f(\mathbf{z}, \bar{\mathbf{z}}) := g(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2\,i}) + i\,h(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2\,i}).$$

The advantage of our view point is that we can use rich techniques of complex hypersurface singularities.

2. Newton boundary and non-degeneracy of mixed functions

2.1. Polar weighted homogeneous polynomials.

2.1.1. *Radial degree and polar degree.* Let $M = \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ be a mixed monomial where $\nu = (\nu_1, \dots, \nu_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and let $P = {}^t(p_1, \dots, p_n)$ be a weight vector. We define the radial degree of M, $\mathrm{rdeg}_P\, M$ and the polar degree of M, $\mathrm{pdeg}_P\, M$ with respect to P by

$$\mathrm{rdeg}_P\, M = \sum_{j=1}^n p_j(\nu_j + \mu_j), \quad \mathrm{pdeg}_P\, M = \sum_{j=1}^n p_j(\nu_j - \mu_j).$$

2.1.2. *Weighted homogeneous polynomials.* I.Recall that a complex polynomial $h(\mathbf{z})$ is called a weighted homogeneous polynomial with weights $P = {}^t(p_1, \dots, p_n)$ if $p_1, \dots, p_n$ are integers and there exists a positive integer d so that

$$f(t^{p_1} z_1, \dots, t^{p_n} z_n) = t^d f(\mathbf{z}), \ t \in \mathbb{C}.$$

The integer d is called the degree of f with respect to the weight vector P.

II. A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^\ell c_i\, \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$ is called a radially weighted homogeneous polynomial if there exist integers $q_1, \dots, q_n \geq 0$ and $d_r > 0$ such that it satisfies the equality:

$$f(t^{q_1} z_1, \dots, t^{q_n} z_n, t^{q_1} \bar{z}_1, \dots, t^{q_n} \bar{z}_n) = t^{d_r} f(\mathbf{z}, \bar{\mathbf{z}}), \quad t \in \mathbb{R}^*.$$

Def. A polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called a polar weighted homogeneous polynomial if there exists a weight vector $(p_1, \dots, p_n)$ and a non-zero integer d_p such that

$$f(\lambda^{p_1} z_1, \dots, \lambda^{p_n} z_n, \bar{\lambda}^{p_1} \bar{z}_1, \dots, \bar{\lambda}^{p_n} \bar{z}_n) = \lambda^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}),\ \lambda \in \mathbb{C}^*,\ |\lambda| = 1$$

where $\gcd(p_1, \dots, p_n) = 1$. This is equivalent to

$$\mathrm{pdeg}_{\,P}\, \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i} = d_p, \quad i = 1, \dots, \ell.$$

Here the weight p_i can be zero or a negative integer. The weight vector $(p_1, \dots, p_n)$ is called the polar weights and d_p is called the polar degree respectively. This notion was first introduced by Ruas-Seade-Verjovsky [20] and Cisneros-Molina [4].

Recall that the radial weights and polar weights define$\mathbb{R}^*$-action and S^1-action on $\mathbb{C}^n$ respectively by

$$t \circ \mathbf{z} = (t^{q_1} z_1, \dots, t^{q_n} z_n), \quad t \circ \bar{\mathbf{z}} = (t^{q_1} \bar{z}_1, \dots, t^{q_n} \bar{z}_n), \quad t \in \mathbb{R}^*$$
$$\lambda \circ \mathbf{z} = (\lambda^{p_1} z_1, \dots, \lambda^{p_n} z_n), \quad \lambda \circ \bar{\mathbf{z}} = \overline{\lambda \circ \mathbf{z}}, \quad \lambda \in S^1 \subset \mathbb{C}$$

In other words, this is an $\mathbb{R}^* \times S^1$ action on $\mathbb{C}^n$.

Lemma 1. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a radially weighted homogeneous polynomial, $V = \{\mathbf{z} \in \mathbb{C}^n \,|\, f(\mathbf{z},\bar{\mathbf{z}}) = 0\}$ and $V^* = V \cap \mathbb{C}^{*n}$. Assume that $V \setminus \{O\}$ (respectively V^*) is smooth and* $\mathrm{codim}_{\mathbb{R}}\mathrm{V} = 2$. *If the radial weight vector is strictly positive, namely $q_j > 0$ for any $j = 1, \dots, n$, the sphere S_r intersects transversely with $V \setminus \{O\}$ (resp. with V^*) for any $r > 0$.*

2.2. **Newton boundary of a mixed function.** A mixed analytic function

$$f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\,\mathbf{z}^\nu \bar{\mathbf{z}}^\mu,\ f(O) = 0, V = f^{-1}(0)$$

We call the variety $V = f^{-1}(0)$ the mixed hypersurface.

The radial Newton polygon $\Gamma_+(f;\mathbf{z},\bar{\mathbf{z}})$ (at the origin) of a mixed function $f(\mathbf{z},\bar{\mathbf{z}})$ is defined by the convex hull of

$$\Gamma_+(f;\mathbf{z},\bar{\mathbf{z}} : \text{ convex hull of } \bigcup_{c_{\nu,\mu}\neq 0} (\nu+\mu) + \mathbb{R}^{+n}$$

$$\Gamma(f) = \partial_{compact}\Gamma^+(f)$$

For a given positive integer vector $P = (p_1, \dots, p_n)$, we associate a linear function ℓ_P on $\Gamma(f)$ defined by $\ell_P(\nu) = \sum_{j=1}^n p_j \nu_j$ for $\nu \in \Gamma(f)$ and let $\Delta(P, f) = \Delta(P)$ be the face where ℓ_P takes its minimal value.

We denote the minimal value of ℓ_P by $d(P; f)$ or simply $d(P)$. Note that

$$d(P; f) = \min\,\{\mathrm{rdeg}_P\, \mathbf{z}^\nu \bar{\mathbf{z}}^\mu \,|\, c_{\nu,\mu} \neq 0\}.$$

For a positive weight P, we define the face function $f_P(\mathbf{z},\bar{\mathbf{z}})$ by

$$f_P(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu+\mu\in\Delta(P)} c_{\nu,\mu}\,\mathbf{z}^\nu \bar{\mathbf{z}}^\mu.$$

f_P is a radially weighted homogeneous polynomial !!.

Example 2. Consider a mixed function $f := z_1^3\bar{z}_1^2 + z_1^2 z_2^2 + z_2^3 \bar{z}_2$. The Newton boundary $\Gamma(f;\mathbf{z},\bar{\mathbf{z}})$ has two faces Δ_1, Δ_2 which have weight vectors $P := {}^t(2,3)$ and $Q := {}^t(1,1)$ respectively. The corresponding invariants are

$$f_P(\mathbf{z},\bar{\mathbf{z}}) = z_1^3\bar{z}_1^2 + z_1^2 z_2^2,\ d(P; f) = 10$$
$$f_Q(\mathbf{z},\bar{\mathbf{z}}) = z_1^2 z_2^2 + z_2^3 \bar{z}_2,\ d(Q; f) = 4.$$

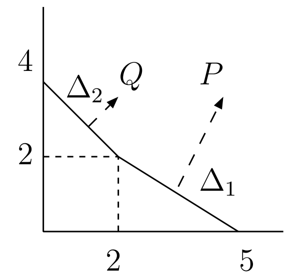

FIGURE 1. $\Gamma(f)$

Definition 3. Let P be a strictly positive weight vector. We say that $f(\mathbf{z},\bar{\mathbf{z}})$ is non-degenerate for P, if the fiber $f_P^{-1}(0)\cap\mathbb{C}^{*n}$ contains no critical point of the mapping $f_P:\mathbb{C}^{*n}\to\mathbb{C}$. In particular, $f_P^{-1}(0)\cap\mathbb{C}^{*n}$ is a smooth real codimension 2 manifold or an empty set. We say that $f(\mathbf{z},\bar{\mathbf{z}})$ is strongly non-degenerate for P if the mapping $f_P:\mathbb{C}^{*n}\to\mathbb{C}$ has no critical points. If $\dim\Delta(P)\ge 1$, we further assume that $f_P:\mathbb{C}^{*n}\to\mathbb{C}$ is surjective onto $\mathbb{C}$.

A mixed function $f(\mathbf{z},\bar{\mathbf{z}})$ is called non-degenerate (respectively strongly non-degenerate) if f is non-degenerate (resp. strongly non-degenerate) for any strictly positive weight vector P.

Consider the function $f(\mathbf{z},\bar{\mathbf{z}})=z_1\bar z_1+\cdots+z_n\bar z_n$. Then $V=f^{-1}(0)$ is a single point $\{O\}$. By the above definition, f is a non-degenerate mixed function.

To avoid such an unpleasant situation, we say that a mixed function $g(\mathbf{z},\bar{\mathbf{z}})$ is a true non-degenerate function if it satisfies further the non-emptiness condition:

(NE) : For any $P\in N^{++}$ with $\dim\Delta(P,g)\ge 1$, the fiber $g_P^{-1}(0)\cap\mathbb{C}^{*n}$ is non-empty.

Example 4. I. Consider the mixed function $f:=z_1^3\bar z_1^2+z_1^2z_2^2+z_2^3\bar z_2$ which we have considered in Example 2. Then f is strongly non-degenerate for each of the weight vectors $P={}^t(2,3)$, $Q={}^t(1,1)$.

II. Consider a mixed function

$$g(\mathbf{z},\bar{\mathbf{z}})=z_1\bar z_1+\cdots+z_r\bar z_r-(z_{r+1}\bar z_{r+1}+\cdots+z_n\bar z_n),\quad 1\le r\le n-1.$$

Then $V=g^{-1}(0)$ is a smooth real codimension one variety and thus it is degenerate for $P={}^t(1,1,\dots,1)$.

III. Consider a mixed function

$$f(\mathbf{z},\bar{\mathbf{z}})=z_1^2+az_1\bar{z}_2+\bar{z}_2^2,\quad a\in\mathbb{C}.$$

Then f is non-degenerate if and only if $a\neq\pm2$.

IV. non-degenerate but not strongly non-degenerate mixed function

$$f(\mathbf{z},\bar{\mathbf{z}})=1/4\,{z_1}^2-1/4\,\bar{z}_1^2+z_1\,\bar{z}_1-(1+i)\,(z_1+z_2)\,(\bar{z}_1+\bar{z}_2)$$

For a complex valued mixed function $f(\mathbf{z},\bar{\mathbf{z}})$, we use the notation ([17]):

$$df(\mathbf{z},\bar{\mathbf{z}})=(\tfrac{\partial f}{\partial z_1},\dots,\tfrac{\partial f}{\partial z_n})\in\mathbb{C}^n,\quad \bar{d}f(\mathbf{z},\bar{\mathbf{z}})=(\tfrac{\partial f}{\partial \bar{z}_1},\dots,\tfrac{\partial f}{\partial \bar{z}_n})\in\mathbb{C}^n$$

We use freely the following convenient criterion for a given point to be a critical point

Proposition 5. (Proposition 1, [17]) *The following two conditions are equivalent. Let $\mathbf{w}\in\mathbb{C}^n$.*

1. (1) $\mathbf{w}$ *is a critical point of* $f:\mathbb{C}^n\to\mathbb{C}$.
2. (2) *There exists a complex number* α *with* $|\alpha|=1$ *such that* $\overline{df(\mathbf{w},\bar{\mathbf{w}})}=\alpha\,\bar{d}f(\mathbf{w},\bar{\mathbf{w}})$.

Let J be a subset of $\{1,\dots,n\}$ and consider *the J-conjugation map* $\iota_J:\mathbb{C}^n\to\mathbb{C}^n$ defined by:

$$\iota_J:\ (z_1,\dots,z_n)\mapsto(w_1,\dots,w_n),\ w_j=\begin{cases}z_j & j\notin J\\ \bar{z}_j & j\in J.\end{cases}$$

Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed function.

$f(\mathbf{z},\bar{\mathbf{z}})$ is *J-conjugate holomorphic* $\iff$ $f\circ\iota_J(\mathbf{z})$:holomorphic function.

Let $M=\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a mixed monomial and let $g(\mathbf{z},\bar{\mathbf{z}})=M\cdot f(\mathbf{z},\bar{\mathbf{z}})$ where $f(\mathbf{z},\bar{\mathbf{z}})$ is a J-conjugate weighted homogeneous polynomial. We say $g(\mathbf{z},\bar{\mathbf{z}})$ is a pseudo J-conjugate weighted homogeneous polynomial if $\mathrm{pdeg}_{P'}g\neq0$ where $P'=\iota_JP$ is the polar weight vector of $f(\mathbf{z},\bar{\mathbf{z}})$. Note that $g\circ\iota_J(\mathbf{z})$ need not to be holomorphic. Further, if $J=\emptyset$, we say that g is a pseudo weighted homogeneous polynomial. Then g takes the form $M\,f(\mathbf{z})$ where f a weighted homogeneous polynomial and M is a mixed monomial.

Example 6. Let $f(\mathbf{z},\bar{\mathbf{z}})=z_1^2+\cdots+z_{n-1}^2+\bar{z}_n^3$. Then f is a J-conjugate weighted homogeneous polynomial of the weight type $(3,\dots,3,2;6)$ with $J=\{n\}$.

Definition 7. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed function. We say that f is a Newton pseudo conjugate weighted homogeneous polynomial if for any $P \in N^{++}$, there exists a subset $J(P) \subset \{1,\dots,n\}$ such that the face function $f_P(\mathbf{z},\bar{\mathbf{z}})$ is a $J(P)$-pseudo conjugate weighted homogeneous polynomial.

Example 8. I. Let $f(\mathbf{z},\bar{\mathbf{z}}) = z_1^5 + z_1^2\bar{z}_2^2 + z_2^m\bar{z}_2^2$ with $m \geq 2$. Then the Newton boundary has two faces and the corresponding weights are $P=(2,3)$ and $Q=(m,2)$. The face functions are

$$f_P(\mathbf{z},\bar{\mathbf{z}}) = z_1^2(z_1^3+\bar{z}_2^2),\; f_Q(\mathbf{z},\bar{\mathbf{z}}) = \bar{z}_2^2(z_1^2+z_2^m)$$

and f is a Newton pseudo conjugate weighted homogeneous polynomial if $m \neq 2$. Note that for $m=2$, the polar degree of $f_Q(\mathbf{z},\bar{\mathbf{z}})$ is 0.

3. Isolatedness of the singularities

Let $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$. $O \in f^{-1}(0)$. Put $V = f^{-1}(0) \subset \mathbb{C}^n$.

3.1. **Mixed singular points.** We say that $\mathbf{w} \in V$ is a mixed singular point if $\mathbf{w}$ is a critical point of the mapping $f : \mathbb{C}^n \to \mathbb{C}$. We say that V is mixed non-singular if it has no mixed singular points. If V is mixed non-singular, V is smooth variety of real codimension two.

Note that a singular point of V (as a point of a real algebraic variety) is a mixed singular point of V but the converse is not necessarily true.

3.2. **Non-vanishing coordinate subspaces.** For a subset $J \subset \{1, 2, \dots, n\}$, we consider the subspace $\mathbb{C}^J$ and the restriction $f^J := f|_{\mathbb{C}^J}$. Consider the set

$$\mathcal{NV}(f) = \{I \subset \{1, \dots, n\} \mid f^I \not\equiv 0\}.$$

We call $\mathcal{NV}(f)$ the set of non-vanishing coordinate subspaces for f. Put

$$V^\sharp = \bigcup_{I\in\mathcal{NV}(f)} V \cap \mathbb{C}^{*I}.$$

Theorem 9. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a true non-degenerate mixed function. Then there exists a positive number r_0 such that the following properties are satisfied.*

1. (Isolatedness of the singularity) *The mixed hypersurface $V^\sharp \cap B_{r_0}$ is mixed non-singular. In particular,* $\mathrm{codim}_{\mathbb{R}} V^\sharp = 2$.
2. (Transversality) *The sphere S_r with $0 < r \le r_0$ intersects $V^\sharp$ transversely.*

We say that f is k-convenient if $J \in \mathcal{NV}(f)$ for any $J \subset \{1, \dots, n\}$ with $|J| = n-k$. We say that f is convenient if f is $(n-1)$-convenient. Note that $V^\sharp = V \setminus \{O\}$ if f is convenient. For a given ℓ with $0 < \ell \le n$, we put $W(\ell) = \{\mathbf{z} \in \mathbb{C}^n \mid |I(\mathbf{z})| \le \ell\}$ where $I(\mathbf{z}) = \{i | z_i = 0\}$. Thus $W(n-1) = \mathbb{C}^{*n}$. If f is ℓ-convenient, $V \cap W(\ell) \subset V^\sharp$.

Corollary 10. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient true non-degenerate mixed polynomial. Then $V = f^{-1}(0)$ has an isolated mixed singularity at the origin.*

Remark 11. *The assumption "true" is to make sure that $V^* = f^{-1}(0) \cap \mathbb{C}^{*n}$ is non-empty.*

4. Resolution of the singularities

We consider a mixed analytic function $f(\mathbf{z},\bar{\mathbf{z}})$ and the corresponding mixed hypersurface $V = f^{-1}(0)$. We assume that $O \in V$ is an isolated mixed singularity, unless otherwise stated.

If f is complex analytic, a "resolution of f" is usually understood as a proper holomorphic mapping $\varphi : X \to \mathbb{C}^n$ so that

(i) $E := \varphi^{-1}(O)$ is a union of smooth (complex analytic) divisors which intersect transversely and $\varphi : X - E \to \mathbb{C}^n - \{O\}$ is biholomorphic,

(ii) the divisor $(\varphi^* f)$ is a union of smooth divisors intersecting transversely and we can write $(\varphi^* f) = \widehat{V} \cup E$ where $\widehat{V}$ is the strict transform of V (= the closure of $\varphi^{-1}(V - \{O\})$),

(iii) for any point $P \in E_I^* \cap \widehat{V}$ with $I = \{i_1, \dots, i_s\}$, there exists an analytic coordinate chart $(u_1, \dots, u_n)$ so that the pull-back of f is written as $U \times u_1^{m_1} \cdots u_j^{m_j}$ where U is a unit in a neighborhood of P, $E_{i_k} = \{u_k = 0\}$ $(k = 1, \dots, s-1)$ and $\widehat{V} = \{u_s = 0\}$. Here $E_I^* := \cap E_{i \in I} \setminus \cup_{j \notin I} E_j$.

For a mixed hypersurface, a resolution of this type does not exist in general. The main reason is that there is no complex structure in the tangent space of V. Nevertheless we will show that a suitable toric modification partially resolves such singularities.

4.1. Toric modification and resolution of complex analytic singularities. For the reader's convenience, we recall some basic facts about the toric modifications at the origin. We use the notations and the terminologies of [14, 15, 16] and §2.2.

4.1.1. *Toric modification.* Let $A = (a_{i,j}) \in GL(n, \mathbb{Z})$ with $\det A = \pm 1$. We call such a matrix *a unimodular matrix*. We associate to A a birational morphism

$$\psi_A : \mathbb{C}^{*n} \to \mathbb{C}^{*n}$$

which is defined by

$$\psi_A(\mathbf{z}) = (z_1^{a_{1,1}} \cdots z_n^{a_{1,n}}, \dots, z_1^{a_{n,1}} \cdots z_n^{a_{n,n}}).$$

If the coefficients of A are non-negative, ψ_A can be defined on $\mathbb{C}^n$. Note that ψ_A is a group homomorphism of the algebraic group $\mathbb{C}^{*n}$ and we have

$$\psi_A^{-1} = \psi_{A^{-1}}, \quad \psi_A \circ \psi_B = \psi_{AB}.$$

Suppose that Σ^* is a regular fan. Let $\mathcal{S}$ be the set of n-dimensional cones and let $\mathcal{V}^+$ be the set of strictly positive vertices. For simplicity, we assume that the vertices of Σ^* are the union of $\{E_1, \dots, E_n\}$ and $\mathcal{V}^+$. For each $\sigma \in \mathcal{S}$, we consider a copy of a complex Euclidean space $\mathbb{C}^n_\sigma$ with coordinates $\mathbf{u}_\sigma = (u_{\sigma 1}, \dots, u_{\sigma n})$ and the morphism $\pi_\sigma : \mathbb{C}^n_\sigma \to \mathbb{C}^n$ defined by $\pi_\sigma(\mathbf{u}_\sigma) = \psi_\sigma(\mathbf{u}_\sigma)$. Taking the disjoint sum $\amalg_{\sigma\in\mathcal{S}}\mathbb{C}^n_\sigma$, we glue together $\amalg_{\sigma\in\mathcal{S}}\mathbb{C}^n_\sigma$ under the following equivalence relation:

$$\mathbf{u}_\sigma \sim \mathbf{u}_\tau \quad \text{if} \quad \psi_{\tau^{-1}\sigma} \text{ is well-defined at } \mathbf{u}_\sigma \text{ and } \psi_{\tau^{-1}\sigma}(\mathbf{u}_\sigma) = \mathbf{u}_\tau.$$

We denote the quotient space $\amalg_{\sigma\in\mathcal{S}}\mathbb{C}^n_\sigma/\sim$ by X_{Σ^*}. Then X_{Σ^*} is a complex manifold of dimension n and the morphisms $\pi_\sigma : \mathbb{C}^n_\sigma \to \mathbb{C}^n$, $\sigma \in \mathcal{S}$ are compatible with the identification and thus they define a birational proper holomorphic mapping

$$\widehat{\pi} : X_{\Sigma^*} \to \mathbb{C}^n.$$

The restriction $\widehat{\pi}$ to $X_{\Sigma^*} \setminus \widehat{\pi}^{-1}(0)$ is a biholomorphic onto $\mathbb{C}^n \setminus \{O\}$. We call $\widehat{\pi} : X_{\Sigma^*} \to \mathbb{C}^n$ the toric modification associated with the regular fan Σ^* [14, 16]. The irreducible exceptional divisors correspond bijectively to the vertices $P \in \mathcal{V}^+$ and we denote it by $\widehat{E}(P)$. Then $\widehat{\pi}^{-1}(O) = \bigcup_{P\in\mathcal{V}^+}\widehat{E}(P)$.

The easiest non-trivial case is when $\mathcal{V}^+ = \{P = {}^t(1, \dots, 1)\}$. In this case, X_{Σ^*} is nothing but the ordinary blowing-up at the origin of $\mathbb{C}^n$.
Example

$$\Sigma_2^* = \left\{ E_1, P = \begin{pmatrix}1\\1\end{pmatrix}, Q = \begin{pmatrix}2\\3\end{pmatrix}, R = \begin{pmatrix}1\\2\end{pmatrix}, E_2 \right\}$$

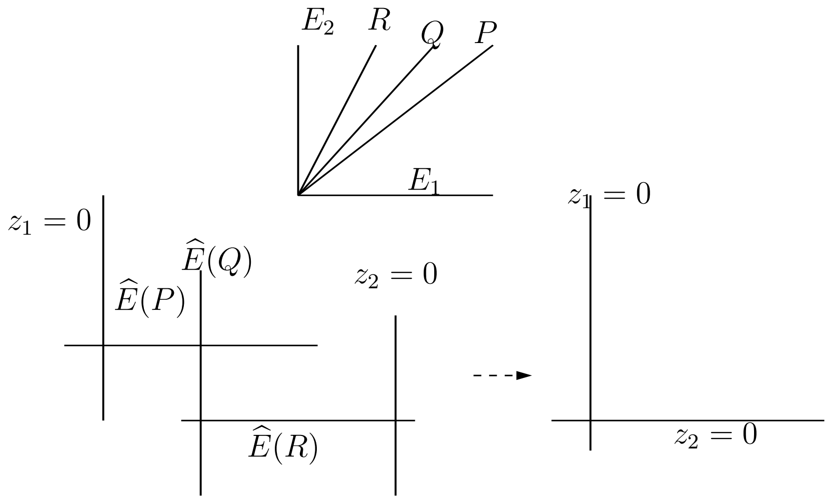

FIGURE 2. Blowing up of a cusp

4.1.2. *Dual Newton diagram and admissible toric modifications.* Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a germ of mixed function in n variables $z_1, \dots, z_n$. We introduce an equivalence relation in $N_{\mathbb{R}}^+$ by

$$P \sim Q,\ P, Q \in N_{\mathbb{R}}^+ \iff \Delta(P; f) = \Delta(Q; f).$$

The set of equivalence classes gives an open polyhedral cone subdivision of $N_{\mathbb{R}}^+$ and we denote it as $\Gamma^*(f; \mathbf{z})$ and we call it the dual Newton diagram. Let Σ^* be a regular fan which is a regular simplicial cone subdivision of $\Gamma^*(f)$. If Σ^* is a regular simplicial cone subdivision of $\Gamma^*(f)$, the toric modification $\widehat{\pi} : X_{\Sigma^*} \to \mathbb{C}^n$ is called admissible for $f(\mathbf{z}, \bar{\mathbf{z}})$. The basic fact for non-degenerate holomorphic functions is:

Theorem 12. *([14, 15, 16]) Assume that $f(\mathbf{z})$ be a non-degenerate convenient analytic function with an isolated singularity at the origin. Let $\widehat{\pi} : X_{\Sigma^*} \to \mathbb{C}^n$ be an admissible toric modification. Then it is a good resolution of the mapping $f : \mathbb{C}^n \to \mathbb{C}$ at the origin.*

4.2. **Blowing up examples.** We consider some examples.

Example 13. A. Let

$$C_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 - z_2^2 = 0\}\}$$
$$V_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid f_1(\mathbf{z}, \bar{\mathbf{z}}) = \bar{z}_1^2 - z_2^2 = 0\}$$
$$V_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid f_2(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1 - z_2^2 = 0\}$$

C_1 is a union of two smooth complex line, V_1 is a union of two smooth real planes, $\bar{z}_1 \pm z_2 = 0$ and V_2 is an irreducible variety. Consider

$$\widehat{\pi}_1 : X_1 \to \mathbb{C}^2$$

where $\widehat{\pi}_1 : X_1 \to \mathbb{C}^2$ is the toric modification associated with the regular fan generated by vertices

$$\Sigma_1^* = \left\{ E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Geometrically, $\widehat{\pi}_1$ is an ordinary blowing up. Note that for the complex curve C_1, the two components are separated by a single blowing up $\widehat{\pi}_1$. We will see what happens to the two other mixed curves V_1, V_2. In the toric coordinate $\mathbb{C}^2_\sigma$ with $\sigma = \mathrm{Cone}(\mathrm{P}, \mathrm{E}_2)$ and the toric coordinates (u_1, u_2), the strict transform $\widehat{V}_1$, $\widehat{V}_2$ of V_1, V_2 are defined in the torus $\mathbb{C}^{*2}_\sigma$ as

$$\begin{aligned}
\widehat{C}_1 \cap \mathbb{C}^{*2}_\sigma &= \{(u_1, u_2) \in \mathbb{C}^{*2}_\sigma \mid u_1^2 - u_1^2 u_2^2 = u_1^2(1 - u_2^2) = 0\} \\
\widehat{V}_1 \cap \mathbb{C}^{*2}_\sigma &= \{(u_1, u_2) \in \mathbb{C}^{*2}_\sigma \mid \bar{u}_1^2 - u_1^2 u_2^2 = 0\}, \\
\widehat{V}_2 \cap \mathbb{C}^{*2}_\sigma &= \{(u_1, u_2) \in \mathbb{C}^{*2}_\sigma \mid u_1(\bar{u}_1 - u_1 u_2^2) = 0\}.
\end{aligned}$$

The first expression shows that $\widehat{C}_1$ is already smooth and separated into two pieces. Unlike the case of holomorphic functions, we observe that

$$\{(u_1, u_2) \in \mathbb{C}^2_\sigma \mid \bar{u}_1^2 - u_1^2 u_2^2 = 0\} \supsetneq \widehat{V}_1, \ \{(u_1, u_2) \in \mathbb{C}^2_\sigma \mid \bar{u}_1 - u_1 u_2^2 = 0\} \supsetneq \widehat{V}_2$$

as $\widehat{E}(P) = \{u_1 = 0\} \not\subset \widehat{V}_i$, $i = 1, 2$. In both cases, we see that the 1-sphere $|u_2| = 1$ appears as their intersection with the exceptional divisor $\widehat{E}(P)$.

Thus $\widehat{L}_+ \cap \widehat{L}_-$ is the 1-sphere $|u_2| = 1$ and the ordinary blowing up does not separate the two smooth components.

For $\widehat{V}_2$, we will see later that it has two link components. See §6 for the definition of the link components. This illustrates the complexity of the limit set of the tangent lines in the mixed varieties.

B. We consider an ordinary cusp (complex analytic)

$$C_2 = \{z_2^2 - z_1^3 = 0\}$$
$$V_3 = \{z_2^2 - z_1^2 \bar z_1 = 0\}$$

with the same Newton boundary and an admissible toric blowing up $\widehat\pi : X_2 \to \mathbb{C}^2$ which is associated with the regular simplicial fan:

$$\Sigma_2^* = \left\{ E_1, P = \begin{pmatrix}1\\1\end{pmatrix}, Q = \begin{pmatrix}2\\3\end{pmatrix}, R = \begin{pmatrix}1\\2\end{pmatrix}, E_2 \right\}$$

Let (u_1, u_2) be the toric coordinate of $\mathbb{C}_\sigma^2$ with $\sigma = (Q, R) = \begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$. Then the pull back of the defining polynomials are defined in this coordinate chart as

$$\widehat{C}_2 \cap \mathbb{C}_\sigma^{*2} = \{(u_1, u_2) \in \mathbb{C}_\sigma^{*2} \mid u_1^6 u_2^3 (u_2 - 1) = 0\}$$
$$\widehat{V}_3 \cap \mathbb{C}_\sigma^{*2} = \{(u_1, u_2) \in \mathbb{C}_\sigma^{*2} \mid u_1^4 u_2^2 (u_1^2 u_2^2 - \bar u_1^2 \bar u 2) = 0\}.$$

Observe that $\widehat{C}_2$ is smooth and transverse to the exceptional divisor $\widehat{E}(Q) = \{u_1 = 0\}$. The strict transform $\widehat{V}_3$ is defined by $u_1^2\, u_2^2 - \bar u_1^2\, \bar u_2 = 0$ in $\mathbb{C}_\sigma^{*2}$. We see again that for $\widetilde{V}_3$, a sphere $|u_2| = 1$ appears as the intersection with the exceptional divisor. We observe that $\widehat{V}_3 \cap \widehat{E}(Q) = \{(0, u_2) \| u_2| = 1\}$.

The above examples show that the toric modification does not resolve the singularities of non-degenerate mixed hypersurfaces.

To get a good resolution of a mixed hypersurface singularity, we need to compose a toric modification with a normal real blowing up or a normal polar modification which we introduce below.

4.3. **Normal real blowing up and normal polar blowing up of $\mathbb{C}$.** Consider the complex plane with two coordinate systems $z = x + i\,y$ and $z = r\exp(i\,\theta)$. We can consider the following two modifications.

(I)

$$\iota_{\mathbb{R}} : \mathbb{C} \setminus \{O\} \to \mathbb{C} \times \mathbb{RP}^1$$

defined by $z = x + i\,y \mapsto (z, [x : y])$ and let $\mathcal{R}\mathbb{C}$ be the closure of the image of $\iota_{\mathbb{R}}$. This is called the real blowing up. $\mathcal{R}\mathbb{C}$ is a real two dimensional manifold which has two coordinate charts $(U_0, (\tilde x, t))$ and $(U_1, (s, \tilde y))$. These coordinates are defined by $\tilde x = x, t = y/x$ and $\tilde y = y, s = x/y$. The canonical projection $\omega_{\mathbb{R}} : \mathcal{R}\mathbb{C} \to \mathbb{C}$ is given as $\omega_{\mathbb{R}}(\tilde x, t) = \tilde x(1 + i\,t)$ and $\omega_{\mathbb{R}}(s, \tilde y) = \tilde y(s + i)$. Note that $\omega_{\mathbb{R}}^{-1}(O) = \mathbb{RP}^1$ and $\omega_{\mathbb{R}} : \mathcal{R}\mathbb{C} \setminus \{O\} \times \mathbb{RP}^1 \to \mathbb{C} \setminus \{O\}$ is diffeomorphism.

(II) Consider the polar embedding

$$\iota_p : \mathbb{C} \setminus \{O\} \to \mathbb{R}^+ \times S^1$$

which is defined by $\iota_p(r\exp(\theta\, i)) = (r, \exp(\theta\, i))$. Here $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$. Let $\mathcal{P}\mathbb{C} = \mathbb{R}^+ \times S^1$ and $\omega_p : \mathcal{P}\mathbb{C} \to \mathbb{C}$ be the projection defined by $\omega_p(r, \exp(\theta\, i)) = r\exp(\theta\, i)$. We can see easily that $\omega_p^{-1}(O) = \{0\} \times S^1$ and $\omega_p : \mathcal{P}\mathbb{C} \setminus \{0\} \times S^1 \to \mathbb{C} \setminus \{O\}$ is a diffeomorphism. Note that $\mathcal{P}\mathbb{C}$ is a manifold with boundary.

4.3.1. *Canonical factorization.* There exists a canonical mapping $\psi : \mathcal{P}X \to \mathcal{R}\mathbb{C}$ which is defined by

$$\psi(r, \exp(\theta\, i)) = \begin{cases} (\tilde{x}, t) = (r\cos\theta, \tan\theta), & \theta \neq \pm\frac{\pi}{2} \\ (s, \tilde{y}) = (\cot\theta, r\sin\theta), & \theta \neq 0, \pi \end{cases}$$

It is obvious that ψ gives the commutative diagram

$$\begin{array}{ccc} \mathcal{P}\mathbb{C} & \xrightarrow{\psi} & \mathcal{R}\mathbb{C} \\ \downarrow \omega_p & & \downarrow \omega_{\mathbb{R}} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

Note that the restriction of ψ over the exceptional sets is a $2:1$ map:

$$\psi : \{O\} \times S^1 \to \{O\} \times \mathbb{RP}^1, \quad \exp(\theta\, i) \mapsto [\cos(\theta) : \sin(\theta)]$$

4.4. **Resolution of a mixed function.** Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function and let $V = f^{-1}(0)$ and we assume that V has an isolated mixed singularity at the origin.

Let Y be a real analytic manifold of dimension $2n$ and let $\Phi : Y \to \mathbb{C}^n$ be a proper real analytic mapping. We say that $\Phi : Y \to \mathbb{C}^n$ is a resolution of a real type (respectively a resolution of a polar type) of the mixed function f if

(1) Let $E = \Phi^{-1}(O)$ and let $E = E_1 \cup \cdots \cup E_r$ be the irreducible components. Each E_j is a real codimension one smooth subvariety.
(2) Y is a real analytic manifold of dimension $2n$. For a resolution of a real type, Y has no boundary while for a resolution of a polar type Y is a real analytic manifold with boundary and $\partial Y = E$.
(3) The restriction $\Phi : Y - E \to \mathbb{C}^n \setminus \{O\}$ is a real analytic diffeomorphism.
(4) Let $\widetilde{V}$ be the strict transform of V (=the closure of $\Phi^{-1}(V \setminus \{O\})$). Then $\widetilde{V}$ is a smooth manifold of real codimension 2 in an open neighborhood of E.

(5) For $I = \{i_1, \dots, i_t\}$, put $E_I^* := \bigcap_{k=1}^{t} E_{i_k} \setminus \bigcup_{j \notin I} E_j$. For $P \in E_I^* \cap \widetilde{V}$, there exists a local real analytic coordinate system $(U, (u_1, \dots, u_{2n}))$ centered at P such that

$$\Phi^* f(\mathbf{u}) = u_1^{m_1} \cdots u_t^{m_t}(u_{t+1} + i\, u_{t+2})$$

so that $U \cap E_{i_j} = \{u_j = 0\}$ for $j = 1, \dots, t$ and $U \cap \widetilde{V} = \{u_{t+1} + i\, u_{t+2} = 0\}$. In the case of a resolution of a polar type, we assume also that $Y \cap U = \{u_1 \geq 0, \dots, u_t \geq 0\}$.

For example, assume that $t = 1$ for simplicity. Then the condition (5) says the following. If we are considering a resolution of a real type,

$$U \cong \mathbb{R}^{2n} \text{ or } B^{2n},\ E_{i_1} = \{u_1 = 0\}, \quad \Phi^* f(\mathbf{u}) = u_1^{m_1}(u_2 + i\, u_3),$$

if we are considering a resolution of polar type,

$$U \cong \mathbb{R}^{2n} \cap \{u_1 \geq 0\},\ E_{i_1} = \{u_1 = 0\},\ \Phi^* f(\mathbf{u}) = u_1^{m_1}(u_2 + i\, u_3).$$

4.4.1. *Normal real blowing up.* Let X be a complex manifold of dimension n with a finite number of smooth complex divisors $E_1, \dots, E_\ell$ such that the union of divisors $E = \bigcup_{i=1}^{\ell} E_i$ has at most normal crossing singularities. Then we can consider the composite of real modifications for the normal complex 1-dimensional subspaces along the divisor $E_1, \dots, E_\ell$. Put it as $\omega_{\mathbb{R}} : \mathcal{R}X \to X$ and we call it the normal real blowing up along E. It is immediate from the definition that

(1) $\mathcal{R}X$ is a differentiable manifold and $\omega_{\mathbb{R}} : \mathcal{R}X \setminus \omega_{\mathbb{R}}^{-1}(E) \to Y \setminus E$ is a diffeomorphism.
(2) Inverse image $\widetilde{E}_j := \omega_{\mathbb{R}}^{-1}(E_j)$ of E_j is a real codimension 1 variety which is fibered over E_j' with a fiber S^1. Here E_j' is the normal real blowing up of E_j along $\bigcup_{i \neq j} E_i \cap E_j$. Putting $E_I^* := \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j$, $\widetilde{E}_I^* := \omega_{\mathbb{R}}^{-1}(E_I^*)$ is fibered over E_I^* with fiber $(S^1)^k$ where $k = |I|$.

Take a point $P \in E_1^*$ and choose a local coordinate $(W, (u_1, \dots, u_n))$ $E_1 = \{u_1 = 0\}$. Then $\omega_{\mathbb{R}}^{-1}(W)$ is isomorphic to $(\mathcal{R}\mathbb{C}) \times \mathbb{C}^{n-1}$ covered by 2 coordinates $W_{\varepsilon_1} = U_{\varepsilon_1} \times \mathbb{C}^{n-m1}$ where $\varepsilon_j = 0$ or 1. For example, W_1 has the coordinates (as a real analytic manifold) $(s_1, \tilde{y}_1, u_2, \dots, u_n)$ so that the projection to the coordinate chart $\mathbf{u} \in W$ is given by

$$u_1 = \tilde{y}_1(s_1 + i)$$

4.4.2. *Normal polar blowing up.* We can also consider the composite of the polar blowing ups along exceptional divisors, which we denote as

$\omega_p : \mathcal{P}X \to X$. In the same coordinate chart $(W, \mathbf{u})$, $\mathbf{u} = (u_1, \dots, u_n)$ as in the previous discussion, $\omega_p^{-1}(W)$ is written as

$$\omega_p^{-1}(W) = (\mathbb{R}^+ \times S^1) \times \mathbb{C}^{n-1}$$

with coordinates $(r_1, \exp(i\,\theta_1), u_2, \dots, u_n)$ and the projection is given by

$$\begin{aligned}&(r_1, \exp(i\,\theta_1), u_2, \dots, u_n) \mapsto (u_1, \dots, u_n),\\ &u_1 = r_1 \exp(i\,\theta_1)\end{aligned}$$

Note that $\mathcal{P}X$ is a manifold with boundary and $\omega_p^{-1}(E_1)$ is the boundary component which is given by $\{r_1 = 0\}$.

4.5. **A resolution of a real type and a resolution of a polar type.** Assume that $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ is a non-degenerate convenient mixed function and consider the mixed hypersurface $V = f^{-1}(0)$.

Step 1. Let $\Gamma(f)$ be the Newton boundary and let $\Gamma^*(f)$ be the dual Newton diagram. Take a regular simplicial cone subdivision in the sense of [16] and let $\widehat{\pi} : X \to \mathbb{C}^n$ be the associated toric modification. Let $\mathcal{V}^+$ be the set of strictly positive vertices of Σ^* and let $\widehat{E}(P), P \in \mathcal{V}^+$ be the exceptional divisors. Put $\widehat{E} = \bigcup_{P\in\mathcal{P}} \widehat{E}(P)$.

Step2. Then we take the normal real blowing-ups $\omega_{\mathbb{R}} : \mathcal{R}X \to X$ along the exceptional divisors of $\widehat{E}$. Then we consider the composite

$$\Phi := \widehat{\pi} \circ \omega_{\mathbb{R}} : \mathcal{R}X \xrightarrow{\omega_{\mathbb{R}}} X \xrightarrow{\widehat{\pi}} \mathbb{C}^n, \quad \xi \mapsto \widehat{\pi}(\omega_{\mathbb{R}}(\xi)).$$

Put $\widetilde{E}(P) := \omega_{\mathbb{R}}^{-1}(\widehat{E}(P))$ with $P \in \mathcal{V}^+$.

Theorem 14. $\Phi := \widehat{\pi} \circ \omega_{\mathbb{R}} : \mathcal{R}X \xrightarrow{\omega_{\mathbb{R}}} X \xrightarrow{\widehat{\pi}} \mathbb{C}^n$ *gives a good resolution of a real type of f at the origin and the exceptional divisors are $\widetilde{E}(P)$ for $P \in \mathcal{V}^+$. The multiplicity of $\widetilde{E}(P)$ of the function $\Phi^* f$ along $\widetilde{E}(P)$ is $d(P; f)$.*

Let $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ be the decomposition of f into the real and the imaginary part. Then the above assertion for the multiplicity is equivalent to: the mutiplicities of $\Phi^* g$, $\Phi^* h$ along $\widetilde{E}(P)$ are the same and equal to $d(P; f)$.

We can also use the normal polar blowing-up $\omega_p : \mathcal{P}X \to X$ along $\widehat{E}(P)$, $P \in \mathcal{V}^+$ and the composite $\Phi_p : \mathcal{P}X \to \mathbb{C}^n$. Put $\widetilde{E}(P) := \Phi_p^{-1}(\widehat{E}(P))$, $P \in \mathcal{V}^+$.

Theorem 15. *Under the same assumption as in Theorem 14, $\Phi_p : \mathcal{P}X \to X$ gives a good resolution of a polar type of $f(\mathbf{z}, \bar{\mathbf{z}})$ where Φ_p is the composite*

$$\Phi_p : \mathcal{P}X \xrightarrow{\omega_p} X \xrightarrow{\widehat{\pi}} \mathbb{C}^n.$$

The multiplicity of $\widetilde{E}(P)$ of the function $\Phi_p^* f$ along $\widetilde{E}(P)$ is $d(P;f)$. There is a canonical factorization $\eta : \mathcal{P}X \to \mathcal{R}X$ so that $\omega_p = \omega_{\mathbb{R}} \circ \eta$ and $\Phi_p = \Phi \circ \eta$.

Example 16. We consider two modifications:

$$\widehat{\pi}_1 : X_1 \to \mathbb{C}^2, \quad \widehat{\pi}_2 : X_2 \to \mathbb{C}^2$$

where $\widehat{\pi}_j : X_j \to \mathbb{C}^2$ is the toric modification associated with the regular fan Σ_j^* $(j = 1, 2)$ which are defined by the vertices as follows.

$$\Sigma_1^* = \left\{ E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$
$$\Sigma_2^* = \left\{ E_1, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, R = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, E_2 \right\}$$

1. Let $V_1 = f(\mathbf{z}, \bar{\mathbf{z}}) = \bar{z}_1^2 - z_2^2 = 0$. This is a union of two smooth real planes $z_2 \pm \bar{z}_1 = 0$. In the toric coordinate chart $\mathbb{C}_\sigma^2$ with $\sigma = \mathrm{Cone(P, E_2)}$, the strict transform $\widetilde{V_1}$ of V_1 is defined in $\mathbb{C}_\sigma^{*2}$ by

$$\widehat{V}_1 : \quad \bar{u}_1^2 - u_1^2 u_2^2 = 0.$$

We have seen that $\widehat{V}_1 \cap \widehat{E}(P) = \{u_1 = 0 \,|\, |u_2| = 1\}$. Now take the normal real blowing up along $\widehat{E}(P)$, $\omega_{\mathbb{R}} : \mathcal{R}X \to X$. The strict transform is defined in $(\mathbb{C}_\sigma^2)_\varepsilon$ as

$$\begin{aligned} \widehat{V}_1 \ &= \{(\tilde{x}_1, t_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, (1 - i\,t_1)^2 - (1 + i\,t_1)^2 u_2^2 = 0\} \\ &= \{(\tilde{y}_1, s_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, (s_1 - i)^2 - (s_1 + i)^2 u_2^2 = 0\} \end{aligned}$$

Note these equations give two smooth components L_ε, $\varepsilon = \pm 1$ which are disjoint:

$$\{(\tilde{x}_1, t_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, (1 - i\,t_1) \pm (1 + i\,t_1) u_2 = 0\}.$$

This expression shows that the strict transform is embedded in the cylinder $|u_2| = 1$. Let us see this in a normal polar modification $\omega_p : \mathcal{P}X \to X$. Now $\mathcal{P}X$ is locally diffeomorphic to the product of $S^1 \times \mathbb{R}^+ \times \mathbb{C}$ and the strict transform is now defined in a simple equation

$$\tilde{V}_1 = \{(r_1, \exp(\theta\, i), u_2) \,|\, u_2 = \mp \exp(-2\,\theta\, i)\}$$

and it has two link components.

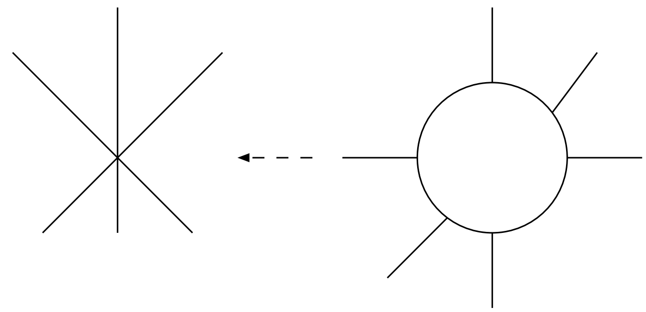

FIGURE 3. Polar modification and Half Real lines

This shows that the strict transform is a product (it does not depend on r_1) and for a fixed r_1, they are parallel torus knots in $S^1 \times S^1 = S^1 \times \{|u_2| = 1\}$. Observe that the direction of twisting is opposite in the first and the second S^1's with respect to the canonical orientation of S^1.

2. Let us consider another mixed curve:

$$V_2 : \{z_1 \bar{z}_1 - z_2^2 = 0\}$$

Equivalently V_2 is defined by

$$\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \,|\, x_1^2 + y_1^2 = x_2^2 - y_2^2,\, x_2 y_2 = 0\}.$$

This can be defined as

$$V_2 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \,|\, y_2 = 0,\, x_2^2 = x_1^2 + y_1^2\}.$$

This curve is real analytically (or real algebraically) irreducible at the origin (see [2] for the definition) but we can see that $V_2 \setminus \{O\}$ has two connected components $z_2 = |z_1|$ and $z_2 = -|z_1|$. Thus for the geometrical study of real analytic varieties, especially for the study of real analytic curves, it is better to see the connected components of $f^{-1}(0) \setminus \{O\}$. We apply the same toric modification $\widehat{\pi}_1$ and we consider its strict transform on the toric chart $\mathrm{Cone}(\mathrm{P}, \mathrm{E}_2)$ where we use the same notation as in Example 16.

$$\widehat{V}_2 : \bar{u}_1 - u_1 u_2^2 = 0.$$

Again we see that $\widehat{V}_2 \cap \widehat{E}(P) = \{(0, u_2) \,|\, |u_2| = 1\}$. Take the normal real blowing up along $\widehat{E}(P)$. The strict transform is defined in $(\mathbb{C}^2_\sigma)_\varepsilon$

as

$$\begin{aligned}\widetilde{V}_2 &= \{(\tilde{x}_1, t_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, (1 - i\,t_1) - (1 + i\,t_1)u_2^2 = 0\} \\ &= \{(\tilde{y}_1, s_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, (s_1 - i) - (s_1 + i)u_2^2 = 0\}\end{aligned}$$

which is non-singular. They have two real analytic components:

$$\{(\tilde{x}_1, t_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, u_2 \pm (1 - i\,t_1)/\sqrt{1 + t_1^2} = 0\} \quad \text{or}$$
$$\{(\tilde{y}_1, s_1, u_2) \in \mathbb{R}^2 \times \mathbb{C} \,|\, u_2 \pm (s_1 - i)/\sqrt{s_1^2 + 1} = 0\}$$

Note that $\sqrt{1 + t_1^2}$ is a real analytic function, although $\sqrt{x_1^2 + y_1^2}$ is not an analytic function at O. The above expression says that $\widetilde{V}_2$ is a product

$$\left\{(t_1, u_2) | \sqrt{1 + t_1^2}\, u_2 \pm (1 - i\,t_1) = 0\right\} \times \mathbb{R}$$

where the second factor is the line with coordinate $\tilde{x}_1$. Using the resolution of a polar type, $\widetilde{V}_2$ is simply written as

$$\widetilde{V}_2 = \{(r_1, \theta_1, u_2) \in \mathbb{R}^+ \times S_1 \times \mathbb{C} \,|\, u_2 \pm \exp(-\,\theta_1\, i) = 0\}.$$

Again we observe that it is a product of torus knots and $\mathbb{R}^+$.

5. Milnor fibration

In this section, we study the Milnor fibration, assuming that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. We have seen in Theorem 9 that there exists a positive number r_0 such that $V = f^{-1}(0)$ is mixed non-singular except at the origin in the ball $B_{r_0}^{2n}$ and the sphere S_r^{2n-1} intersects transversely with V for any $0 < r \le r_0$. The following is a key assertion for which we need the strong non-degeneracy.

Lemma 17. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. For any fixed positive number r_1 with $r_1 \le r_0$, there exists positive numbers $\delta_0 \ll r_1$ such that for any $\eta \ne 0$, $|\eta| \le \delta_0$ and r with $r_1 \le r \le r_0$,* (a) *the fiber $V_\eta := f^{-1}(\eta)$ has no mixed singularity inside the ball $B_{r_0}^{2n}$ and* (b) *the intersection $V_\eta \cap S_r^{2n-1}$ is transverse and smooth.*

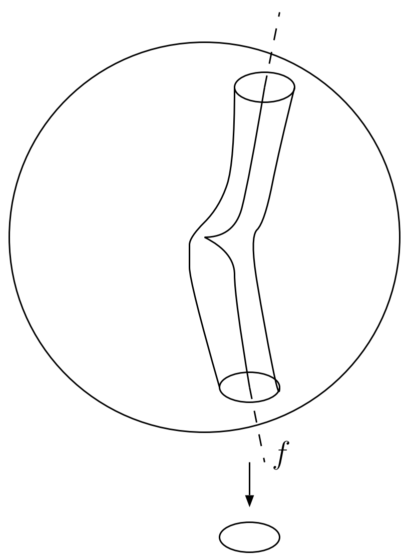

Figure 4. Second Milnor fibering

5.1. Milnor fibration, the second description.

Put

$$D(\delta_0)^* = \{\eta \in \mathbb{C} \mid 0 < |\eta| \le \delta_0\},\ S_{\delta_0}^1 = \partial D(\delta_0)^* = \{\eta \in \mathbb{C} \mid |\eta| = \delta_0\}$$
$$E(r, \delta_0)^* = f^{-1}(D(\delta_0)^*) \cap B_r^{2n},\ \partial E(r, \delta_0)^* = f^{-1}(S_{\delta_0}^1) \cap B_r^{2n}.$$

By Lemma 17 and the theorem of Ehresman ([24]), we obtain the following description of the Milnor fibration of the second type ([8]).

Theorem 18. (The second description of the Milnor fibration) *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient, strongly non-degenerate mixed function. Take positive numbers r_0, r_1 and δ_0 such that $r \le r_0$ and $\delta_0 \ll r_1$ as in Lemma 17. Then*

$$\begin{array}{ccc} E(r,\delta_0)^* & \xrightarrow{f} & D(\delta_0)^* \\ \uparrow & & \uparrow \\ \partial E(r,\delta_0)^* & \xrightarrow{f} & S^1_{\delta_0} \end{array}$$

are locally trivial fibrations and the topological isomorphism class does not depend on the choice of δ_0 and r.

5.2. **Milnor fibration, the first description.** We consider now the original Milnor fibration on the sphere, which is defined as follows:

$$\varphi : S_r^{2n-1} \setminus K_r \to S^1, \quad \mathbf{z} \mapsto \varphi(\mathbf{z}) = f(\mathbf{z},\bar{\mathbf{z}})/|f(\mathbf{z},\bar{\mathbf{z}})|$$

where $K_r = V \cap S_r^{2n-1}$.

For a mixed function $g(\mathbf{z},\bar{\mathbf{z}})$, we use two complex "gradient vectors" defined by

$$dg = (\frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_n}), \quad \bar{d}g = (\frac{\partial g}{\partial \bar{z}_1}, \dots, \frac{\partial g}{\partial \bar{z}_n}).$$

Take a smooth path $\mathbf{z}(t)$, $-1 \le t \le 1$ with $\mathbf{z}(0) = \mathbf{w} \in \mathbb{C}^n \setminus V$ and put $\mathbf{v} = \frac{d\mathbf{z}}{dt}(0) \in T_{\mathbf{w}}\mathbb{C}^n$. Then we have

$$\begin{aligned}
&-\frac{d}{dt}\left(\Re(i \,\log f(\mathbf{z}(t),\bar{\mathbf{z}}(t))\right)_{t=0} \\
&\quad = -\Re\left(\sum_{i=1}^n i\left\{\frac{\partial f}{\partial z_j}(\mathbf{w},\bar{\mathbf{w}})\frac{dz_j}{dt}(0) + \frac{\partial f}{\partial \bar{z}_j}(\mathbf{w},\bar{\mathbf{w}})\frac{d\bar{z}_j}{dt}(0)\right\}/f(\mathbf{w},\bar{\mathbf{w}})\right) \\
&\quad = \Re(\mathbf{v}, i\,\overline{d\log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\bar{\mathbf{v}}, i\,\overline{\bar{d}\log f}(\mathbf{w},\bar{\mathbf{w}})) \\
&\quad = \Re(\mathbf{v}, i\,\overline{d\log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\mathbf{v}, -i\,\bar{d}\log f(\mathbf{w},\bar{\mathbf{w}})) \\
&\quad = \Re(\mathbf{v}, i\,(\overline{d\log f} - \bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})).
\end{aligned}$$

Namely we have

$$-\frac{d}{dt}\left(\Re(i \,\log f(\mathbf{z}(t),\bar{\mathbf{z}}(t))\right)_{t=0} = \Re(\mathbf{v}, i\,(\overline{d\log f} - \bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})).$$

Thus by the same argument as in Milnor [12], we get

Lemma 19. *A point $\mathbf{z} \in S_r^{2n-1} \setminus K_r$ is a critical point of φ if and only if the two complex vectors $i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))$ and $\mathbf{z}$ are linearly dependent over $\mathbb{R}$.*

The key assertion is the following.

Lemma 20. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a strongly non-degenerate mixed function. Then there exists a positive number r_0 such that the two complex vectors $i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}})-\bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))$ and $\mathbf{z}\in S_r\setminus K_r$ are linearly independent over $\mathbb{R}$ for any r with $0<r\le r_0$.*

Observation 21. *Let $\mathbf{w}\in f^{-1}(\eta)$, $\eta\ne 0$ be a smooth point. Then the tangent space $T_{\mathbf{w}}f^{-1}(\eta)$ is the real subspace of $\mathbb{C}^n$ whose vectors are orthogonal in $\mathbb{R}^{2n}$ to the two vectors*

$$i\,(\overline{d\log f}-\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}}),\ \ (\overline{d\log f}+\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}}).$$

Now we are ready to prove the existence of the Milnor fibration of the first description.

Theorem 22. (Milnor fibration, the first description) *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a strongly non-degenerate convenient mixed function. There exists a positive number r_0 such that*

$$\varphi=f/|f|:S_r^{2n-1}\setminus K_r\to S^1$$

is a locally trivial fibration for any r with $0<r\le r_0$.

5.3. **Equivalence of two Milnor fibrations.** Take positive numbers r, δ_0 with $\delta_0\ll r$ as in Theorem 18. We compare the two fibrations

$$f:\partial E(r,\delta_0)\to S^1_{\delta_0},\qquad \varphi:S_r^{2n-1}\setminus K_r\to S^1$$

and we will show that they are isomorphic. However the proof is much more complicated compared with the case of holomorphic functions. The reason is that we have to take care of the two vectors

$$i\,(\overline{d\log f}-\bar{d}\log f),\ \overline{d\log f}+\bar{d}\log f$$

which are not perpendicular. (In the holomorphic case, the proof is easy as the two vectors reduce to the perpendicular vectors $i\,\overline{d\log f}$, $\overline{d\log f}$.) Consider a smooth curve $\mathbf{z}(t)$, $-1\le t\le 1$, with $\mathbf{z}(0)=\mathbf{w}\in B_r^{2n}\setminus V$ and $\mathbf{v}=\frac{d\mathbf{z}(t)}{dt}(0)$. Put $\mathbf{v}=(v_1,\dots,v_n)$. First from (1) and (**??**), we observe that

$$\begin{aligned}\frac{\log f(\mathbf{z}(t),\,\bar{\mathbf{z}}(t))}{dt}\Big|_{t=0}&=\sum_{j=1}^{n}\left(v_j\,\frac{\partial\log f}{\partial z_j}(\mathbf{w},\bar{\mathbf{w}})+\bar{v}_j\,\frac{\partial\log f}{\partial\bar{z}_j}(\mathbf{w},\bar{\mathbf{w}})\right)\\&=\Re(\mathbf{v},\,(\overline{d\log f}+\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}}))+i\,\Re(\mathbf{v},\,i\,(\overline{d\log f}-\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})).\end{aligned}$$

Define two vectors on $\mathbb{C}^n-V$:

$$\begin{aligned}\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}})&=\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}})+\bar{d}\log f(\mathbf{z},\bar{\mathbf{z}})\\\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})&=i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}})-\bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))\end{aligned}$$

The above equality is translated as

$$(1)\quad \frac{\log f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}|_{t=0} = \Re(\mathbf{v}, \mathbf{v}_1(\mathbf{w}, \bar{\mathbf{w}})) + i\,\Re(\mathbf{v}, \mathbf{v}_2(\mathbf{w}, \bar{\mathbf{w}})).$$

Now we are ready to prove the isomorphism theorem:

Theorem 23. *Under the same assumption as in Theorem 22, the two fibrations*

$$f : \partial E(r, \delta_0) \to S^1_{\delta_0}, \qquad \varphi : S_r^{2n-1} \setminus K_r \to S^1$$

are topologically isomorphic.

5.4. **Polar weighted homogeneous polynomial and its Milnor fibration.** Consider a mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ which is a radially weighted homogeneous polynomial of type $(q_1, \dots, q_n; d_r)$ and a polar weighted homogeneous polynomial of type $(p_1, \dots, p_n; d_p)$. Put $V = f^{-1}(0)$ as before. Then

$$f : \mathbb{C}^n \setminus V \to \mathbb{C}^*$$

is a locally trivial fibration [17]. We call it the global fibration. On the other hand, the Milnor fibration of the first type:

$$\varphi := f/|f| : S_r \setminus K_r \to S^1, \quad K_r = f^{-1}(0) \cap S_r$$

always exists for any $r > 0$ and the isomorphism class does not depend on the choice of r. This can be shown easily, using the polar action. We simply use the polar action to show the local triviality:

$$\psi : \varphi^{-1}(\theta) \times (\theta - \pi, \theta + \pi) \to \varphi^{-1}((\theta - \pi, \theta + \pi))$$
$$\psi(\mathbf{z}, \theta + \eta) := (z_1 \exp(i\,p_1\eta/d_p), \dots, z_n \exp(i\,p_n\eta/d_p))$$

Now we have the following assertion which is a generalization of the same assertion for weighted homogeneous polynomials.

Theorem 24. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted polynomial as above. We assume that the radial weight vector ${}^t(q_1, \dots, q_n)$ is strictly positive. Then the two fibrations*

$$f : f^{-1}(S^1_\delta) \to S^1_\delta, \quad \varphi = f/|f| : S_r^{2n-1} - K_r \to S^1,$$

are isomorphic for any $r > 0$ and $\delta > 0$.

The following is an important criterion for the connectivity of the Milnor fiber of a polar weighted mixed polynomial.

Proposition 25. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted mixed polynomial of n variables $\mathbf{z} = (z_1, \dots, z_n)$. We assume that $f^{-1}(0)$ has at least one mixed smooth point. Then the fiber $F := f^{-1}(1) \subset \mathbb{C}^n$ is connected.*

6. Curves defined by mixed functions

In this section, we focus our study to mixed plane curves ($n = 2$).

6.1. **Holomorphic plane curves.** Assume that C is a germ of a complex analytic curve defined by a convenient non-degenerate holomorphic function $f(z_1, z_2)$ and let Δ_j, $j = 1, \dots, r$ be the 1-dimensional faces and $M_0, M_1, \dots, M_{r-1}, M_r$ be the vertices of $\Gamma(f)$ such that $\Delta_j = \overline{M_{j-1}M_j}$ and M_0, M_r are on the coordinate axes. Then each face function f_{Δ_j} can be factorized as

$$f_{\Delta_j}(z_1, z_2) = c_j\, z_1^{a_j}\, z_2^{b_j} \prod_{i=1}^{\nu_j} (z_1^{p_j} + \alpha_{j,i}\, z_2^{q_j}), \quad \gcd(p_j, q_j) = 1$$

where $\alpha_{j,1}, \dots, \alpha_{j,\nu_j}$ are mutually distinct.

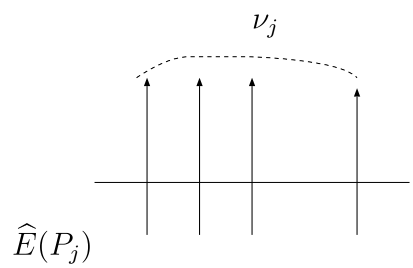

Figure 5. irreducible components

Then any toric modification with respect to a regular simplicial cone subdivision Σ^* of the dual Newton diagram $\Gamma^*(f)$ gives a good resolution of $f : (\mathbb{C}^2, O) \to (\mathbb{C}, 0)$. Let P_j be the weight vector of the face Δ_j. Each vertex P of Σ^* gives an exceptional divisor $\widehat{E}(P)$ and the strict transform $\widetilde{C}$ intersects with $\widehat{E}(P)$ if and only if $P = P_j$ for some $j = 1, \dots, r$. In the case $P = P_j$, $\widehat{E}(P_j) \cap \widetilde{C}$ is ν_j point which corresponds to irreducible components associated with f_{Δ_j}. The vertices $M_1, \dots, M_{r-1}$ do not contribute to the irreducible components. The number of irreducible components of (C, O) is given by $\sum_{i=1}^r \nu_i$. Note that $1 + \sum_{i=1}^r \nu_i$ is the number of integral points on $\Gamma(f)$ ([16]). The situation for mixed polynomials is more complicated as we will see later.

6.2. **Mixed curves.** Now we consider curves defined by a mixed function with the same Newton boundary as in the previous subsection. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a non-degenerate convenient mixed function with two variables $\mathbf{z}=(z_1,z_2)$ and let $C=f^{-1}(0)$. Let

$$\varphi : Y \xrightarrow{\omega} X \xrightarrow{\widehat{\pi}} \mathbb{C}^2$$

($Y=\mathcal{R}X$, $\omega=\omega_{\mathbb{R}}$ or $\mathcal{P}X$ and $\omega=\omega_p$) be the resolution map, described in Theorem 14 and Theorem 15. Let $\widetilde{E}(P)=\omega^{-1}(\widehat{E}(P))$ for a vertex P of Σ^*.

6.2.1. *Simple vertices.* A vertex $M=(a,b)\in\Gamma(f)$ is called simple if f_M contains only a single monomial $z_1^{a_1}\,z_2^{b_1}\,\bar{z}_1^{a_2}\,\bar{z}_2^{b_2}$ such that $a=a_1+a_2$, $b=b_1+b_2$. Otherwise we say M is a multiple vertex of $\Gamma(f)$.

Example 26. Let

$$f(\mathbf{z},\bar{\mathbf{z}})=z_1^3+t\,z_1^2\,\bar{z}_1+z_2^2$$

Then $\Gamma(f)$ has one face with edge vertices $M_1=(3,0)$ and $M_2=(0,2)$. $f(\mathbf{z},\bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial of type $(2,3;6)$. The vertex M_1 is a multiple vertex as $f_{M_1}(\mathbf{z},\bar{\mathbf{z}})=z_1^3+t\,z_1^2\bar{z}_1$.

Lemma 27. *Suppose $M=(n,0)$ and let $f_M(z_1,\bar{z}_1)=\sum_{j=0}^n c_j\,z_1^j\bar{z}_1^{n-j}$. Consider the factorization $f_M(z_1,\bar{z}_1)=c\prod_{j=1}^n(z_1-\alpha_j\,\bar{z}_1)$. Then $V^*:=\{z_1\in\mathbb{C}^*\mid f_M(z_1,\bar{z}_1)=0\}$ is empty if and only if $|\alpha_j|\neq 1$ for any $j=1,\dots,n$.*

Note that $f_M(z_1,\bar{z}_1)$ is non-degenerate if and only if $V^*=\emptyset$. For an inside vertex M_j (namely, M_j is not on the axis), the criterion for non-degeneracy of the function $f_{M_j}(\mathbf{z},\bar{\mathbf{z}})$ is not so simple.

Example 28. Consider

$$C:=\{\mathbf{z}\in\mathbb{C}^2\mid f_M(\mathbf{z},\bar{\mathbf{z}})=t\,z_1z_2\,+\,z_1\bar{z}_2\,+\,\bar{z}_1z_2\}.$$

We assert that

Assertion 29. *$f_M^{-1}(0)\subset\mathbb{C}^{*2}$ is non-empty if and only if $|t|\le 2$. f_M is non-degenerate if and only if $|t|>2$ or $0<|t|<2$.*

6.2.2. *Link components.* Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed function with two variables $\mathbf{z}=(z_1,z_2)$ and let $C=f^{-1}(0)$. The link components at the origin are the components of $S_\varepsilon^3\cap C$ for a sufficiently small ε. We are interested in finding out how to compute the number of the link components of C at the origin. Let us denote this number by $\mathrm{lkn}(C,O)$ and we call $\mathrm{lkn}(C,O)$ the link component number. Let us denote the number of components which are not the coordinate axes $z_1=0$ or $z_2=0$ by $\mathrm{lkn}^*(C,0)$. In the case of f being a holomorphic function, $\mathrm{lkn}(C,O)$

is equal to the number of irreducible components of (C, O), which is a combinatorial invariant, provided f is Newton non-degenerate, as we have seen in the previous section §6.1. However for a generic mixed function, $\mathrm{lkn}(C, O)$ might be strictly greater than the number of irreducible components (see Example 16 for example).

Theorem 30. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient non-degenerate mixed polynomial of two variables $\mathbf{z} = (z_1, z_2)$ and let $C = f^{-1}(0)$. Let $\mathcal{F}$ be the set of 1-faces of $\Gamma(f)$. Assume that the vertices of $\Gamma(f)$ are simple. Then the number of the link components $\mathrm{lkn}(C, O)$ is given be the formula:*

$$\mathrm{lkn}(C, O) = \sum_{\Delta \in \mathcal{F}} \mathrm{lkn}^*(f_\Delta^{-1}(0), O).$$

Now our interest is finding out how we can compute $\mathrm{lkn}^*(f_\Delta^{-1}(0), O)$. In general, it is not so easy to compute this number but there is a class for which the link number is easily computed.

6.2.3. *Good Newton polar boundary.* We say that $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is a good polar weighted polynomial if $\dim \widehat{\Delta} = 1$ and $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ factors as

$$f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = c\, \mathbf{z}^{\mathbf{m}}\, \bar{\mathbf{z}}^{\mathbf{n}} \prod_{j=1}^{k} (z_2^a\, \bar{z}_2^{a'} - \lambda_j\, z_1{}^b\, \bar{z}_1^{b'})^{\mu_j} \tag{2}$$

with $a \neq a'$, $b \neq b'$ and $\gcd(a, a', b, b') = 1$. Note that in this case, $p_1(b - b') = p_2(a - a')$ and non-zero. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ has a good Newton polar boundary if for every face Δ of $\Gamma(f)$, $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is a good polar weighted polynomial.

Lemma 31. *Assume that $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is a good polar weighted polynomial and assume that a factorization of $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is given as (2). Then $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is non-degenerate if and only if $\mu_1 = \cdots = \mu_k = 1$.*

6.2.4. *Good binomial polar weighted polynomial.* A polynomial

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_2^a \bar{z}_2^{a'} - \lambda z_1^b \bar{z}_1^{b'}$$

with $a \neq a'$, $b \neq b'$, $\lambda \neq 0$ and $\gcd(a, a', b, b') = 1$ is called an irreducible binomial polar weighted homogeneous polynomial. It is irreducible as a mixed polynomial. By Lemma 31, this is a basic polar weighted polynomial for our purpose. Then the associated Laurent polynomial in the sense of [17] is

$$g(z_1, z_2) = z_2^{c_2} - \lambda z_1^{c_1},\ c_1 = b - b',\ c_2 = a - a'.$$

Let $C = \{f = 0\}$ and $C' = \{g = 0\}$. Note that $c_1, c_2 \neq 0$ by the polar weightedness.

Lemma 32. *We have the equality:*

$$\mathrm{lkn}^*(C,O) = \gcd(c_1, c_2) = \sharp(C')$$

where $\sharp(C')$ *is the number if irreducible components of* C'*.*

Corollary 33. *Let* $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ *be a good polar weighted polynomial which is factored as*

$$f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = c\,\mathbf{z}^\nu \bar{\mathbf{z}}^\mu \prod_{j=1}^{k} (z_2^a \bar{z}_2^{a'} - \lambda_j\, {z_1}^b \bar{z}_1^{b'})$$

with $\gcd(a, a', b, b') = 1$, $a \neq a'$, $b \neq b'$ *as in Lemma 31 and let* $C = f_\Delta^{-1}(0)$*. Then* $\mathrm{lkn}^*(C) = k \gcd(a - a', b - b')$*.*

6.2.5. *Example of a radially weighted homogeneous polynomial with a non-simple vertex.* The link number for a radially weighted homogeneous polynomial with a non-simple vertex is more complicated, as is seen by the next example. Consider the radially weighted homogeneous polynomial

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 + c\, z_1 \bar{z}_1^2 - z_2^3$$

and put $C = f^{-1}(0)$. Then $\Gamma(f)$ consists of a single face with vertices $(3, 0)$, $(0, 3)$. It is easy to see that f is non-degenerate if and only if $|c| \neq 1$. The vertex (3,0) is not simple.

For $|c| < 1$, we have

$$z_2 = z_1 \omega^j \left(1 + c \exp(-4\,\theta\, i)\right)^{1/3}, \ j = 0, 1, 2$$

where $\omega = \exp(2\,\pi\, i/3)$, $z_1 = r \exp(\theta\, i)$ and $\mathrm{lkn}(C,O) = 3$. The function $(1 + c \exp(-4\,\theta\, i))^{1/3}$ is a well-defined single-valued function of c, z_1 with $|c| < 1$ so that it takes value 1 for $c = 0$. Considering the family $f(\mathbf{z}, \bar{\mathbf{z}}, t) = z_1^3 + c\, t\, z_1 \bar{z}_1^2 - z_2^3$ for $0 \le t \le 1$, we see that this curve is topologically the same as $z_1^3 + z_2^3 = 0$.

Assume that $|c| > 1$. Then $(1 + c \exp(-4\,\theta\, i))^{1/3}$ is not a single valued function as a function of $0 \le \theta \le 2\pi$. However we have a better expression. Put $z_1 = r \exp(\theta\, i)$ and $c = s \exp(\eta\, i)$.

$$z_2 = s^{1/3}\, r\, \omega^j \exp(i\, \frac{-\theta + \eta}{3}) \left(1 + \frac{\exp(4\,\theta\, i)}{c}\right)^{1/3}, \ j = 1, 2, 3$$

where $0 \le \theta \le 2\pi$. Note that $f^{-1}(0) \setminus \{O\}$ is a 3-sheeted covering over $\{z_1 \neq 0\}$ and three points over $\theta = 0$ are cyclically permuted by the monodromy $\theta : 0 \to 2\pi$. Thus this expression shows that $\mathrm{lkn}(C,O) = 1$. It is also easy to see that this knot is topologically the same with $z_1 |z_1|^2 - z_2^3 = 0$. Thus we observe that the topology of a mixed singularities is not a combinatorial invariant of $\Gamma(f)$.

7. Resolution of a polar type and the zeta function

In this section, we will study the relation between a resolution of a polar type and the Milnor fibration of the second type. We expect a similar formula like the formula of A'Campo ([1]) or the formula of Varchenko [23]. We will restrict ourselves to the case of mixed curves.

7.1. **Polar weighted case.** Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial of n variables $z_1,\dots,z_n$ and let $(q_1,\dots,q_n;d_r)$ and $(p_1,\dots,p_n;d_p)$ be the radial and polar weight types. We assume that $d_p>0$.

$$f:\mathbb{C}^{*n}-f^{-1}(0)\to\mathbb{C}^*$$

is a fibration. Put $F_s^*=f^{-1}(s)\cap\mathbb{C}^{*n}$ for $s\in\mathbb{C}^*$. Then the monodromy map $h:F_s^*\to F_s^*$ is given by the polar action as

$$h(z_1,\dots,z_n)=(z_1\omega^{p_1},\dots,z_n\omega^{p_n}),\ \omega=\exp(\frac{2\pi i}{d_p})$$

Put $F^*=F_1^*$ and let $\chi(F^*)$ be the Euler characteristic of F^*. Then the monodromy has the period d_p and the set of the fixed points of $h^j:F^*\to F^*$ is empty if $j\not\equiv 0$ modulo d_p, where $h^j=h\circ\cdots\circ h$ (j-times). Thus using the formula of the zeta function for a periodic mapping ([12]), we get

Lemma 34. *Under the above assumption, the zeta-function of $h:F^*\to F^*$ is given as*

$$\zeta(t)=(1-t^{d_p})^{-\chi(F^*)/d_p}.$$

The zeta function of the global fibration $f:\mathbb{C}^n\setminus f^{-1}(0)\to\mathbb{C}^*$ can be obtained by patching the data for each torus stratum.

Let us do this for curves $(n=2)$. Let $f(\mathbf{z})$ be a non-degenerate polar weighted homogeneous polynomial of type $(p_1,p_2;d_p)$. The signs of p_1,p_2 are chosen so that $d_p>0$. Suppose that the two edge vertices of $\Gamma(f)$ are simple. Assume that the two end monomials are

$$z_2^{\mu_2}\bar{z}_2^{\nu_2},\quad z_1^{\mu_1'}\bar{z}_1^{\nu_1'}$$

with $\mu_1+\nu_1<\mu_1'+\nu_1'$ and $\mu_2+\nu_2>\mu_2'+\nu_2'$.

Let $F=f^{-1}(1)\subset\mathbb{C}^2$, $F_{z_1}=F\cap\{z_2=0\}$ and $F_{z_2}=F\cap\{z_1=0\}$. Note that

$$F_{z_1}=\{(z_1,0)\,|\,z_1^{\mu_1'-\nu_1'}=1\},\quad F_{z_2}=\{(0,z_2)\,|\,z_2^{\mu_2-\nu_2}=1\}.$$

The monodromy map is defined by

$$h:F\to F,\quad (z_1,z_2)\mapsto(z_1\omega^{p_1},z_2\omega^{p_2}),\ \omega=\exp(\frac{2\pi i}{d_p})$$

Note that $p_1(\mu'_1 - \nu'_1) = p_2(\mu_2 - \nu_2) = d_p$. Therefore the fixed points set $\mathrm{Fix}(h^j)$ of h^j is non-empty only for $j = |\mu'_1 - \nu'_1|, |\mu_2 - \nu_2|$, or d_p and their multiples. Thus using the calculation through $\exp\zeta(t)$ as in [12], we get

Lemma 35. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted convenient polynomial as above. Let $z_1^{\mu'_1}\bar{z}_1^{\nu'_1}$, $z_2^{\mu_2}\bar{z}_2^{\nu_2}$ be the end monomials and let d_p be the polar degree. Then the Euler-Poincaré characteristic $\chi(F)$ and the zeta function of the monodromy $h : F \to F$ are given as*

$$\chi(F) = \chi(F^*) + |\mu'_1 - \nu'_1| + |\mu_2 - \nu_2|,\ \mu = 1 - \chi(F)$$

$$\zeta(t) = \frac{(1 - t^{d_p})^{-\chi(F^*)/d_p}}{(1 - t^{|\mu'_1 - \nu'_1|})\,(1 - t^{|\mu_2 - \nu_2|})}$$

7.1.1. *Simplicial polar weighted polynomial.* Let

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{m} c_j\, \mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j}.$$

The associated Laurent polynomial $g(\mathbf{z})$ is defined by

$$g(\mathbf{z}) = \sum_{j=1}^{m} c_j\, \mathbf{z}^{\mu_j - \nu_j}.$$

Recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is called simplicial polar weighted homogeneous if $m = n$ and the two matrices have a non-zero determinant [17]:

$$M = \begin{pmatrix} \mu_{11} + \nu_{11} & \dots & \mu_{1n} + \nu_{1n} \\ \vdots & \dots & \vdots \\ \mu_{n1} + \nu_{n1} & \dots & \mu_{nn} + \nu_{nn} \end{pmatrix}, \quad N = \begin{pmatrix} \mu_{11} - \nu_{11} & \dots & \mu_{1n} - \nu_{1n} \\ \vdots & \dots & \vdots \\ \mu_{n1} - \nu_{n1} & \dots & \mu_{nn} - \nu_{nn} \end{pmatrix}$$

where $\mu_j = (\mu_{j1}, \dots, \mu_{jn})$ and $\nu_j = (\nu_{j1}, \dots, \nu_{jn})$, $j = 1, \dots, n$ respectively. If f is a simplicial polar weighted homogeneous polynomial, we have shown that the two fibrations defined by $f(\mathbf{z}, \bar{\mathbf{z}})$ and $g(\mathbf{w})$:

$$f : \mathbb{C}^{*n} \setminus f^{-1}(0) \to \mathbb{C}^*, \quad g : \mathbb{C}^{*n} \setminus g^{-1}(0) \to \mathbb{C}^*$$

are equivalent ([17]). Thus the topology of the Milnor fibration is determined by the mixed face $\widehat{\Delta}$ where Δ is the unique face of $\Gamma(f)$. In particular, the zeta function of $h : F^* \to F^*$ is given as $\zeta(t) = (1 - t^{d_p})^{(-1)^n d/d_p}$ where $d = |\det(N)|$ ([17]). On the other hand, if f is not simplicial, the topology is not even a combinatorial invariant of $\widehat{\Delta}$ (§6.2.5). Therefore there does not exist any direct connection with the topology of the associated Laurent polynomial $g(\mathbf{z})$. However here is a useful lemma.

Lemma 36. *Suppose that $f_t(\mathbf{z},\bar{\mathbf{z}})$, $0\le t\le 1$ is a family of convenient, non-degenerate polar weighted homogeneous polynomials with the same radial and the polar weights, and assume that $\Gamma(f_t)$ is constant. Then the Milnor fibration $f_t:\mathbb{C}^n\setminus f_t^{-1}(0)\to\mathbb{C}^*$ and its restriction $\mathbb{C}^{*n}\setminus f_t^{-1}(0))\to\mathbb{C}^*$ are homotopically equivalent to $f_0:\mathbb{C}^n\setminus f_0^{-1}(0)\to\mathbb{C}^*$ and $f_0:\mathbb{C}^{*n}\setminus f_0^{-1}(0)\to\mathbb{C}^*$ respectively.*

Example 37. Consider the family of polar weighted mixed polynomials in two variables:

$$f_t(\mathbf{z},\bar{\mathbf{z}})=-2z_1^2\,\bar z_1+z_2^2\,\bar z_2+t\,z_1^2\,\bar z_2,\ t\in\mathbb{C}$$

and let $C_t=f_t^{-1}(0)$. The radial and polar weight types are $(1,1;3)$ and $(1,1;1)$ respectively. Thus the critical points of $f_t:\mathbb{C}^2\to\mathbb{C}$ are the solutions of

$$|\alpha|=1,\quad \begin{cases} -4z_1\,\bar z_1+2\bar t\,\bar z_1\,z_2=-2\alpha\,z_1^2\\ 2z_2\,\bar z_2=\alpha\,(z_2^2+t\,z_1^2)\\ -2z_1^2\,\bar z_1+z_2^2\,\bar z_2+t\,z_1^2\,\bar z_2=0.\end{cases} \tag{3}$$

By further calculation, we can see that

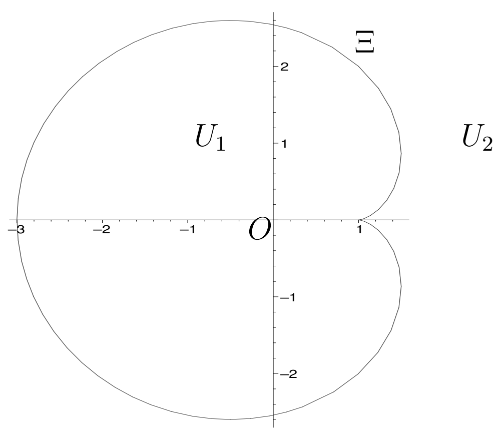

FIGURE 6. Degeneration locus Ξ

$$\mathrm{lkn}(C_t)=1,\chi(F)=1,\chi(F^*)=-1,\quad t\in U_1$$
$$\mathrm{lkn}(C_t)=3,\chi(F)=-1,\chi(F^*)=-3,\quad t\in U_2.$$

7.2. **Zeta function of non-degenerate mixed curves.** Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a convenient non-degenerate mixed polynomial and let $\Delta_1,\dots,\Delta_s$ be the faces of $\Gamma(f)$. Let $Q_j={}^t(q_{j1},q_{j2})$ be the weight vector of Δ_j for $j=1,\dots,s$. Assume that each face function f_{Δ_j} is also polar weighted and the inside monomials corresponding to the vertices $M_j = \Delta_j \cap \Delta_{j+1}$, $j = 1, \dots, s-1$ are polar admissible. Let $(a_1+2b_1, 0)$, $(0, a_2+2b_2)$ be the vertices of $\Gamma(f)$ on the coordinate axes which come from the monomials $z_1^{a_1}|z_1|^{2b_1}$ and $z_2^{a_2}|z_2|^{2b_2}$ respectively. We call a_1, a_2 the polar sections of $\Gamma(f)$ on the respective coordinate axes $z_2 = 0$ and $z_1 = 0$. Let $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$ be the face function of Δ_i and assume that $(p_{i1}, p_{i2}; m_i)$ is the polar weight type of $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$. Let $F_i^* = \{\mathbf{z} \in \mathbb{C}^{*2} \mid f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}}) = 1\}$. Then we have the following.

Theorem 38. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a non-degenerate convenient mixed polynomial such that its face functions $f_{\Delta_j}(\mathbf{z}, \bar{\mathbf{z}})$, $j = 1, \dots, s$ are polar weighted polynomials. Then the Euler-Poincaré characteristic of the Milnor fiber F of f and the zeta function of the monodromy $h : F \to F$ are given as follows.*

$$\chi(F) = \sum_{i=1}^{s} \chi(F_i^*) \,+\, |a_1| + |a_2|$$

$$\zeta(t) = \frac{\prod_{i=1}^{s}(1 - t^{m_i})^{-\chi(F_i^*)/m_i}}{(1 - t^{|a_1|})\,(1 - t^{|a_2|})}$$

where a_1, a_2 are the respective polar sections and m_j is the polar degree of the face function $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$, $j = 1, \dots, s$ as above ($m_j > 0$).

7.2.1. *Resolution of a polar type and the Milnor fibration.* Let us consider an admissible toric modification $\widehat{\pi} : X \to \mathbb{C}^2$ with respect to the regular fan Σ^* with vertices $\{P_0, P_1, \dots, P_{\ell+1}\}$ and we assume that $Q_j = P_{\nu_j}, j = 1, \dots, s$ and $P_0 = E_1 = {}^t(1, 0)$ and $P_{\ell+1} = E_2 = {}^t(0, 1)$. Then we take the polar modification $\omega_p : \mathcal{P}X \to X$ along $\widehat{E}(P_1), \dots, \widehat{E}(P_\ell)$. Put $\Phi_p : \mathcal{P}X \to \mathbb{C}^2$ be the composite with $\widehat{\pi} : X \to \mathbb{C}^2$. Consider the second Milnor fibration

$$f \circ \Phi_p : \Phi_p^{-1}(E(r, \delta)^*) \to D(\delta)^*$$

on the resolution space $\mathcal{P}X$. Take P_j for $1 \le j \le \ell$. There are two toric coordinate charts of X which contain the vertex P_j:

$$\sigma_{j-1} = \mathrm{Cone(P_{j-1}, P_j)} \quad \text{gives the coordinate chart } (\mathrm{U_{j-1}, (u_{j-1}, v_{j-1})})$$
$$\sigma_j = \mathrm{Cone(P_j, P_{j+1})} \quad \text{gives the coordinate chart } (\mathrm{U_j, (u_j, v_j)}).$$

Put $M = (P_j, P_{j+1})^{-1}(P_{j-1}, P_j)$. It takes the form:

$$M = \begin{pmatrix} \gamma_j & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the two coordinate systems are connected by the relation

$$u_j = u_{j-1}^{\gamma_j} v_{j-1}, \quad v_j = u_{j-1}^{-1}. \tag{4}$$

Put $P_j = {}^t(c_j, d_j)$, $j = 1, \dots, \ell$. The inverse image $\widetilde{U}_j := \omega_p^{-1}(U_j)$ has the polar coordinates $(r_j, \theta_j, s_j, \eta_j)$ which corresponds to (u_j, v_j) with $u_j = r_j \exp(i\,\theta_j)$ and $v_j = s_j \exp(i\,\eta_j)$. The relation (4) says that

$$s_j = r_{j-1}^{-1}, \qquad \eta_j = -\theta_{j-1}. \tag{5}$$

We do not take a normal polar modification along the two non-compact divisors $u_0 = 0$ and $v_\ell = 0$. Thus the coordinates of $\widetilde{U}_0$ and $\widetilde{U}_\ell$ are (u_0, s_0, η_0) and $(r_\ell, \theta_\ell, v_\ell)$ respectively. Recall that the exceptional divisor $\widetilde{E}(P_j)$ is defined by $r_j = 0$ in $\widetilde{U}_j$ and by $s_{j-1} = 0$ in $\widetilde{U}_{j-1}$ for $1 \le j \le \ell$. Note that $u_0 = 0$ in U_0 corresponds bijectively to the axis $z_1 = 0$ in the base space $\mathbb{C}^2$ and

$$(P_0, P_1) = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix},\ d_1 = 1,\ z_1 = u_0 v_0^{c_1},\ z_2 = v_0.$$

Similarly on $\widetilde{U}_\ell$, $v_\ell = 0$ corresponds to $z_2 = 0$ and

$$z_1 = u_\ell,\ z_2 = u_\ell^{d_\ell} v_\ell,\ \ c_\ell = 1.$$

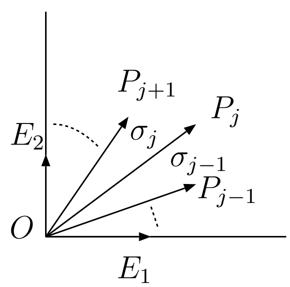

Figure 7. Regular fan Σ^*

7.3. **Decomposition of the fiber.** Recall that

$$E(r,\delta)^* = \{(z_1, z_2) \mid 0 < |f(z_1, z_2, \bar{z}_1, \bar{z}_2)| \le \delta,\ \|(z_1, z_2)\| \le r\}$$
$$\phi(\mathbf{z}) := \sqrt{|z_1|^2 + |z_2|^2}, \quad \widetilde{B}_r = \phi^{-1}(B_r)$$
$$F_\delta = \{(z_1, z_2) \mid f(z_1, z_2, \bar{z}_1, \bar{z}_2) = \delta,\ (z_1, z_2) \in B_r\} : \text{ Milnor fiber.}$$

We denote the pull-back of a function h on $\mathbb{C}^2$ to $\mathcal{P}X$ by $\tilde{h}$ for simplicity. On $\mathcal{P}X$, we consider the subsets

$$W_j(r,\rho) = \{\widetilde{\mathbf{x}} = (r_j,\theta_j,s_j,\eta_j) \in \widetilde{U}_j \mid 1/\rho \geq s_j \geq \rho\}$$
$$T_{j-1}(\rho) = \{(r_{j-1},\theta_{j-1},s_{j-1},\eta_{j-1}) \in \widetilde{U}_{j-1} \,|\, r_{j-1} \leq \rho,\ s_{j-1} \leq \rho\}$$
$$WT_j(\rho) = \{(r_j,\theta_j,s_j,\eta_j) \in \widetilde{U}_j \,|\, s_j = \rho,\ r_j \leq \rho\}$$
$$TW_j(\rho) = \{(r_{j-1},\theta_{j-1},s_{j-1},\eta_{j-1}) \in \widetilde{U}_{j-1} \,|\, r_{j-1} = \rho,\ s_{j-1} \leq \rho\}$$

and

$$T_0(\rho) := \{(u_0,s_0,\eta_0) \in \widetilde{U}_0 \,|\, |u_0| \leq \rho,\ s_0 \leq \rho\}$$
$$W_0(r,\rho) := \{(u_0,s_0,\eta_0) \in \widetilde{U}_0 \,|\, \tilde{\phi}(u_0,s_0,\eta_0) \leq r,\ |u_0| \geq \rho,\ s_0 \geq \rho\}$$
$$T_\ell(\rho) := \{(r_\ell,\theta_\ell,v_\ell) \in \widetilde{U}_\ell \,|\, r_\ell \leq \rho,\ |v_\ell| \leq \rho\}$$
$$W_\ell(r,\rho) := \{(r_\ell,\theta_\ell,v_\ell) \in \widetilde{U}_\ell, |\, r_\ell \geq \rho,\ |v_\ell| \geq \rho,\ \widetilde{\phi}(r_\ell,\theta_\ell,v_\ell) \leq r\}$$

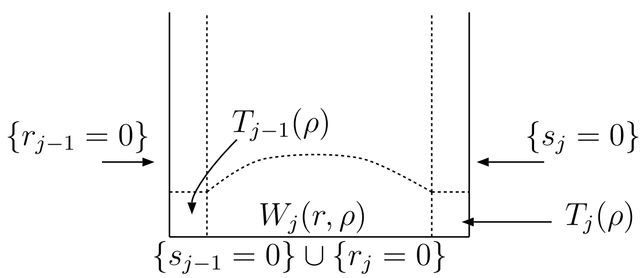

FIGURE 8. Decomposition of $\mathcal{P}X$

Note that

$$\widetilde{\phi}(u_0,s_0,\eta_0) = s_0\sqrt{1+|u_0|^2 s_0^{2c_1-2}} = s_0 + o(s_0)$$
$$\widetilde{\phi}(r_\ell,\theta_\ell,v_\ell) = r_\ell\sqrt{1+|v_\ell| r_\ell^{2d_\ell-2}} = r_\ell + o(r_\ell)$$

Here $o(s_0)$ implies $o(s_0)/s_0 \to 0$ when $s_0 \to 0$. Put

$$A(r,\rho) = \bigcup_{j=0}^{\ell+1} W_j(r,\rho) \cup \bigcup_{j=0}^{\ell} T_j(\rho).$$

Put $\widetilde{E}(r,\delta)^* = \Phi_p^{-1}(E(r,\delta)^*)$ with $\delta \ll r$ and $A(r,\rho,\delta)^* = A(r,\rho) \cap \widetilde{f}^{-1}(D_\delta^*)$ with $\delta \ll r,\rho$. It is easy to see that $A(r,\rho,\delta)^* = \widetilde{E}(r,\delta)^*$ as long as $\rho \ll r$ and $\delta \ll \rho, r$. We see that the choice of ρ does not give any effect on $A(r,\rho,\delta)^*$, as long as $\delta \ll \rho \ll r$. Thus we can use $A(r,\rho,\delta)^*$ as the total space of the Milnor fibration: $\widetilde{f} : A(r,\rho,\delta)^* \to D_\delta^*$. We decompose $A(r,\rho,\delta)^*$ into monodromy invariant subspaces as follows.

$$A(r,\rho,\delta)^* \cap W_j(r,\rho),\ A(r,\rho,\delta)^* \cap T_j(\rho)$$
$$A(r,\rho,\delta)^* \cap TW_j(\rho),\ A(r,\rho,\delta)^* \cap WT_j(\rho), \quad j = 0,\dots,\ell.$$

7.3.1. *Transversality.* Assume that $\Delta(P_j) = \Delta_t \cap \Delta_{t+1} = \{M_t\}$ and that M_t comes from the monomial $z_1^{\alpha_{t1}}\,|z_1|^{2\beta_{t1}}\,z_2^{\alpha_{t2}}\,|z_2|^{2\beta_{t2}}$. By the definition we can write

$$\widetilde{f}(r_j,\theta_j,s_j,\eta_j) \equiv r_j^{d(P_j)}\, s_j^{d(P_{j+1})} \exp((\alpha_{t1}\,c_j + \alpha_{t2}\,d_j)\theta_j\, i)$$
$$\times \exp((\alpha_{t1}\,c_{j+1} + \alpha_{t2}\,d_{j+1})\eta_j\, i) + O(r_j^{d(P_j)+1}).$$

Thus it is easy to see that $\widetilde{f}^{-1}(\xi)$, $|\xi| = \delta$ intersects transversely with $WT_j(\rho)$ if δ is sufficiently small and $\delta \ll r,\rho$. Similarly $\widetilde{f}^{-1}(\xi)$ intersects transversely with $TW_j(\rho)$ under the same assumptions.
Fix such r,δ,ρ. Under the above decomposition of $A(r,\rho,\delta)^*$, the Milnor fiber $\widetilde{F}_\delta := \widetilde{f}^{-1}(\delta) \cap \widetilde{B}$ decomposes into the following strata:

$$\widetilde{F}_\delta \cap W_j(r,\rho),\ \widetilde{F}_\delta \cap T_j(\rho),\ \widetilde{F}_\delta \cap WT_j(\rho),\ \widetilde{F}_\delta \cap TW_j(\rho),\ j = 0,\dots,\ell.$$

By the above transversality, we see that (after choosing a suitable vector field to define the characteristic diffeomorphisms) $\widetilde{F}_\delta \cap W_j(r,\rho)$, $\widetilde{F}_\delta \cap T_j(\rho)$, $\widetilde{F}_\delta \cap TW_j(\rho)$ and $\widetilde{F}_\delta \cap WT_j(\rho)$ are invariant by the monodromy $h : \widetilde{F}_\delta \to \widetilde{F}_\delta$. Now the proof of Theorem 38 follows from the following observations.

(1) The zeta functions of h restricted on $\widetilde{F}_\delta \cap T_j(\rho)$ are trivial for $1 \le j \le \ell - 1$.
(2) The zeta functions of h restricted on $\widetilde{F}_\delta \cap W_j(r,\rho)$ with $j \ne \nu_1,\dots,\nu_s$ are trivial.
(3) The zeta functions of h restricted on $\widetilde{F}_\delta \cap WT_j(\rho)$ and $\widetilde{F}_\delta \cap TW_j(\rho)$ are trivial.
(4) The zeta functions of h on $\widetilde{F}_\delta \cap T_0(\rho)$ and $\widetilde{F}_\delta \cap T_\ell(\rho)$ are respectively given by
$$\frac{1}{(1-t^{|a_2|})},\quad \frac{1}{(1-t^{|a_1|})}.$$
(5) (Face contribution) The zeta function of $h : \widetilde{F}_\delta \cap W_{\nu_j}(\rho)$ is $(1-t^{m_j})^{-\chi(F_j^*)/m_j}$ where $F_j^* = f_{\Delta_j}^{-1}(1) \cap \mathbb{C}^{*2}$ and m_j is the polar degree of f_{Δ_j}.

7.4. **Topology of a polar weighted polynomial and Kouchnirenko type formula.** Let $(p_1, p_2; m_\Delta)$ be the polar weight type. Let $F_\Delta = f_\Delta^{-1}(1)$ be the fiber of the global fibration, $F_\Delta^* = F_\Delta \cap \mathbb{C}^{*2}$ and let $K_\Delta = f_\Delta^{-1}(0) \cap S^3$. Note that F_Δ is diffeomorphic to the fiber of the Milnor fibration We consider the Wang sequence of the Milnor fibration:

$$0 \to H_2(S^3 - K_\Delta) \to H_1(F_\Delta) \xrightarrow{h_* - \mathrm{id}} H_1(F_\Delta) \to H_1(S^3 - K_\Delta) \to \mathbb{Z} \to 0.$$

Put $r_\Delta^* = \mathrm{lkn}^*(f_\Delta^{-1}(0))$. Let μ_Δ and μ'_Δ be the multiplicities of the factor $(t-1)$ in $P_1(t)$ and $\zeta(t)$ respectively. Then

$$\mu_\Delta = \mu'_\Delta + 1, \quad \mu'_\Delta = -\chi(F_\Delta^*)/m_\Delta - 2 + \varepsilon(\Delta).$$

On the other hand by the Alexander duality, we have the isomorphism:

$$H_2(S^3 - K_\Delta) \cong H^1(S^3, K_\Delta) \cong \widetilde{H}^0(K_\Delta).$$

As the monodromy map h_* is periodic, we have

$$r_\Delta^* + \varepsilon(\Delta) - 1 = \dim \mathrm{Ker}\,\{h_* - \mathrm{id} : H_1(F_\Delta) \to H_1(F_\Delta)\} = \mu_\Delta.$$

Thus we obtain

Lemma 39. *The Euler-Poincaré characteristic and the link component number satisfy the following equality:*

$$r_\Delta^* = -\chi(F_\Delta^*)/m_\Delta.$$

Usually it is easier to compute r_Δ^* and we can compute $\chi(F_\Delta^*)$ by Lemma 39. Now we can state our Kouchnirenko type formula:

Theorem 40. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate convenient mixed polynomial as in Theorem 38. Let $\Delta_1, \dots, \Delta_s$ be faces of $\Gamma(f)$ and we assume that $f_{\Delta_j}(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial with polar degree m_j. Let $r_j = \mathrm{lkn}^*(f_{\Delta_j}^{-1}(0))$ for $j = 1, \dots, s$. Then the Milnor number $\mu(F) = b_1(F)$ is given by the formula:*

$$\mu(F) = \sum_{j=1}^{s} r_j\, m_j \,-\, |a_1| \,-\, |a_2| \,+\, 1.$$

Here m_j is the polar degree of f_{Δ_j} and we assume that $m_j > 0$. a_1, a_2 are the polar sections of $\Gamma(f)$ on the respective coordinate axes.

As a special case, the following is a formula for a good polar weighted mixed polynomial (see §6.2.3 for the definition) which corresponds to the Orlik-Milnor formula [13] for a weighted homogeneous isolated singularity.

Corollary 41. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a good polar weighted polynomial which is factored as*

$$f(\mathbf{z},\bar{\mathbf{z}}) = c\prod_{j=1}^{k}(z_2^a\,|z_2|^{2a'} - \lambda_j\,{z_1}^b\,|z_1|^{2b'}),\quad c\neq 0 \tag{6}$$

with $a\neq 0$, $b\neq 0$. Let $r=\gcd(|a|,|b|)$. The polar weight is given by $P={}^t(p_1\varepsilon_1,p_2\varepsilon_2)$ where $p_1=|a|/r$, $p_2=|b|/r$, $\varepsilon_1=b/|b|,\varepsilon_2=a/|a|$ and the polar degree d_p is given as $d_p=|a|\,|b|\,k/r$, $\mathrm{lkn}(f^{-1}(0))=r\,k$ and

$$\mu = |a|\,|b|\,k^2 - k\,(|a|+|b|)+1 = (k\,|a|-1)\,(k\,|b|-1) \text{ and}$$

$$\zeta(t)=\frac{(1-t^{d_p})^{rk}}{(1-t^{|a|})(1-t^{|b|})}.$$

Conjecture.

1. Is the Milnor fiber F of a non-degenerate mixed function $(n-1)$-dimensional CW complex?

2. Is F $(n-2)$-connected?

For detail, See "M. Oka: Non-degenerate mixed functions", to appear in Kodai J. Math.

http://www.ma.kagu.tus.ac.jp/ oka

References

[1] N. A'Campo. La fonction zeta d'une monodromie. *Comm. Math. Helv.*, 50:539–580, 1975.

[2] R. Benedetti and J.-J. Risler. *Real algebraic and semi-algebraic sets*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990.

[3] A. Bodin and A. Pichon. Meromorphic functions, bifurcation sets and fibred links. *Math. Res. Lett.*, 14(3):413–422, 2007.

[4] J. L. Cisneros-Molina. Join theorem for polar weighted homogeneous singularities. In *Singularities II*, volume 475 of *Contemp. Math.*, pages 43–59. Amer. Math. Soc., Providence, RI, 2008.

[5] S. M. Guseĭn-Zade, I. Luengo, and A. Melle-Hernández. Zeta functions of germs of meromorphic functions, and the Newton diagram. *Funktsional. Anal. i Prilozhen.*, 32(2):26–35, 95, 1998.

[6] S. M. Guseĭn-Zade, I. Luengo, and A. Melle-Hernández. On the topology of germs of meromorphic functions and its applications. *Algebra i Analiz*, 11(5):92–99, 1999.

[7] H. Hamm. Lokale topologische Eigenschaften komplexer Räume. *Math. Ann.*, 191:235–252, 1971.

[8] H. A. Hamm and D. T. Lê. Un théorème de Zariski du type de Lefschetz. *Ann. Sci. École Norm. Sup. (4)*, 6:317–355, 1973.

[9] M. Kato and Y. Matsumoto. On the connectivity of the Milnor fiber of a holomorphic function at a critical point. In *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, pages 131–136. Univ. Tokyo Press, Tokyo, 1975.

[10] A. G. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32:1–31, 1976.

[11] F. Michel, A. Pichon, and C. Weber. The boundary of the Milnor fiber of Hirzebruch surface singularities. In *Singularity theory*, pages 745–760. World Sci. Publ., Hackensack, NJ, 2007.

[12] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.

[13] J. Milnor and P. Orlik. Isolated singularities defined by weighted homogeneous polynomials. *Topology*, 9:385–393, 1970.

[14] M. Oka. On the resolution of the hypersurface singularities. In *Complex analytic singularities*, volume 8 of *Adv. Stud. Pure Math.*, pages 405–436. North-Holland, Amsterdam, 1987.

[15] M. Oka. Principal zeta-function of nondegenerate complete intersection singularity. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 37(1):11–32, 1990.

[16] M. Oka. *Non-degenerate complete intersection singularity*. Hermann, Paris, 1997.

[17] M. Oka. Topology of polar weighted homogeneous hypersurafces. *Kodai Math. J.*, 31(2):163–182, 2008.

[18] A. Pichon. Real analytic germs $f\overline{g}$ and open-book decompositions of the 3-sphere. *Internat. J. Math.*, 16(1):1–12, 2005.

[19] A. Pichon and J. Seade. Real singularities and open-book decompositions of the 3-sphere. *Ann. Fac. Sci. Toulouse Math. (6)*, 12(2):245–265, 2003.

[20] M. A. S. Ruas, J. Seade, and A. Verjovsky. On real singularities with a Milnor fibration. In *Trends in singularities*, Trends Math., pages 191–213. Birkhäuser, Basel, 2002.

[21] J. Seade. Open book decompositions associated to holomorphic vector fields. *Bol. Soc. Mat. Mexicana (3)*, 3(2):323–335, 1997.

[22] J. Seade. On the topology of hypersurface singularities. In *Real and complex singularities*, volume 232 of *Lecture Notes in Pure and Appl. Math.*, pages 201–205. Dekker, New York, 2003.

[23] A. N. Varchenko. Zeta-function of monodromy and Newton's diagram. *Invent. Math.*, 37:253–262, 1976.

[24] J. A. Wolf. Differentiable fibre spaces and mappings compatible with Riemannian metrics. *Michigan Math. J.*, 11:65–70, 1964.