0-stable singularities in weighted homogeneous map germs from $(C^{n+1}, 0)$ to $(C^n, 0)$, $n = 2, 3$

M. J. Saia ICMC USP São Carlos Brasil
Joint work with: A. J. Miranda and E. C. Rizziolli

1 Introduction

The study of stable singularities of differentiable maps was initiated by Whitney in 1955:

The singularities which appear in any stable map from the plane to the plane are the cusps and double points.

These isolated singularities, called 0-stable singularities, are also very important in the study of the non stable maps, in special for the class of the finitely determined maps, since they have the interesting property that for these maps, they are preserved for any stable deformation. The type and also the number of such singularities is very relevant because they hide information about the local geometric behavior of such maps.

T. Gaffney, D. M. Q. Mond in *Cusps and double folds of germs of analytic maps* $\mathbb{C}^2 \to \mathbb{C}^2$, showed how to determine algebraically the number of cusps and double points of finitely determined map germs from the plane to the plane.

Moreover, they showed that these numbers are topological invariants in families of such germs.

The class of quasi homogeneous maps is known as some of great interest, since for this class there are several results showing how to compute these numbers in terms of weights and degrees.

We say that a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $f = (f_1, \ldots, f_p)$, is **quasi-homogeneous** of type

$$(\omega_1, \ldots, \omega_n;\ d_1, \ldots, d_p)$$

if there are positive numbers $\omega_1, \omega_2, \ldots, \omega_n$ in $\mathbb{Q}$ called weights, and $d_1, \ldots, d_p$, called degrees, such that $f_i(\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_n} x_n) = \lambda^{d_i} f_i$ for all $i = 1, \ldots, p$, $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$,

For the finitely determined quasi homogeneous map germs from the plane to the plane, again Gaffney and Mond in *Weighted homogeneous maps from plane to plane* obtained formulae to compute the number of cusps and double-folds that involve only weights and degrees.

W.L. Marar, J.A. Montaldi and M.A.S. Ruas in *Multiplicities of zero-schemes in quasi-homogeneous co-rank 1 singularities* $\mathbb{C}^n \to \mathbb{C}^n$ described algebraically all 0-stable singularities that appear in any co-rank one stable map germ.

Following Arnold's notation, these singularities are the $A_{\mathcal{P}}$ singularities, for any partition $\mathcal{P}$ of n.

They showed how to compute the number of such $A_{\mathcal{P}}$ singularities for quasi homogeneous map germs in terms of the weights and degrees.

These numbers are computed as the complex dimension (as vector space) of some algebras of type $\frac{\mathcal{O}_s}{J}$.

In this case J is an ideal which defines an Isolated Complete Intersection Singularity, (**ICIS**) and $\mathcal{O}_s$ denotes the local ring of holomorphic germs of functions in s variables for some s.

They applied the Bezout theorem which shows how to compute such dimensions for ICIS.

Here: We are interested in the 0-stable singularities that appear in the discriminant of a stable deformation of weighted homogeneous finitely determined map germ from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$ with $n = 2$ or 3 also in the co-rank one case.

The main difference from the case of map germs from $(\mathbb{C}^n, 0)$ **to** $(\mathbb{C}^n, 0)$**:** The corresponding defining ideals of the 0-stable singularities, which are in the multiple point set of f, are in general not isolated singularities.

First we study the mono germs singularities, which can be obtained as the the complex dimension (as vector space) of some algebras of type $\frac{\mathcal{O}_n}{J}$, but the ideal J does not define an ICIS in a general situation.

We apply the Hilbert's Syzygy Theorem, where it is shown how to compute the free resolution given by the Syzygy modules of the ideal J.

To apply this theorem it is necessary to find a convenient filtration in $\mathcal{O}_n$ and we obtain this filtration from the weights given by the weighted homogenous germ.

To describe the singularities which are multi germs we apply the recent results given by A. J. Miranda in its Ph.D. Thesis, were it is shown how to compute the multi germs in terms of the mono germs and also in terms of algebras defined by the Fitting ideals associated to the discriminant.

2 Points A_n and the Iterated Jacobian ideals

In this section we describe algebraically the mono germs, or points A_n, in terms of the iterated Jacobian ideals, as defined by Morin and described by T. Fukui, J. J. Nuño Balesteros and M. J. Saia in *On the number of singularities of generic deformations of map germs*, J. of The London Mathematical Society, London (2) **58**, 141–152, (1998).

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map-germ and $I \subset \mathcal{O}_n$ a finite co-length ideal generated by the system $g_1, \ldots, g_r$.

For each $t \in \{1, \ldots, n\}$ the **jacobian extension of the rank** t for the pair (f, I) is defined by $\Delta_t(f, I) := I + I_t(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$, where $I_t(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$ denotes the ideal generated by minors of size $t \times t$ from the jacobian matrix of the $f_1, \ldots, f_p, g_1, \ldots, g_r$ $(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$.

If $i = (i_1, \ldots, i_k)$ is a Boardman number, the **Iterated Jacobian Ideal** for i, $J_i(f)$ is inductively defined in the following manner:

$$J_i(f) = \begin{cases} \Delta_{n-i_1+1}(f, \{0\}) & \text{se } k = 1 \\ \Delta_{n-i_k+1}(f, J_{i_1,\ldots,i_k}(f)) & \text{se } k > 1. \end{cases}$$

Co-rank one map germs from $(\mathbb{C}^{2+\ell}, 0)$ to $(\mathbb{C}^2, 0)$

The iterated Jacobian ideal associated to the points A_2 is the ideal $J_{\ell+1,1}(f)$.

Gaffney and Mond showed the construction of this ideal when $\ell = 0$ and f has any co-rank:

Weighted homogeneous maps from plane to plane, Math. Proc. Camb. Phil. Soc., **109**, 451–470, (1991).

We describe here the case $\ell > 0$ for co-rank one map germs $f : (C^{2+\ell}, 0) \rightarrow (C^2, 0)$ written as $f(x, y, z_1, \dots, z_\ell) = (x, g(x, y, z_1, \dots, z_\ell))$.

To obtain the ideal $J_{\ell+1,1}(f)$, first we consider the jacobian matrix of f, which is the $2 \times (2+\ell)$ matrix

$$d(f) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ g_x & g_y & g_{z_1} & \dots & g_{z_\ell} \end{pmatrix},$$

where g_w denotes the partial derivative of g with respect to the variable $w \in \{x, y, z_1, \dots, z_\ell\}$.

Now consider the ideal $J_{\ell+1}(f) = I_2(d(f)) = \langle g_y, g_{z_1}, \dots, g_{z_\ell} \rangle$ to get the ideal

$$J_{\ell+1,1}(f) = J_{\ell+1}(f) + I_{\ell+2}(D(f, g_y, g_{z_1}, \dots, g_{z_\ell})).$$

Therefore $J_{\ell+1,1}(f) = \langle g_y, g_{z_1}, \dots, g_{z_\ell}, M \rangle$, where M denotes the determinant of the matrix of order $\ell + 1$ given by the second partial derivatives of g with respect to the variables $(y, z_1, \dots, z_\ell)$.

Then

$$\#A_2 = dim_{\mathbb{C}} \frac{\mathcal{O}_{\ell+2}}{\langle g_y, g_{z_1}, \dots, g_{z_\ell}, M \rangle}.$$

We remark that in this case, $n = 2$, the ideal $J_{\ell+1,1}(f)$ is an ICIS and this occurs because the map germ is of co-rank one, if we consider the co-rank 2 case, this ideal in general is not an ICIS.

Co-rank one map germs from $(\mathbb{C}^{3+\ell}, 0)$ to $(\mathbb{C}^3, 0)$

The Iterated Jacobian Ideal associated to the points A_3 is the ideal $J_{\ell+1,1,1}(f)$.

We describe here the case $\ell = 1$, which is our main subject of study.

Let f be a finitely determined co-rank one map germ from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$ written as

$$f(x, y, u, v) = (x, y, g(x, y, u, v)).$$

The Jacobian matrix of f is the 3×4 matrix

$$d(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ g_x & g_y & g_u & g_v \end{pmatrix},$$

and g_z denotes a partial derivative of the g with respect to a variable z.

To obtain the Jacobian iterated ideal for $i = (2, 1, 1)$, first we get $J_2(f) = I_3(d(f)) = \langle g_u, g_v \rangle$
Then

$$J_{(2,1)} = J_2(f) + I_4(D(f, g_u, g_v)) = \langle g_u, g_v, H \rangle,$$

where $H := g_{uu}g_{vv} - (g_{uv})^2$ and finally

$$J_{(2,1,1)}(f) = J_{(2,1)} + I_4(d(f, g_u, g_v, H)) = \langle g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u \rangle.$$

Therefore

$$\#A_3 = dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\langle g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u \rangle}.$$

We remark that if the map germ f in $\mathcal{O}_{(4,3)}$ is a suspension of a map germ from $\mathbb{C}^3$ to $\mathbb{C}^3$ it can be written as $f(x, y, u, v) = (x, y, g(x, y, u) + v^2)$, therefore the iterated Jacobian ideal is generated by $\{g_u, g_{uu}, g_{uuu}, v\}$ and in this case we have

$$\#A_3 = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle g_u, g_{uu}, g_{uuu} \rangle}.$$

3 Points A_n and the The Syzygy Modules

3.1 The resolution via Syzygy Modules

The first step in the method to compute the complex dimension of any algebra $\frac{\mathcal{O}_n}{\mathcal{I}}$ for any finite co-length ideal I in $\mathcal{O}_n$ is to get a free resolution for such ideal. The existence of such free resolution is shown in the Hilbert's Syzygy Theorem, which we recover here.

Theorem 3.1. *(Hilbert's Syzygy Theorem) Let $>$ be any monomial ordering on $K[x] = K[x_1, \ldots, x_n]$ and $R = K[x]_>$ be the associated ring. Then any finitely generated R-module M has a free resolution*

$$0 \to F_m \to F_{m-1} \to \ldots \to F_0 \to M \to 0,$$

of length $m \leq n$, where the F_i are free R-modules.

The algorithm to get a such free resolution can be found in the book:
A Singular Introduction to Commutative Algebra, Second Edition, Springer, (2007). p.165, G. M. Greuel, G. Pfister.

The construction of such free resolution is based in the description of the morphisms between the modules F_{m-r}, called syzygies, and they are the main tool to fix the right filtration in the ring R (in our case $R = \mathcal{O}_{n+\ell}$) which we need to get the formulae for the number of points A_n.

We recover now the definition of the syzygies.

Let R be an arbitrary ring. A **syzygy** between k elements $h_1, \ldots, h_k$ of an R-module M is a k-uple $(r_1, \ldots, r_k) \in R^k$ satisfying $\sum_{i=1}^{k} r_i h_i = 0$.

The set of all syzygies between $h_1, \ldots, h_k$, denoted by $syz(h_1, \ldots, h_k)$, is a submodule of R^k, it is the kernel of the ring homomorphism $\phi : F_1 := \bigoplus_{i=1}^{k} R\varepsilon_i \to M$, with $\varepsilon_i \mapsto r_i$, where $\{\varepsilon_1, \ldots, \varepsilon_k\}$ denotes the canonical basis of R^k.

If $I = \langle h_1, \ldots, h_k \rangle_R$, define $syz(I) := syz(h_1, \ldots, h_k) := ker(\phi)$, the module of syzygies of I with respect to the generators $\{h_1, \ldots, h_k\}$.

The $k - th$ syzygy module is defined inductively: $syz_k(I) := syz(syz_{k-1}(I))$, setting $syz_0(I) := I$.
The existence of such syzygies is guaranteed by the Buchberger's criterion for the existence of a standard basis of $syz(I)$.

The next step is to define an appropriate filtration in the ring $\mathcal{O}_{n+\ell}$ which gives a graduation for the exact sequence of the free resolution given in the Theorem above in such a way that the morphisms are zero degree.

From this graded exact sequence we get the Poincaré series (or the Hilbert Samuel polynomial) and the evaluation of this polynomial at 1 gives us the desired formula.

We shall show next how to obtain such resolution and the formulae for the cases which we are interested.

3.2 On $\#A_2$ for co-rank one map germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^2, 0)$

Now we show how to compute the number of points $\#A_2$ of any finitely determined co-rank one weighted homogeneous map germ from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^2, 0)$.

Let $f(x, u, v) = (x, g(x, u, v))$ be such germ and suppose that $g(x, u, v)$ is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3; d)$.

Then $J_{(2,1)}(f) = \langle g_u, g_v, g_{u^2}.g_{v^2} - (g_{uv})^2 \rangle$ and $\#A_2 = dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{J_{(2,1)}(f)}$.

Applying the "Hilbert's Syzygy Theorem", we obtain the free resolution for $J_{(2,1)}(f)$ in $\mathcal{O}_3$.

$$0 \to \mathcal{O}_3 \overset{\alpha_3}{\to} \bigoplus_{i=1}^{3} \mathcal{O}_3 \overset{\alpha_2}{\to} \bigoplus_{i=1}^{3} \mathcal{O}_3 \overset{\alpha_1}{\to} \mathcal{O}_3 \overset{\pi}{\to} \frac{\mathcal{O}_3}{J_{(2,1)}(f)} \to 0,$$

where $\alpha_i = syzygy(\alpha_{i-1})$ denotes the associated matrix of the i-th syzygy module and π is the natural projection.

We remember that in this case the ideal $J_{(2,1)}(f)$ defines an ICIS in $\mathbb{C}^3$.

Now, from the weights and degrees of the germ g, we can compute the weighted degree for each generator of $J_{(2,1)}(f)$ and then we can write this resolution as an exact sequence of graded modules with homomorphisms of degree zero, as follows:

$$0 \to \mathcal{O}_3[-C] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-B_i] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-A_i] \to \mathcal{O}_3 \to \frac{\mathcal{O}_3}{J_{(2,1)}(f)} \to 0,$$

where $A_1 = d - \omega_2,\ A_2 = d - \omega_3,\ A_3 = 2(d - \omega_2 - \omega_3),\ \ B_1 = 2d - \omega_2 - \omega_3,\ B_2 = 3d - 3\omega_2 - 2\omega_3,\ B_3 = 3d - 2\omega_2 - 3\omega_3$ and $C = 4d - 3\omega_2 - 3\omega_3$.

Now we remember that for each $\mathcal{O}_3[-r]$, the associated Poincaré series is defined as:

$$P_{\mathcal{O}_3[-r]}(t) = \frac{t^r}{(1-t^{\omega_1})(1-t^{\omega_2})(1-t^{\omega_3})}.$$

Therefore, as the sequence above is exact, we have that the Poincaré series of $\dfrac{\mathcal{O}_3}{J_{(2,1)}(f)}$ is the alternate sum of the Poincaré series of each $\mathcal{O}_3[-r]$ in the sequence, hence

$$P_{\frac{\mathcal{O}_3}{J_{(2,1)}(f)}}(t) = \frac{1 - (\sum_{i=1}^{3} t^{A_i}) + (\sum_{i=1}^{3} t^{B_i}) - (t^C)}{(1-t^{\omega_1})(1-t^{\omega_2})(1-t^{\omega_3})}.$$

Therefore,

$$\#A_2 = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{(2,1)}(f)} = P_{\frac{\mathcal{O}_3}{J_{(2,1)}(f)}}(1) = \lim_{t \to 1} P_{\frac{\mathcal{O}_3}{J_{(2,1)}(f)}}(t) = \frac{1}{\omega_1\omega_2\omega_3}.\{P_3 d^3 + P_2 d^2 + P_1 d + P_0\},$$

where $P_3 = 2,\ \ P_2 = -4\omega_2 - 4\omega_3,\ \ P_1 = 2{\omega_2}^2 + 6\omega_2.\omega_3 + 2{\omega_3}^2,\ \ P_0 = -2{\omega_2}^2\omega_3 - 2\omega_2{\omega_3}^2$.

Example: For each number $k \in \{2,3,4,5,6,7\}$, define F_k from $(\mathbb{C}^3,0)$ to $(\mathbb{C}^2,0)$ as:

$$F_k(x,u,v) := (x, G_k(x,u,v)), \qquad \text{where} \qquad G_k(x,u,v) := xu + uv^2 + u^k.$$

Each map germ F_k is finitely determined and quasi-homogeneous of type $(k-1, \frac{k-1}{2}, 1; k)$.

Applying the result above, for any $k \in \{2,3,4,5,6,7\}$ we obtain

$$\#A_2(F_k) = k+1.$$

3.3 On $\#A_3$ for co-rank one map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$

In this case we need to split our calculation in two different situations.

1. The map germ is a suspension from a map germ from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^3, 0)$.
2. The general case.

The main difference is the number of generators, which gives rise to different resolutions.

For suspensions, the associated iterated Jacobian ideal defines ICIS in $\mathbb{C}^4$, hence we can apply directly the Besout theorem for ICIS.

In the general case the ideals are not ICIS, neither determinantals.

Now we compute the number of points A_3 for map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$ which are not suspensions

Let f be a finitely determined co-rank one map germ from $(C^4, 0)$ to $(C^3, 0)$.

So f can be written as $f(x, y, u, v) = (x, y, g(x, y, u, v))$.

Suppose that g is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3, \omega_4; d)$, then f is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3, \omega_4; \omega_1, \omega_2, d)$

We have

$$J_{(2,1,1)}(f) = \langle g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u \rangle.$$

Call $g_1 := g_u, g_2 := g_v, g_3 := H, g_4 := g_{uu}H_v - g_{uv}H_u, g_5 := g_{uv}H_v - g_{vv}H_u$.

Observe that each generator of $J_{(2,1,1)}$ is weighted homogeneous polynomial of degrees $d_1 := d - \omega_3,\ d_2 := d - \omega_4,\ d_3 := 2d - 2\omega_3 - 2\omega_4,\ d_4 := 3d - 4\omega_3 - 3\omega_4$ and $d_5 := 3d - 3\omega_3 - 4\omega_4$ respectively.

In this case, we obtain the free resolution for $\dfrac{\mathcal{O}_4}{J_{(2,1,1)}(f)}$,

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4 \to^{M_4} \bigoplus_{i=1}^{7} \mathcal{O}_4 \to^{M_3} \bigoplus_{i=1}^{9} \mathcal{O}_4 \to^{M_2} \bigoplus_{i=1}^{5} \mathcal{O}_4 \to^{M_1} \mathcal{O}_4 \to^{\pi} \frac{\mathcal{O}_4}{J_{(2,1,1)}(f)} \to 0,$$

where $M_i = syz(M_{i-1})$ and π the natural projection.

We can write this resolution as an exact sequence of graded modules with homomorphisms of degree zero, as follows: if $\mathcal{O}_4[-r]$ denotes $\mathcal{O}_4$ with its grading shifted by r (so that 1 has degree $-r$) then associated to the above sequence we have the graded resolution,

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4[-D_i] \to \bigoplus_{i=1}^{7} \mathcal{O}_4[-C_i] \to \bigoplus_{i=1}^{9} \mathcal{O}_4[-B_i] \to \bigoplus_{i=1}^{5} \mathcal{O}_4[-A_i] \to \mathcal{O}_4 \to \frac{\mathcal{O}_4}{J_{(2,1,1)}(f)} \to 0,$$

where

$$Ai = d_i, \ \forall i \in \{1, \dots, 5\}$$

$$B_1 = 5d - 7\omega_3 - 5\omega_4, \quad B_2 = 5d - 5\omega_3 - 7\omega_4, \quad B_3 = 4d - 3\omega_3 - 5\omega_4,$$

$$B_4 = 3d - 2\omega_3 - 3\omega_4, \quad B_5 = 4d - 4\omega_3 - 4\omega_4, \quad B_6 = 2d - \omega_3 - \omega_4,$$

$$B_7 = 4d - 4\omega_3 - 4\omega_4, \quad B_8 = 3d - 3\omega_3 - 2\omega_4, \quad B_9 = 4d - 5\omega_3 - 3\omega_4$$

$$C_1 = 5d - 5\omega_3 - 4\omega_4, \quad C_2 = 4d - 3\omega_3 - 3\omega_4, C_3 = 5d - 4\omega_3 - 4\omega_4,$$

$$C_4 = 6d - 5\omega_3 - 8\omega_4, \quad C_5 = 6d - 6\omega_3 - 7\omega_4, \quad C_6 = 6d - 7\omega_3 - 6\omega_4, C_7 = 6d - 8\omega_3 - 5\omega_4$$

and $D_1 = 7d - 6\omega_3 - 8\omega_4, \quad D_2 = 7d - 8\omega_3 - 6\omega_4.$

The Poincaré series of the graded resolution of $\frac{\mathcal{O}_4}{J_{(2,1,1)}(f)}$ is equal to

$$P_{\frac{\mathcal{O}_4}{J_{(2,1,1)}(f)}}(t) = \frac{1 - (\sum_{i=1}^{5} t^{A_i}) + (\sum_{i=1}^{9} t^{B_i}) - (\sum_{i=1}^{7} t^{C_i}) + (\sum_{i=1}^{2} t^{D_i})}{(1-t^{\omega_1})(1-t^{\omega_2})(1-t^{\omega_3})(1-t^{\omega_4})}.$$

Therefore,

$$\#A_3 = dim_{\mathbb{C}} \frac{\mathcal{O}_4}{J_{(2,1,1)}(f)} = P_{\frac{\mathcal{O}_4}{J_{(2,1,1)}(f)}}(1) = \lim_{t \to 1} P_{\frac{\mathcal{O}_4}{J_{(2,1,1)}(f)}}(t)$$

Thus,

$$\#A_3 = \frac{1}{\omega_1\omega_2\omega_3\omega_4}.\{P_4 d^4 + P_3 d^3 + P_2 d^2 + P_1 d + P_0\},$$

where $P_4 = 16$, $P_3 = -3 - 55\omega_4 - 55\omega_3$, $P_2 = \frac{15}{2}\omega_3 + 63{\omega_3}^2 + 63{\omega_4}^2 + 139\omega_3\omega_4 + \frac{15}{2}\omega_4$,

$$P_1 = -24{\omega_3}^3 - 108{\omega_3}^2\omega_4 - 24{\omega_4}^3 - 12\omega_3\omega_4 - 108\omega_3{\omega_4}^2 - \frac{9}{2}{\omega_3}^2 - \frac{9}{2}{\omega_4}^2,$$

and

$$P_0 = 24{\omega_3}^3\omega_4 + 24\omega_3{\omega_4}^3 + 45{\omega_3}^2{\omega_4}^2 + \frac{9}{2}{\omega_3}^2\omega_4 + \frac{9}{2}\omega_3{\omega_4}^2.$$

Example 3.2. Let F_1 and F_2 be two finitely determined co-rank one map germs from $(C^4, 0)$ to $(C^3, 0)$ defined by, respectively:

$$F_1(x, y, u, v) := (x, y, G_1(x, y, u, v)), \text{ with } G_1(x, y, u, v) := yu + xv + uv^2 + u^3,$$

And,

$$F_2(x, y, u, v) := (x, y, G_2(x, y, u, v)), \text{ with } G_2(x, y, u, v) := yu + xv + u^3 + v^3,$$

Note that both maps are quasi-homogeneous of type $(2, 2, 1, 1; 3)$. Also, $J_{(2,1,1)}(F_1)$ and $J_{(2,1,1)}(F_2)$ have the same standard, indeed, $I = \langle y, x, v^2, uv, u^2\rangle$. Applying the resulted above, substituting $d = 3$, $\omega_1 = 2 = \omega_2$ and $\omega_3 = 1 = \omega_4$, we obtain

$$\#A_3(F_1) = 3 = \#A_3(F_2).$$

We can check that this number is correct in two different ways: we can apply the method described by Gaffney and calculate using the multiplicity of pairs of modules or using the SINGULAR software, calculating the codimension of the standard base of the Jacobian Iterated Ideal $J_{2,1,1}(f)$.

4 The 0-stable multiple points and Fitting ideals.

For any germ $f : (\mathbb{C}^{n+\ell}, 0) \to (\mathbb{C}^n, 0)$, Mond and Pellikan showed in [10] that the Fitting ideals of the discriminant of f are very important to compute the 0-stable multiple points of f.

For map germs from $\mathbb{C}^{2+\ell}$ to $\mathbb{C}^2$ with $\ell \geq 0$, the Fitting ideals which appear in the discriminant curve $\Delta(f) = f(\Sigma(f)) \subset \mathbb{C}^2$ of f are two, denoted by $\mathcal{F}_0$ and $\mathcal{F}_1$. They define in $\Delta(f)$ the following sets:

1. $V(\mathcal{F}_0) := \Delta(f)$, or $\mathcal{F}_0$ is the defining ideal of the discriminant curve.
2. $V(\mathcal{F}_1) = \cup A_{1,1} \cup A_2$, union of the isolated singularities in the target.

For the special case of map germs from the plane to the plane, or when $\ell = 0$, Gaffney and Mond in [2] showed the formula which relates these ideals and the 0-stable singularities.

$$\#A_2 + \#A_{1,1} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{\mathcal{F}_1}$$

We remark that such formula also holds for the general case of co-rank one map germs from $\mathbb{C}^{2+\ell}$ to $\mathbb{C}^2$ with $\ell \geq 1$, this is a direct consequence of the results given by Mond Pellikan in [10].

From this formula and applying the formula to compute the number of points A_2 given before, we get a formula to compute the number of points $A_{1,1}$.

Now, we consider the case of map germs from $\mathbb{C}^4$ to $\mathbb{C}^3$, here the Fitting ideals which appear in the discriminant $\Delta(f) = f(\Sigma(f)) \subset \mathbb{C}^3$ of f are $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ and they define in $\Delta(f)$ the following sets:

1. $V(\mathcal{F}_0) := \Delta(f)$ here $\mathcal{F}_0$ is the defining ideal of the discriminant.
2. $V(\mathcal{F}_1) = f(D(f)) \cup f(\Sigma^{2,1}(f))$, union of the curves of double points and the cuspidal edges of f.
3. $V(\mathcal{F}_2) = \cup A_{1,2} \cup A_{1,1,1} \cup A_3$, union of the isolated singularities in the target.

From this we get the formulae which relates these ideals and the 0-stable singularities, recently shown in [6], which are:

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^3,0)}}{\mathcal{F}_2} = \#A_{1,2} + \#A_{1,1,1} + \#A_3 \qquad (*)$$

$$\#A_{1,1,1} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(\mathcal{F}_2 : I(f(\Sigma^{\ell+1,1}(f))))} \qquad (**)$$

If f is of co-rank one we also have:

$$\#A_{1,2} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2} - \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(\mathcal{F}_2 : I(f(\Sigma^{\ell+1,1}(f))))} - \dim_{\mathbb{C}} \frac{\mathcal{O}_{\ell+3}}{I(A_3)}. \qquad (***)$$

5 On $\#A_{1,1,1}$ and $\#A_{2,1}$ for map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$

In this section we give an example to show how to compute the number of points $\#A_{1,1,1}$ and $\#A_{2,1}$ of a finitely determined co-rank one map germ $f(x, y, u, v) = (x, y, g(x, y, u, v))$, which is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3, \omega_4; d)$, d denotes the weighted degree of g with respect to the weights $(\omega_1, \omega_2, \omega_3, \omega_4)$.

The main tools to compute these numbers are the Fitting ideals associate to the discriminant of f and the relationship between these numbers, the dimension of the algebras associated to these ideals and the number of points $\#A_3$ of f given in recently in [3].

For any $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$, associated to the discriminant of f there are three Fitting ideals denoted $\mathcal{F}_0$, with $V(\mathcal{F}_0)$ is the discriminant of f, $\mathcal{F}_1$ and $\mathcal{F}_2$.

If f is weighted homogeneous, we can show that the generators of $\mathcal{F}_1$ and $\mathcal{F}_2$ are also weighted homogenous with respect to the fixed weights of f.

Another ideal which is needed to compute these relations is the ideal $J_{2,1}(f)$ and this ideal is also weighted homogeneous. Therefore we can compute these numbers in terms of the weights and the degree of f, as we see in the example below.

Example 5.1. Let $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$, defined as $f(x, y, z, w) = (x, y, w^4 + z^2 w + yz + xw)$.

The first step to compute the numbers $\#A_{1,1,1}$ and $\#A_{2,1}$ of f is the calculation of the number of points $\#A_3$ of f and from the results shown below we get that $\#A_3 = 5$.

The next step is the calculation of the Fitting ideals of the discriminant.

In this case we have that

$$\mathcal{F}_0 := \quad < 3125Y^8 - 6912X^5Y^2 - 36000X^2Y^4Z - 6912X^4Z^2 - 102400XY^2Z^3 - 65536Z^5 >$$

$$\begin{aligned} \mathcal{F}_1 := \quad & < 27X^4Z + 75XY^2Z^2 + 160Z^4, 375XY^4 - 432X^3Z + 1600Y^2Z^2, 90X^3Y^2 + 125Y^4Z + \\ & +144X^2Z^2, 27X^4Y + 75XY^3Z + 160YZ^3, 625Y^6 - 2640X^2Y^2Z - 3072XZ^3 > \\ \mathcal{F}_2 := \quad & < Z^3,\ YZ^2,\ XZ^2,\ Y^2Z,\ XYZ,\ X^2Z,\ XY^2,\ X^2Y,\ X^3,\ Y^4 > . \end{aligned}$$

We remark that these ideals define in the discriminant $\Delta(f) = f(\Sigma(f)) \subset \mathbb{C}^3$ of f the following sets:

1. $V(\mathcal{F}_0) := \Delta(f)$ here $\mathcal{F}_0$ is the defining ideal of the discriminant.
2. $V(\mathcal{F}_1) = f(D(f)) \cup f(\Sigma^{2,1}(f))$, union of the curves of double points and the cuspidal edges of f.
3. $V(\mathcal{F}_2) = \cup A_{1,2} \cup A_{1,1,1} \cup A_3$, union of the isolated singularities in the target.

The next step is to apply the formulae given in the Ph.D. Thesis of A. J. Miranda which are:

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^3,0)}}{\mathcal{F}_2} = \#A_{1,2} + \#A_{1,1,1}\#A_3 \qquad (*)$$

$$\#A_{1,1,1} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(\mathcal{F}_2 : I(f(\Sigma^{n+1,1}(f))))} \qquad (**)$$

If f is of co-rank one we also have:

$$\#A_{1,2} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2} - \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(\mathcal{F}_2 : I(f(\Sigma^{n+1,1}(f))))} - \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+3}}{I(A_3)}.(***)$$

From the fact that $\mathcal{F}_2$ and $I(f(\Sigma^{n+1,1}(f)))$ are weighted homogeneous ideals we can compute the dimensions of the algebras above in an analogous way to the section below to get:

$$\#A_{1,1,1} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(\mathcal{F}_2 : I(f(\Sigma^{n+1,1}(f))))} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{< X, Y, Z >} = 1.$$

From the equation $(**)$ we get $\#A_{1,2} := 5$

6 Appendix

Constructing a standard bases for the first syz(I).

If $R = K[x]$, $f \in R^r \setminus \{0\}$ and $>$ denote a module ordering, then we can written uniquely as $f = cx^{\alpha}e_i + f^*$, with $c \in K \setminus \{0\}$ and $x^{\alpha}e_i > x^{\alpha^*}e_j$ for any non-zero term $c^*x^{\alpha^*}e_j$ of f^*. So we define: the leading monomial by $LM(f) = x^{\alpha}e_i$, leading coefficient by $LC(f) = c$, the leading term by $LT(f) = cx^{\alpha}e_i$ and the tail of f by $tail(f) := f - LT(f)$. Also, if $G \subset R^r$ we define the leading submodule of $\langle G\rangle$ by $L(G) = \langle LM(g) : g \in G \setminus \{0\}\rangle_R$.

Now, let $G = \{f_1, \dots, f_k\}$ be a standard base (i.e., $G \subset I$ and the leading ideals coincide: $L(I) = L(G)$; we note that in the global case, it is also called Grobner base) of $I = \langle f_1, \dots, f_k\rangle$, with $f_i \in R^r \setminus \{0\}$, $\forall i \in \{1, \dots, k\}$. For each $i \neq j$ such that f_i and f_j have their leading terms in the same component, i.e., $LM(f_i) = x^{\alpha_i}e_{\nu}$, $LM(f_j) = x^{\alpha_j}e_{\nu}$, we define the monomial $m_{ji} = x^{\lambda-\alpha_i} \in R$, where $\lambda := lcm(\alpha_i, \alpha_j) := (max({\alpha_i}^1, {\alpha_j}^1), \dots, max({\alpha_i}^r, {\alpha_j}^r))$ is the least common multiple of α_i and α_j, $c_i = LC(f_i)$, $c_j = LC(f_j)$. Then the s-polynomial of f_i and f_j is given by $spoly(f_i, f_j) = m_{ji}f_i - \frac{c_i}{c_j}m_{ij}f_j$. We can assume that the $spoly(f_i, f_j)$ has a standard representation: $m_{ji}f_i - \frac{c_i}{c_j}m_{ij}f_j = \sum_{\nu-1}^{k} a_{\nu}^{(ij)}f_{\nu}$, $a_{\nu}^{(ij)} \in R$.

Now, for $i < j$ such that $LM(f_i)$ and $LM(f_j)$ involve the same component, define

$$s_{ij} = m_{ji}\epsilon_i - \frac{c_i}{c_j}m_{ij}\epsilon_j - \sum_{\nu-1}^{k} a_{\nu}^{ij}f_{\nu}$$

It is possible to show that $s_{ij} \in syg(I)$.

With these notation the construction of a base for $syz(I)$ is described below.

Theorem 6.1. [4] . *Let $G = \{h_1, \dots, h_k\}$ be a set of generators of $I \subset R^r$. Let $P := \{(i, j)\ 1 \leq i < j \leq k$ such that the leading terms of the r_i and r_j involve in the same component and let $J \subset P$.*

Assume that $NF(spoly(h_i, h_j) \mid G_{ij}) = 0$ for some $G_{ij} \subset G$, $(i, j) \in J$ and for $i = 1, \dots, r$ we have the equality

$$\langle\{m_{ij}\varepsilon_i|(i, j) \in J\}\rangle = \langle\{m_{ij}\varepsilon_i|(i, j) \in P\}\rangle.$$

Then the following statements hold: G is a standard basis of I (Buchberger's criterion) and *$S := \{s_{ij}|(i, j) \in J\}$ is a standard basis of $syz(I)$.*

Example 6.2. Let F be the finitely determined co-rank one map germ from $(C^4, 0)$ to $(C^3, 0)$ defined by:

$$F(x, y, u, v) := (x, y, G(x, y, u, v)), \text{ with } G(x, y, u, v) := yu + xv + uv^2 + u^3.$$

Note that this map is quasi-homogeneous of type $(2, 2, 1, 1; 3)$ and $J_{(2,1,1)}(F)$ has the standard base $I = \langle y, x, v^2, uv, u^2\rangle$.

We are going to get the resolution free applying the Hilbert's Syzygy Theorem.

The first Syzygy module, M_1, is given for $M_1 = \begin{pmatrix} y & x & v^2 & uv & u^2 \end{pmatrix} \in M_{1\times 5}(\mathcal{O}_4)$.

Now, we can numbering the elements of G: $g_1 = y, g_1 = x$, $g_3 = v^2$, $g_4 = uv$ and $g_5 = u^2$. We are admitting a monomial ordering, $>$, such that $LM(g_1) > LM(g_2) > LM(g_3) > LM(g_4 > LM(g_5))$.

The respective monomials $m_{ij}\epsilon_i$, $1 \leq i < j \leq 5$, are given the following table

$i\diagdown j$	2	3	4	5
1	$x\epsilon_1$	$v^2\epsilon_1$	$uv\epsilon_1$	$u^2\epsilon_1$
2	$-$	$v^2\epsilon_2$	$uv\epsilon_2$	$u^2\epsilon_2$
3	$-$	$-$	$u\epsilon_3$	$u^2\epsilon_3$
4	$-$	$-$	$-$	$u\epsilon_4$

Hence, we may choose

$$J := \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(4,5)\}$$

and compute

$$\begin{aligned}
s_{1,2} &= m_{2,1}\epsilon_1 - m_{1,2}\epsilon_2 - spoly(g_1,g_2) = x\epsilon_1 - y\epsilon_2 - 0 = (x,-y,0,0,0),\\
s_{1,3} &= m_{3,1}\epsilon_1 - m_{1,3}\epsilon_3 - spoly(g_1,g_3) = v^2\epsilon_1 - y\epsilon_3 - 0 = (v^2,0,-y,0,0),\\
s_{1,4} &= m_{4,1}\epsilon_1 - m_{1,4}\epsilon_4 - spoly(g_1,g_4) = uv\epsilon_1 - y\epsilon_4 - 0 = (uv,0,0,-y,0),\\
s_{1,5} &= m_{5,1}\epsilon_1 - m_{1,5}\epsilon_5 - spoly(g_1,g_5) = u^2\epsilon_1 - y\epsilon_5 - 0 = (u^2,0,0,0,-y),\\
s_{2,3} &= m_{3,2}\epsilon_2 - m_{2,3}\epsilon_3 - spoly(g_2,g_3) = v^2\epsilon_2 - x\epsilon_3 - 0 = (0,v^2,-x,0,0),\\
s_{2,4} &= m_{4,2}\epsilon_2 - m_{2,4}\epsilon_4 - spoly(g_2,g_4) = uv\epsilon_2 - x\epsilon_4 - 0 = (0,uv,0,-x,0),\\
s_{2,5} &= m_{5,2}\epsilon_2 - m_{2,5}\epsilon_5 - spoly(g_2,g_5) = u^2\epsilon_2 - x\epsilon_5 - 0 = (0,u^2,0,0,-x),\\
s_{3,4} &= m_{4,3}\epsilon_3 - m_{3,4}\epsilon_4 - spoly(g_3,g_4) = u\epsilon_3 - v\epsilon_4 - 0 = (0,0,u,-v,0),\\
s_{4,5} &= m_{5,4}\epsilon_4 - m_{4,5}\epsilon_5 - spoly(g_4,g_5) = u\epsilon_4 - v\epsilon_5 - 0 = (0,0,0,u,-v).
\end{aligned}$$

The set $S := \{s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}, s_{2,3}, s_{2,4}, s_{2,5}, s_{3,4}, s_{4,5}\}$ is a inter-reduced standard basis for $Syz(I) := M_2$. Therefore, by 6.1, the second Syzygy module is generated from the each column the matrix below:

$$M_2 = syz(I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x & v^2 & uv & u^2 \\ 0 & 0 & v^2 & uv & u^2 & -y & 0 & 0 & 0 \\ 0 & u & -x & 0 & 0 & 0 & -y & 0 & 0 \\ u & -v & 0 & -x & 0 & 0 & 0 & -y & 0 \\ -v & 0 & 0 & 0 & -x & 0 & 0 & 0 & -y \end{pmatrix} \in M_{5\times 9}(\mathcal{O}_4).$$

Now, we can numbering the generated of $Syz(I)$: $h_1 = (0,0,0,u,-v)$, $h_2 = (0,0,u,-v,0)$, $h_3 = (0,v^2,-x,0,0)$, $h_4 = (0,uv,0,-x,0)$, $h_5 = (0,u^2,0,0,-x)$, $h_6 = (x,-y,0,0,0)$, $h_7 = (v^2,0,-y,0,0)$, $h_8 = (uv,0,0,-y,0)$, $h_9 = (u^2,0,0,0,-y)$. So, we see that the set M of pairs (i,j), $1 \le i < j \le 9$, such that the leading monomials of the i-th an j-th element of S involve the same components consists of 7 elements: $M = \{(3,4),(4,5),(6,7),(6,8),(6,9),(7,8),(8,9)\}$, since that, the respective monomials $m_{ij}\epsilon_i$, $1 \le i < j \le 9$, are given the following tables

$i\diagdown j$	4	5
3	$u\epsilon_3$	~~$u^2\epsilon_3$~~
4	$-$	$u\epsilon_4$
5	$-$	$-$

,

$i\diagdown j$	7	8	9
6	$v^2\epsilon_6$	$uv\epsilon_6$	$u^2\epsilon_6$
7	$-$	$u\epsilon_7$	~~$u^2\epsilon_7$~~
8	$-$	$-$	$u\epsilon_8$
9	$-$	$-$	$-$

We compute,

$s^{(1)}_{3,4} = m_{4,3}\epsilon_3 - m_{3,4}\epsilon_4 - spoly(h_3, h_4) = u\epsilon_3 - v\epsilon_4 - (-x)\epsilon_2 = (0, x, u, -v, 0, 0, 0, 0, 0),$
$s^{(1)}_{4,5} = m_{5,4}\epsilon_4 - m_{4,5}\epsilon_5 - spoly(h_4, h_5) = u\epsilon_4 - v\epsilon_5 - (-x)\epsilon_1 = (x, 0, 0, u, -v, 0, 0, 0, 0),$
$s^{(1)}_{6,7} = m_{7,6}\epsilon_6 - m_{6,7}\epsilon_7 - spoly(h_6, h_7) = v^2\epsilon_6 - x\epsilon_7 - (-y)\epsilon_3 = (0, 0, y, 0, 0, v^2, -x, 0, 0),$
$s^{(1)}_{6,8} = m_{8,6}\epsilon_6 - m_{6,8}\epsilon_8 - spoly(h_6, h_8) = uv\epsilon_6 - x\epsilon_8 - (-y)\epsilon_4 = (0, 0, 0, y, 0, uv, 0, -x, 0),$
$s^{(1)}_{6,9} = m_{9,6}\epsilon_6 - m_{6,9}\epsilon_9 - spoly(h_6, h_8) = u^2\epsilon_6 - x\epsilon_9 - (-y)\epsilon_5 = (0, 0, 0, 0, y, u^2, 0, 0, -x),$
$s^{(1)}_{7,8} = m_{8,7}\epsilon_7 - m_{7,8}\epsilon_8 - spoly(h_7, h_8) = u\epsilon_7 - v\epsilon_8 - (-y)\epsilon_2 = (0, y, 0, 0, 0, 0, u, -v, 0),$
$s^{(1)}_{8,9} = m_{9,8}\epsilon_8 - m_{8,9}\epsilon_9 - spoly(h_8, h_9) = u\epsilon_8 - v\epsilon_9 - (-y)\epsilon_1 = (y, 0, 0, 0, 0, 0, 0, u, -v).$

The set $S^{(1)} := \{s^{(1)}_{3,4}, s^{(1)}_{4,5}, s^{(1)}_{6,7}, s^{(1)}_{6,8}, s^{(1)}_{6,9}, s^{(1)}_{7,8}, s^{(1)}_{8,9}\}$ is a interreduced standard basis for $Syz(Syz(I)) = Syz(M_2) := M_3$. Therefore, by 6.1, the third Syzygy module is generated from the each column the matrix below:

$$M_3 = syz(M_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & y & u & 0 & 0 & 0 \\ 0 & y & 0 & -v & 0 & u & 0 \\ y & 0 & 0 & 0 & 0 & -v & 0 \\ u^2 & uv & v^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & u & 0 & 0 \\ 0 & -x & 0 & 0 & -v & 0 & u \\ -x & 0 & 0 & 0 & 0 & 0 & -v \end{pmatrix} \in M_{9\times 7}(\mathcal{O}_4).$$

Now, numbering the generators of $Syz(Syz(I))$: $l_1 = (0, 0, 0, 0, y, u^2, 0, 0, -x)$,
$l_2 = (0, 0, 0, y, 0, uv, 0, -x, 0)$, $l_3 = (0, 0, y, 0, 0, v^2, -x, 0, 0)$, $l_4 = (0, x, u, -v, 0, 0, 0, 0, 0)$,
$l_5 = (0, y, 0, 0, 0, 0, u, -v, 0)$, $l_6 = (x, 0, 0, u, -v, 0, 0, 0, 0)$, $l_7 = (y, 0, 0, 0, 0, 0, 0, u, -v)$.
So, we see that the set N of pairs (i, j), $1 \le i < j \le 7$, such that the leading monomials of the i-th an j-th element of $S^{(1)}$ involve the same components consists of 2 elements: $N = \{(4, 5), (6, 7)\}$, since that, the respective monomials $m_{ij}\epsilon_i$, $1 \le i < j \le 7$, are given the following tables

$i \diagdown j$	5
4	$y\epsilon_4$
5	–

,

$i \diagdown j$	7
6	$y\epsilon_6$
7	–

We compute,

$s^{(2)}_{4,5} = m_{5,4}\epsilon_4 - m_{4,5}\epsilon_5 - spoly(l_4, l_5) = y\epsilon_4 - x\epsilon_5 + u\epsilon_3 - v\epsilon_2 = (0, -v, u, y, -x, 0, 0),$
$s^{(2)}_{6,7} = m_{7,6}\epsilon_6 - m_{6,7}\epsilon_7 - spoly(l_6, l_7) = y\epsilon_6 - x\epsilon_7 + u\epsilon_2 - v\epsilon_1 = (-v, u, 0, 0, 0, y, -x),$

The set $S^{(2)} := \{s^{(2)}_{4,5}, s^{(2)}_{6,7}\}$ is a inter-reduced standard basis for $Syz(Syz(Syz(I))) = Syz(M_3) := M_4$. Therefore, by 6.1, the fourth Syzygy module is generated from the each column the matrix below:

The fourth Syzygy module $M_4 = syz(M_3) = \begin{pmatrix} 0 & -v \\ -v & u \\ u & 0 \\ y & 0 \\ -x & 0 \\ 0 & y \\ 0 & -x \end{pmatrix} \in M_{7\times 2}(\mathcal{O}_4)$.

Thus, we obtained the following free resolution for $\dfrac{\mathcal{O}_4}{J_{(2,1,1)}(f)}$,

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4 \to^{M_4} \bigoplus_{i=1}^{7} \mathcal{O}_4 \to^{M_3} \bigoplus_{i=1}^{9} \mathcal{O}_4 \to^{M_2} \bigoplus_{i=1}^{5} \mathcal{O}_4 \to^{M_1} \mathcal{O}_4 \to^{\pi} \frac{\mathcal{O}_4}{J_{(2,1,1)}(f)} \to 0,$$

where $M_i = syz(M_{i-1})$ and π the natural projection.

References

[1] T. Fukui, J. J. Nuño Balesteros, M. J. Saia, *On the number of singularities of generic deformations of map germs*, J. of The London Mathematical Society, London (2) **58**, 141–152, (1998).

[2] T. Gaffney, D. M. Q. Mond, *Cusps and double folds of germs of analytic maps* $\mathbb{C}^2 \to \mathbb{C}^2$,

[3] T. Gaffney, D. M. Q. Mond, *Weighted homogeneous maps from plane to plane*, Math. Proc. Camb. Phil. Soc., **109**, 451–470, (1991).

[4] G. M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, Second Edition, Springer, (2007).

[5] W.L. Marar, J.A. Montaldi, M.A.S. Ruas, *Multiplicities of zero-schemes in quasi-homogeneous corank-1 singularities* $\mathbb{C}^n \to \mathbb{C}^n$, Singularity Theory, London Math. Soc. Lecture Note Ser., **263**, 353–367, Cambrigde Univ. Press, Cambrigde, (1999).

[6] A. J. Miranda, *Invariantes de Germes de Aplicaes de* $\mathbb{C}^{n+m}$ *em* $\mathbb{C}^m$ *e Ideais de Fitting* Tese de doutorado, ICMC-USP, 2009.

[7] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space*, Topology, vol. 26 No. 3, 1987, 361-383.

[8] D. Mond, *Vanishing cycles for analytic maps*, Singularity theory and applications, Warwick 1989, Springer Lecture Notes in Mathematics 1462, 1991, 221-234.

[9] D. Mond, *The number of vanishing cycles for a quasihomogeneous mapping from* $\mathbb{C}^2$ *to* $\mathbb{C}^3$, Quart. J. Math., Oxford (2), **42**, 335–345, (1991).

[10] D. Mond and R. Pellikan, *Fitting ideals and multiple points of analytic mappings.* Algebraic geometry and complex analysis (Pátzcuaro, 1987), 107–161, Lecture Notes in Math., 1414, Springer, Berlin, 1989.

[11] Morim,

[12] T. Gaffney, D. M. Q. Mond, *Cusps and double folds of germs of analytic maps* $\mathbb{C}^2 \to \mathbb{C}^2$,

[13] T. Gaffney, D. M. Q. Mond, *Cusps and double folds of germs of analytic maps* $\mathbb{C}^2 \to \mathbb{C}^2$,

[14] SINGULAR *A Computer Algebra System for Polynomial Computations*, version 3-0-3, 2007, by: G.-M. Greuel, G. Pfister, H. Schoenemann, FB Mathematik der Universitaet, D-67653, Kaiserslautern, (2007).

Universidade Federal de Alfenas, Campus de Varginha, Varginha, M.G.
E-mail: aldicio@icmc.usp.br
Departamento de Matemática, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista "Júlio Mesquita Filho" - Campus de Rio Claro, Caixa Postal 178, 13506-700 Rio Claro, SP, Brazil.
E-mail: eliris@rc.unesp.br

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil.
E-mail: mjsaia@icmc.usp.br'