On the Morse Complex of Complexified Arrangements

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June 25, 2009

Anatoly Libgober 60th Birthday
Jaca, Spain

We consider the following situation:

Let $\mathcal{A} := \{H\}$ be an *arrangement of hyperplanes* (locally finite, affine).

Let

$$\mathcal{M}(\mathcal{A}) \ := \ \mathbb{C}^n \ \setminus \cup_{H \in \mathcal{A}} H$$

Motivations:

- singularities (e.g., simple singularities), Milnor fibers
- Braid groups and generalizations (Artin groups, Coxeter groups)
- $\mathcal{M}_{0,n}$
- Configuration Spaces
- Combinatorics
- Root systems, splines, partition functions, hypergeometric functions,
- etc.

Some well known facts

some well-known facts

1. $H^*(M(\mathcal{A}); \mathbb{Z})$ = Orlik-Solomon algebra $\dfrac{E}{I}$
   it is a free $\mathbb{Z}$-module

   associate lattice of intersections $L(\mathcal{A})$

   then: $H^*(M(\mathcal{A}); \mathbb{Z})$ depends only on $L(\mathcal{A})$.

   i.e. $L(\mathcal{A}) \cong L(\mathcal{A}') \Rightarrow H^*(M(\mathcal{A}); \mathbb{Z}) \cong H^*(M(\mathcal{A}'); \mathbb{Z})$

2. $L(\mathcal{A}) \cong L(\mathcal{A}') \not\Rightarrow M(\mathcal{A}) \sim M(\mathcal{A}')$

   (counterexample by Rybnikov, where $\pi_1(M(\mathcal{A})) \not\cong \pi_1(M(\mathcal{A}'))$)

[rem $\Rightarrow$ true for special lattices]

3. Deligne thm: assume:

   (i) all $H \in \mathcal{A}$ are defined over $\mathbb{R}$

   (ii) the connected components of $\mathbb{R}^n \setminus \bigcup H$ (chambers) are simplicial cones

   then: $M(\mathcal{A})$ is a $K(\pi, 1)$ - space

__Local systems on $\mathcal{M}(\mathcal{A})$__ $\longleftrightarrow$ $\pi_1(\mathcal{M}(\mathcal{A}))$ - modules R

Try to calculate $\boxed{H^*(\mathcal{M}(\mathcal{A}); R)}$

abelian local systems factor through H_1 $(\cong \mathbb{Z}^{|\mathcal{A}|})$

so they are given by a collection $t = \{t_H \ , H \in \mathcal{A}, \ t_H \in \mathrm{Aut}(R)\}$

__Thm__ Assume $\mathcal{A}$ real. Then for generic $t = \{t_H \in \mathbb{C}^*\}$

$$H^k(\mathcal{M}(\mathcal{A}); \mathbb{C}_t) = 0 \quad \text{if } k < n$$

$$\dim H^n(\mathcal{M}(\mathcal{A}); \mathbb{C}_t) = \# \text{ bounded chambers}$$

__Characteristic varieties:__ $t \in (\mathbb{C}^*)^{|\mathcal{A}|}$ s.t. dimensions jump

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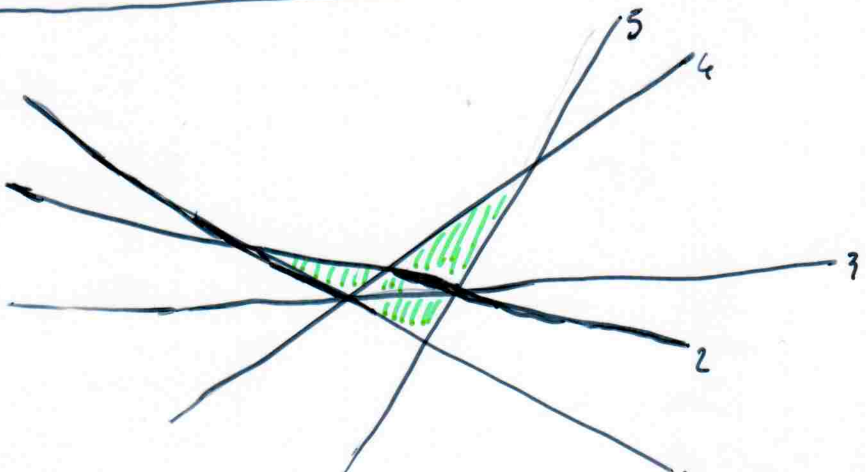


$\dim H_2(\mathcal{M}(\mathcal{A}), \mathbb{C}_t) = 4$ for generic $t = (t_1, t_2, t_3, t_4, t_5)$

Among local systems, it is particularly interesting the case of the π_1-module $R[t^{\pm 1}]$, R any ring, with action:

$$\boxed{\text{elem. loop around hyperplane} \longrightarrow t \text{. multiplication}}$$

In fact

$$H^*(\mathcal{M}(\mathcal{A}); R[t^{\pm 1}]) \cong H^*(F; R) \qquad F: \text{Milnor fiber}$$

as $R[t^{\pm}]$-modules, where t-action on the right is given by the classical monodromy.

Many computations where done for <u>Coxeter arrangements</u>

In this case one has <u>orbit space</u>

$$\mathcal{M}(\mathcal{A})/W \quad , W \text{ reflection group}$$

and $$\pi_1\left(\mathcal{M}(\mathcal{A})/W\right) = \text{Artin group of type } W$$

$$\underset{\substack{\uparrow \\ \text{braid group}}}{Br_n} = \pi_1\left(\mathcal{M}(\mathcal{A})/\Sigma_n\right) \qquad \mathcal{A} = \{z_i = z_j\}$$

Case A_n : classical works by Arnold, Brieskorn, ecc.

over $\mathbb{Q}$ more recently F. Cohen, Vainshtein, Frenkel, Salvetti '94

- D. Cohen-Suciu; DeConcini-[redacted]-Procesi-Sal. - '02 (Topol)
- F. Callegaro (2006) Alg. Geom. Topol (complete calculation over $\mathbb{Z}$)

Case B_n, D_n : De Concini-Procesi-Salvetti '02

Exceptional groups : Callegaro-Sal. '04

Case $\tilde{A}_n$: Callegaro-Moroni-Sal. '08 Trans. Amer. Math. Soc

Case $\tilde{B}_n$: Callegaro-Moroni-Sal,
"The $K(\pi,1)$-problem for Artin groups of type $\tilde{B}_n$ and related cohomologies"
to appear in JEMS

- A. Dimca - S. Papadima "Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements", Ann. of Math. (2003)

- R. Randell, "Morse theory, Milnor fibers and minimality of a complex hyperplane arrangement", Proc. Amer. Math. Soc. (2002)

- M. Yoshinaga, "Hyperplane arrangements and Lefschetz's hyperplane section theorem", Kodai Math. J. (2007)

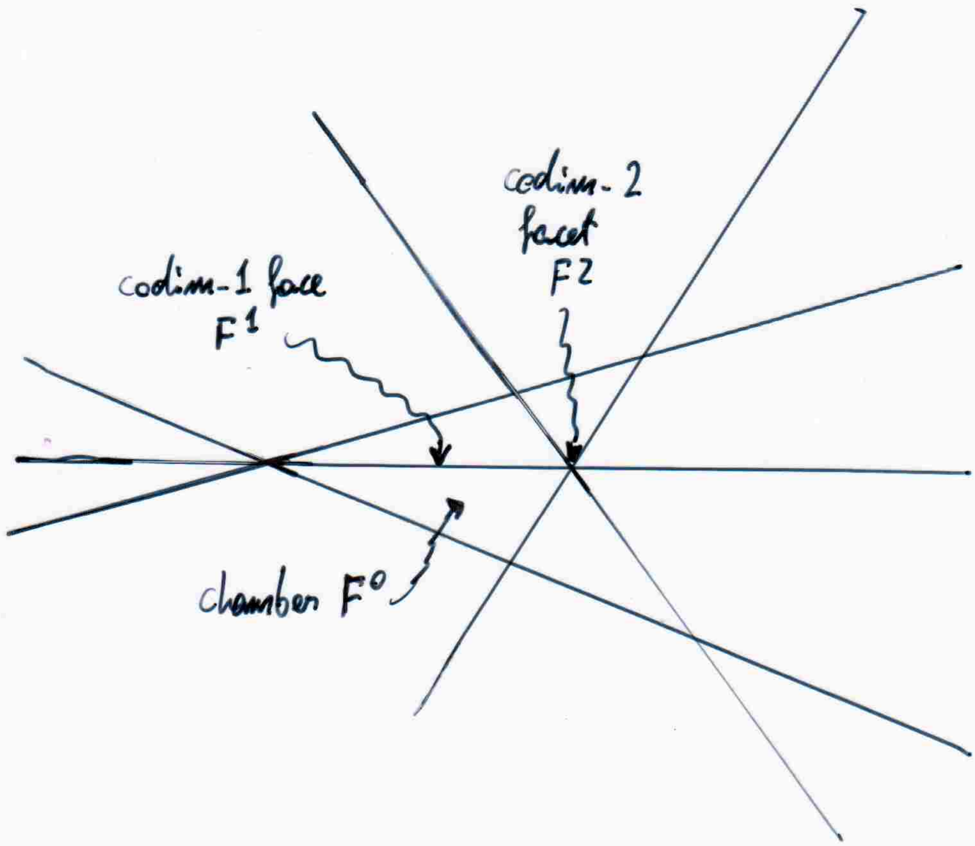
- Sal. - S. Settepanella "Combinatorial Morse theory and minimality of hyperplane arrangements", Geom. & Topol., 2007

- G. Gaiffi - Sal. "The Morse complex of a line arrangement", Jour. of Alg., '09

- E. Delucchi "Shelling-type orderings of Regular CW-complexes and Acyclic Matchings of the Salvetti complex" IMRN, '08

- E. Delucchi, S. Settepanella "Combinatorial polar orderings and follow-up arrangements", to appear in J. Comb.

If $\mathcal{A}$ is defined over $\mathbb{R}$, there is more structure: so called "oriented matroid". [Thm $OM(\mathcal{A}) \cong OM(\mathcal{A}') \Rightarrow M(\mathcal{A}_{\mathbb{C}}) \cong M(\mathcal{A}'_{\mathbb{C}})$]



$\mathcal{S} = (\{F^k\}, \prec)$ stratification of $\mathbb{R}^n$ induced by the arrangement $\mathcal{A} = \{H\}$

$$F \prec G \quad \text{iff} \quad G \subset \operatorname{clos}(F)$$

We want to find a discrete Morse function or a discrete gradient vector field over the Salvetti complex $S \simeq M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$ with minimal numbers of critical cells.

Recall:

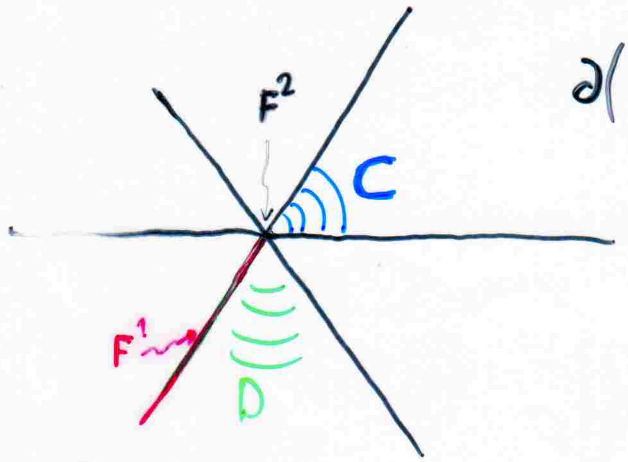
1) $\{k\text{-cells of } S\} \longleftrightarrow \{\text{pairs } [C \prec F^k]\}$

2) a cell $[D \prec G]$ is in the boundary of $[C \prec F]$ iff

i) $G \prec F$ and [redacted]

ii) $D = C.G$

where $C.G$ means the unique chamber of $\mathcal{A}$ containing G in its closure and being in the same chamber of $\mathcal{A}_{|G|} = \{H \in \mathcal{A} \mid G \subset H\}$ as the chamber C



$$\partial([C \prec F^2]) \supset [D \prec F^1]$$

2-cell   1-cell

Discrete Morse functions [redacted]

Let $K = \{\sigma\}$ be finite regular CW-complex.

<u>def</u>. a discrete Morse function over K is a function

$$f : K \longrightarrow \mathbb{R}$$

satisfying $\forall \sigma^{(p)} \in K$

$$\text{(i)} \ \# \{ \tau^{(p+1)} > \sigma^{(p)} \mid f(\tau^{(p+1)}) \le f(\sigma^{(p)}) \} \le 1$$

$$\text{(ii)} \ \# \{ \tau^{(p-1)} < \sigma^{(p)} \mid f(\sigma^{(p)}) \le f(\tau^{(p-1)}) \} \le 1$$

[redacted]

[<u>rmk</u>: at least one of (i), (ii) is 0, $\forall \sigma$]

Gradient vector fields.

f discrete Morse function on K. The

<u>def</u> discrete gradient vector field V_f of f is

$$V_f = \{ (\sigma^{(p)}, \tau^{(p+1)}) \mid \sigma^{(p)} < \tau^{(p+1)}, \ f(\tau^{(p+1)}) \le f(\sigma^{(p)}) \}$$

<u>rmk</u> each cell belongs to at most one pair of V_f.

more generally:

<u>def</u> A discrete vector field Φ on K is a collection of pairs

$$(\sigma^{(p)}, \tau^{(p+1)}) \in K \times K \quad , \ \sigma^{(p)} < \tau^{(p+1)}$$

such that $\forall \sigma \in K$, σ belongs to at most one pair of Φ.

<u>def</u> A Φ-path is a sequence:

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \tau_2^{(p+1)}, \ \text{-----}, \ \tau_r^{(p+1)}, \sigma_{r+1}^{(p)}$$

such that $\forall i$: $\quad (\sigma_i^{(p)}, \tau_i^{(p+1)}) \in \Phi$ and

$$\sigma_i^{(p)} \neq \sigma_{i+1}^{(p)} < \tau_i^{(p+1)}.$$

<u>Thm</u> A discrete vector field Φ is the gradient vector field of a discrete Morse function iff there are no nontrivial closed Φ-paths.

R. Forman '98

critical point of index p

$\rightsquigarrow$ $\boxed{\text{critical cell } \sigma^p \text{ of dimension } p}$

σ^p is critical $\Leftrightarrow$ $\sigma^p \notin V_f$ $\Leftrightarrow$ $f(\sigma^{p+1}) > f(\sigma^p)$ if $\sigma^p < \sigma^{p+1}$

$f(\sigma^{p-1}) < f(\sigma^p)$ if $\sigma^{p-1} < \sigma$

Classical Morse theory is reproduced, in particular:

- the homotopy type of the level set $f^{-1}(-\infty, a]$ changes only when a crosses a critical value (= value of critical cell)

- the level set is obtained by attaching critical cells up to homotopy

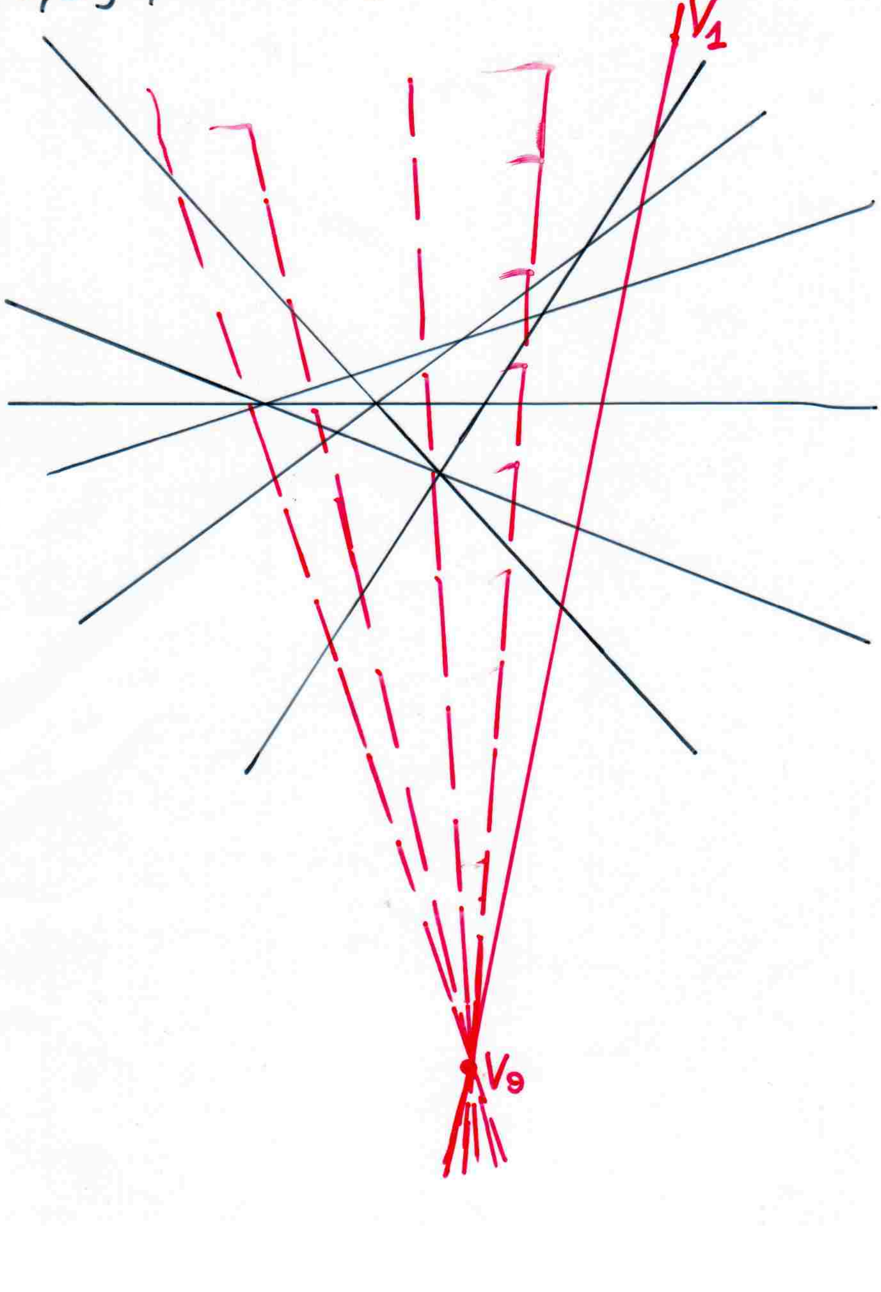
- the homology and cohomology are obtained by a "Morse complex"

Take a complete **flag**

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \quad , \dim V_i = i$$

which is **generic** w.r.t. $\mathcal{A}$:

it means that moving V_i inside a 1-parameter family of subspaces $V_i(\theta) \subset V_{i+1}$, around V_{i-1} , $\theta \in [0, \delta]$, remains transverse to the arrangement.



$V = V_0, \dots, V_n$ generic flag gives a system of

polar coordinates $P \equiv (\theta_0, \dots, \theta_{n-1})$

where $\theta_0 = \|OP\|$ and

θ_i are suitable defined angles.

Proposition. For all $F \in \mathcal{S}$ it is well-defined the first point of F, $P(F) \in F$.

Theorem [redacted] To the generic polar system of coordinates is associated a total ordering of the facets $\mathcal{S}$

$F \triangleleft G$ iff either $P(F) \neq P(G)$ and the coordinates of $P(F)$ are lower than those of $P(G)$ (w.r.t. anti-lex ordering)

or $P(F) = P(G)$ but "moving" the subspace $V_i(\theta)$ which contains $P(F)$, $F \cap V(\theta+\varepsilon) \triangleleft G \cap V(\theta+\varepsilon)$.

Call $\triangleleft$ the polar ordering of $\mathcal{S}$

Def [Polar Gradient] Define a discrete v.f. Φ over S by: in dimension $j+1$ it is given by all pairs

$$([C \prec F^j], [C \prec F^{j+1}]) \qquad F^j \prec F^{j+1} \quad \text{(same } C\text{)}$$

where

- $F^{j+1} \vartriangleleft F^j$
- $\forall\, F^{j-1} \prec F^j$ s.t. $C \prec F^{j-1}$ the pair $([C \prec F^{j-1}], [C \prec F^j]) \notin \Phi_j$

Theorem [II] (i) Φ is a combinatorial vector field on S which is the gradient of a discrete Morse function;

(ii) Φ is given (non-recursively) in terms of $\prec, \vartriangleleft$ by:

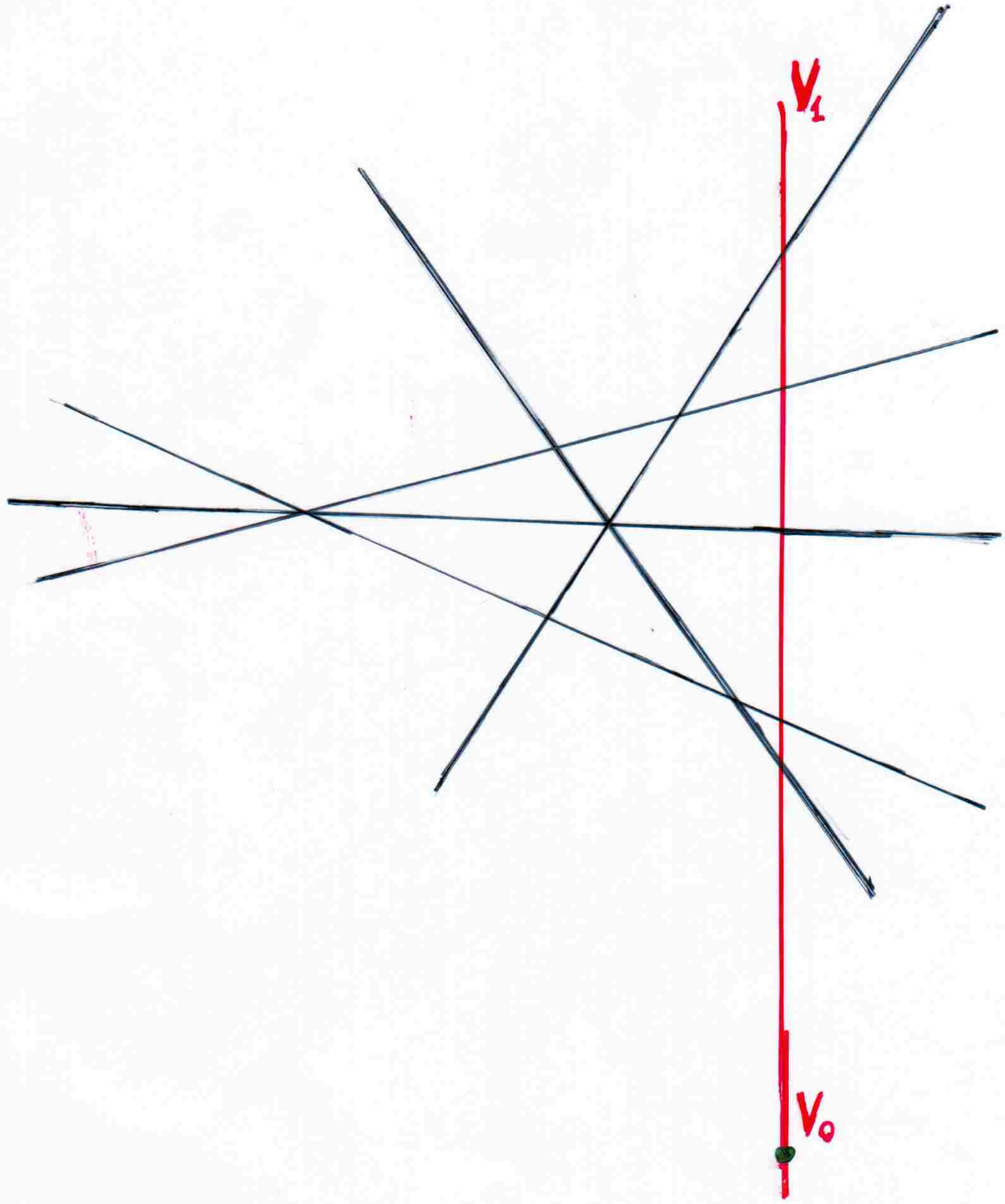
the pair $([C \prec F^j], [C \prec F^{j+1}])$, $F^j \prec F^{j+1}$ belongs to Φ iff:

(a) $F^{j+1} \vartriangleleft F^j$

(b) $\forall\, F^{j-1}$ s.t. $C \prec F^{j-1} \prec F^j$ one has $F^{j-1} \vartriangleleft F^j$.

(iii) The set of k-dimensional singular cells is

$$\mathrm{Sing}_k(S) = \{ [C \prec F^k] \mid \text{(i) } F^k \vartriangleleft F^{k+1},\ \forall F^{k+1} \text{ s.t. } F^k \prec F^{k+1} \\ \text{(ii) } F^{k-1} \vartriangleleft F^k,\ \forall F^{k-1} \text{ s.t. } C \prec F^{k-1} \prec F^k \}$$



critical 2-cells : $[C<P]$ s.t. P is maximal among all cel
of $\overline{C}$

critical 1-cells : $[C<F]$ F 1-dim facet which intersects V

critical 0-cell : only $V_0 = [C_0 < C_0]$.

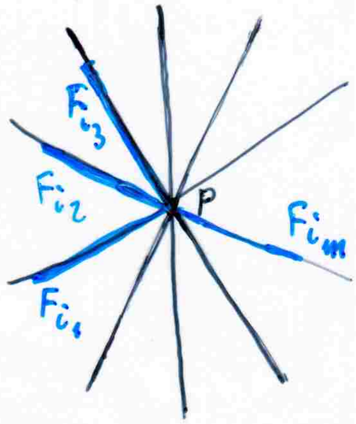
Def Given $P \in \mathcal{S}$ define a block as a sequence

$$\mathcal{F}(P) = (F_{i_1}, \dots, F_{i_{m-1}}, F_{i_m})$$

of 1-facets s.t. $F_{i_j} \prec P$ and $P \lhd F_{i_j}$ $1 \leq j \leq m-1$

$F_{i_m} \lhd P$

and $F_{i_1} \lhd \dots \lhd F_{i_{m-1}}$



We can compose two blocks:

$$\mathcal{F}(P)\,\mathcal{F}(P') = (F_{i_1}, \dots, F_{i_m}, F'_{j_1}, \dots, F'_{j_{m'}})$$

when $F_{i_m} \prec P'$

Def. Given a critical 2-cell $[C \prec P]$ and critical 1-cell $[D \prec G]$ define an admissible 1-sequence for the given pair a sequence

$$\mathcal{F} = (F_{i_1}, \dots, F_{i_h}) \quad \text{obtained by}$$

composition of blocks $\mathcal{F}(P_{j_1}) \cdots \mathcal{F}(P_{j_s})$ such that

(a) $P_{j_1} = P$; (b) $F_{i_h} = G$ and $C . F_{i_1} . \cdots . F_{i_h} = D$

(c) $\forall j$ $C . F_{i_1} . \cdots . F_{i_j} \lhd F_{i_j}$

We want to describe explicitly the Morse complex.

In case of $\mathbb{Z}$-coefficients, the theory says that it is given by:

$$\to \mathrm{Sing}_{n+1}(S) \xrightarrow{\partial_{n+1}} \mathrm{Sing}_n(S) \xrightarrow{\partial_n} \mathrm{Sing}_{n-1}(S) \to$$

where

$$\partial_n(e^n) = \sum_{f^{n-1} \in \mathrm{Sing}_{n-1}} \left(\sum_u (-1)^{|u|} \right) . f^{n-1}$$

where u is an alternating path between e^n and f^{n-1}, and $|u|$ is the # pairs in Φ contained in u.

For non-trivial cohomology, one has to modify by

$$\partial_n(e^n) = \sum_{f^{n-1}} \left(\sum_u (-1)^{|u|} u_* \right) . f^{n-1}$$

Standard to associate to an admissible sequence homotopy classes of paths in $\mathcal{M}(\mathcal{A})$:

$$s = (F_{i_1}, \ldots, F_{i_h}) \rightsquigarrow u(s) \in \pi_1(\mathcal{M}(\mathcal{A}), P_0)$$

(Sal-Settepanella)

<u>Thm</u> [redacted] The boundary operator ∂_K [redacted] is

$$\partial_K(\ell . e_{[C \prec F^k]}) = \sum A^{[C \prec F^k]}_{[D \prec G^{k-1}]}(\ell) . e_{[D \prec G^{k-1}]}$$

where the coefficient is

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \sum_{s \in \text{Adm. Seq}} (-1)^{\ell(s) - b(s)} \, u(s)_*$$

Here coefficients are over a $\pi_1(\mathcal{M}(\mathcal{A}))$-module L and $\ell \in L$, (so we are computing

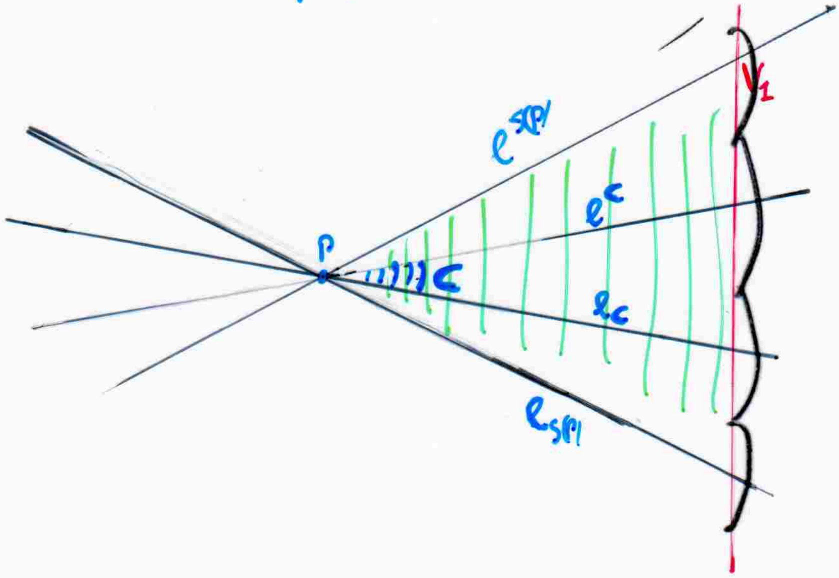
$$H_*(\mathcal{M}(\mathcal{A}); L)$$

<u>Rmk</u> 1). Thm holds for any local system.

2) For abelian local systems there are "a priori" semplifications.

__Notations__ - $S(P) := \{ \ell \in \mathcal{A} \mid P \in \ell \}$

- $\mathrm{Cone}(P)$:= closed cone delimited by the two lines $\ell_{S(P)}, \ell^{S(P)}$ with minimal and maximal index resp., and which intersects V_1 in bounded cham



- if $[C \prec P]$ in a critical 2-cell, let

$$U(C) := \{ \ell_i \in S(P) \mid i \geq \text{index of } \ell^C \}$$

$$L(C) := \{ \ell_i \in S(P) \mid i \leq \text{index of } \ell_C \}$$

- $U(P) := \{ \ell \in \mathcal{A} \mid P$ is "below" $\ell \}$

Let now $L = L(t_1, \dots, t_n)$ be an abelian $\pi_1(M(\mathcal{A}))$-mod. where $t_i \in \mathrm{Aut}(L)$ is associated to an elementary loop around the line ℓ_i (well ordered).

(Journ. of Algebra, '03)

Thm [Caiffi - S.] Let $[C_0 < F_1], [C_1 < F_2], \dots, [C_{n-1} < F_n]$ be the critical 1-cells of C_*, where $\forall i \;\; F_i \subset \ell_i$.

Given a critical 2-cell $[C \prec P]$, the 2-boundary of the Morse complex is given by:

$$\partial_2\left(\ell . \, e_{[C \prec P]}\right) =$$

$$= \sum_{\substack{[C_{j-1} \prec F_j] \\ \text{s.t. } |F_j| \in S(P)}} \Bigg(\prod_{\substack{i<j \text{ s.t.} \\ \ell_i \in U(P)}} t_i\Bigg)\Bigg[\prod_{\substack{i \text{ s.t.} \\ \ell_i \in [C \to |F_j|]}} t_i - \prod_{\substack{i<j \text{ s.t.} \\ \ell_i \in S(P)}} t_i\Bigg](\ell) . \, e_{[C_{j-1} \prec F_j]}$$

$$+ \sum_{\substack{[C_{j-1} \prec F_j] \text{ s.t.} \\ |F_j| \in U(P) \\ F_j \subset Cone(P)}} \Bigg[\prod_{\substack{i<j \text{ s.t.} \\ \ell_i \in U(P)}} t_i\Bigg]\Bigg(1 - \prod_{\substack{i<j \text{ s.t.} \\ \ell_i \in L(C)}} t_i\Bigg)\Bigg(\prod_{\substack{i<j \text{ s.t.} \\ \ell_i \in U(C)}} t_i - \prod_{\substack{i \text{ s.t.} \\ \ell_i \in U(C)}} t_i\Bigg)(\ell) . \, e_{[C_{j-1} \prec F_j]}$$

Here $[C \to |F_j|]$ means

$$\{\ell_k \in U(C) \mid k < j\} \quad \text{if } |F_j| \in U(C)$$

$$\{\ell_k \in S(P) \mid k < j\} \cup U(C) \quad \text{if } |F_j| \in L(C)$$

[[illegible]: if $\prod$ is empty then it is 1 by convention]

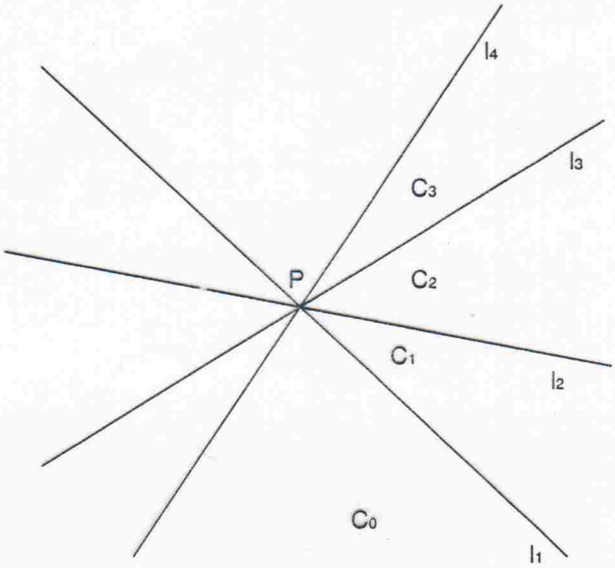
In $\mathcal{S}$ we find three 2-cells:

$$[C_1 \prec P], [C_2 \prec P], [C_3 \prec P]$$

four 1-cells:

$$[C_0 \prec F_1], [C_1 \prec F_2], [C_2 \prec F_3], [C_3 \prec F_4]$$

and the 0-cell $[C_0 \prec C_0]$.



$$[\partial_2] = \begin{array}{c} \begin{array}{ccc} [C_1 \prec P] & [C_2 \prec P] & [C_3 \prec P] \end{array} \\ \begin{pmatrix} t_2t_3t_4 - 1 & t_3t_4 - 1 & t_4 - 1 \\ 1 - t_1 & t_1t_3t_4 - t_1 & t_1t_4 - t_1 \\ t_2 - t_1t_2 & 1 - t_1t_2 & t_1t_2t_4 - t_1t_2 \\ t_2t_3 - t_1t_2t_3 & t_3 - t_1t_2t_3 & 1 - t_1t_2t_3 \end{pmatrix} \end{array}$$

$$\simeq \begin{pmatrix} t_2t_3t_4 - 1 & t_3t_4 - 1 & t_4 - 1 \\ 1 - t_1t_2t_3t_4 & 0 & 0 \\ 0 & 1 - t_1t_2t_3t_4 & 0 \\ 0 & 0 & 1 - t_1t_2t_3t_4 \end{pmatrix}$$

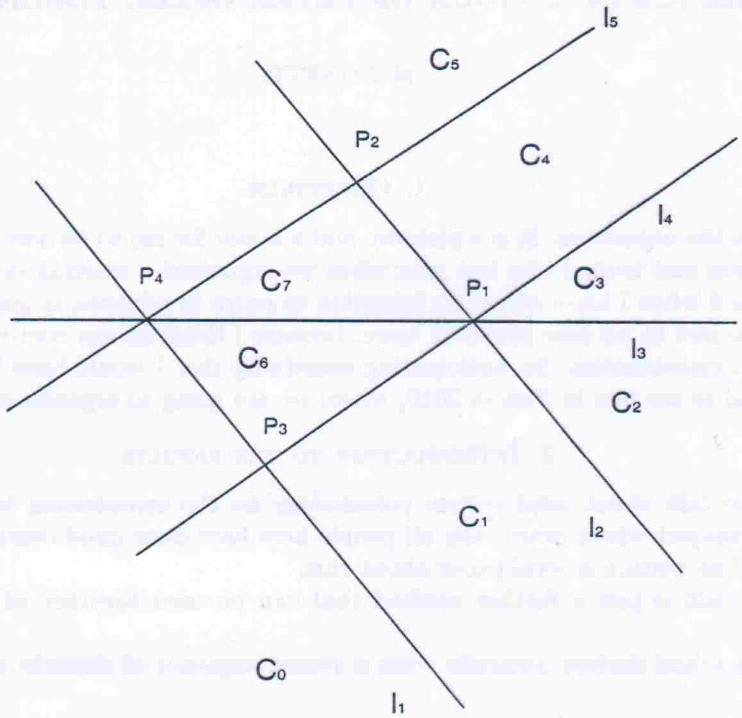
The characteristic variety $V_1^1(\mathcal{A})$ is therefore equal to

$$V_1^1(\mathcal{A}) = \{(t_1, t_2, t_3, t_4) \in \mathbb{C}^* \mid t_1t_2t_3t_4 = 1\}$$

$H_2(M(\mathcal{A}); \mathbb{Z}[t^{\pm 1}]) \cong 0$; $H_1(M(\mathcal{A}); \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z} \oplus \left(\frac{\mathbb{Z}[t^{\pm 1}]}{(t^n - 1)}\right)^{n-2}$

$H_0(M(\mathcal{A}); \mathbb{Z}[t^{\pm 1}]) \cong \frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)} \cong \mathbb{Z}$

3. Example: deconing A_3



$$
\begin{array}{cccccc}
[C_1 \prec P_3] & [C_2 \prec P_1] & [C_3 \prec P_1] & [C_4 \prec P_2] & [C_6 \prec P_4] & [C_7 \prec P_4] \\
\end{array}
$$

$$
\begin{pmatrix}
t_4 - 1 & 0 & 0 & 0 & t_3t_5 - 1 & t_5 - 1 \\
1 - t_1 - t_4 + t_1t_4 & t_3t_4 - 1 & t_4 - 1 & t_5 - 1 & 1 - t_3t_5 - t_1 + t_1t_3t_5 & 1 - t_5 - t_1 + t_1t_5 \\
t_2(1 - t_1 - t_4 + t_1t_4) & 1 - t_2 & t_2(t_4 - 1) & 0 & t_2(1 - t_1) & t_1t_2(t_5 - 1) \\
t_2t_3(1 - t_1) & t_3(1 - t_2) & 1 - t_2t_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - t_2 & t_2t_3(1 - t_1) & t_2(1 - t_1t_3)
\end{pmatrix}
$$

The five irreducible components of the characteristic variety $V_1^1(\mathcal{A})$ are: $t_1 = t_2 = t_3 = 1$, $t_2 = t_4 = t_1t_3t_5 = 1$, $t_1 = t_5 = t_2t_3t_4 = 1$, $t_3 = t_4 = t_5 = 1$ and

$$(t_1 = t_4) \cap (t_2 = t_5) \cap (t_3t_4t_5 = 1).$$

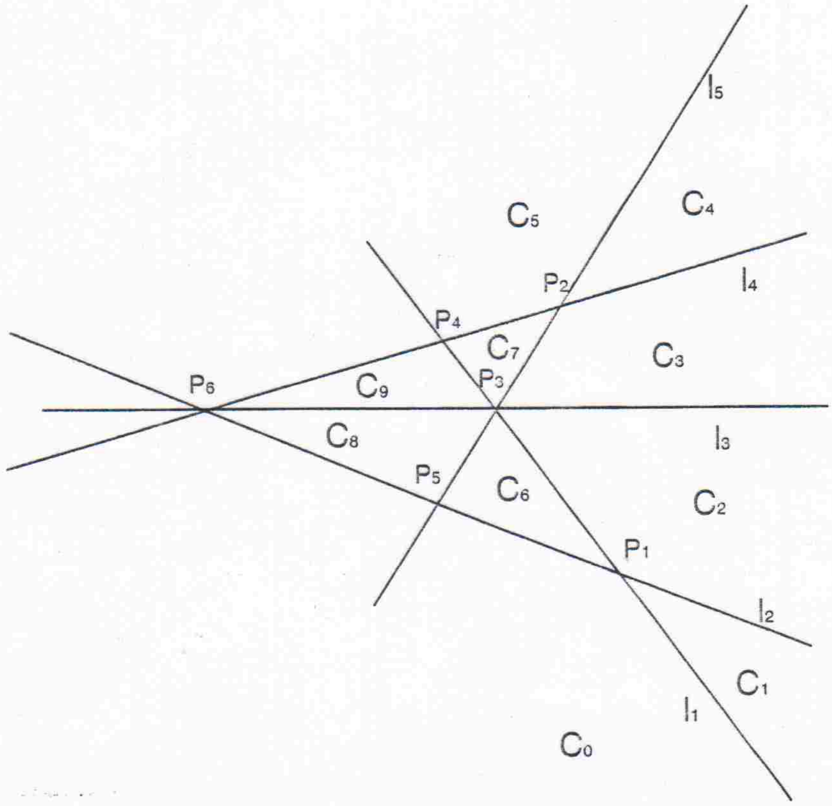
Specializing to the case when $t_1 = t_2 = \cdots = t_n = t$ we obtain:

$$H_2(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \left(\mathbb{Z}[t^{\pm 1}]\right)^2$$

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \left(\frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)}\right)^3 \oplus \frac{\mathbb{Z}[t^{\pm 1}]}{(t^3-1)} \cong \mathbb{Z}^3 \oplus \frac{\mathbb{Z}[t^{\pm 1}]}{(t^3-1)}$$

$$H_0(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)} \cong \mathbb{Z}$$

2. Example: two connected triple points



$$\begin{pmatrix}
[C_1 \prec P_1] & [C_2 \prec P_3] & [C_3 \prec P_3] & [C_4 \prec P_2] & [C_6 \prec P_5] & [C_7 \prec P_4] & [C_8 \prec P_6] & [C_9 \prec P_6] \\
t_2 - 1 & t_3 t_5 - 1 & t_5 - 1 & 0 & 0 & t_4 - 1 & 0 & 0 \\
1 - t_1 & 0 & 0 & 0 & t_1(t_5 - 1) & 0 & t_1(t_3 t_4 - 1) & t_1(t_4 - 1) \\
0 & 1 - t_1 & t_1(t_5 - 1) & 0 & t_1\ c(t_2, t_5) & 0 & t_1(1 - t_2) & t_1 t_2(t_4 - 1) \\
0 & t_3\ a(t_1, t_5) & b(t_1, t_3, t_5) & t_5 - 1 & t_1 t_3\ c(t_2, t_5) & 1 - t_1 & t_1 t_3(1 - t_2) & t_1(1 - t_2 t_3) \\
0 & t_3 t_4(1 - t_1) & t_4(1 - t_1 t_3) & 1 - t_4 & t_1 t_3 t_4(1 - t_2) & 0 & 0 & 0
\end{pmatrix}$$

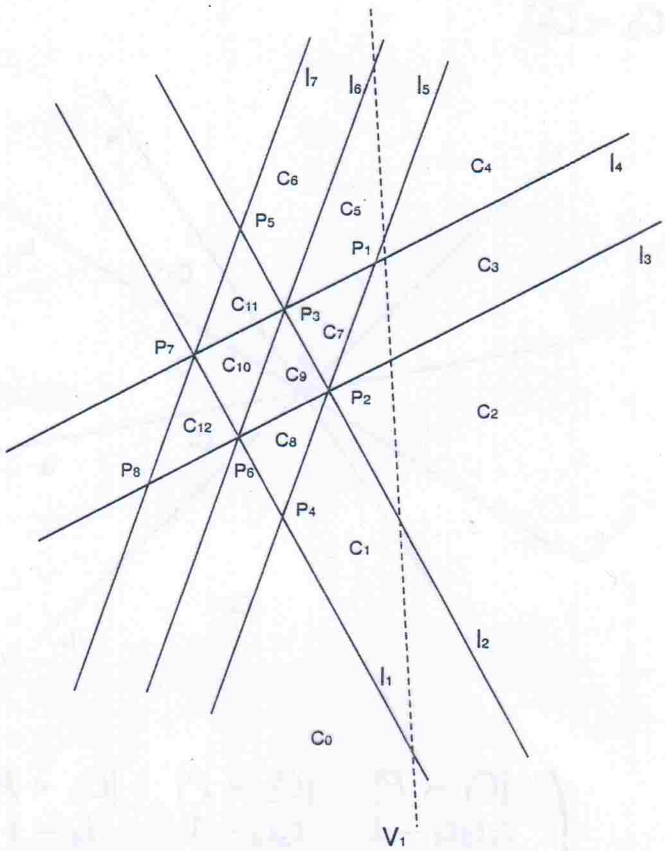
where $a(t_1, t_5) = (1 - t_1)(1 - t_5)$, $b(t_1, t_3, t_5) = (1 - t_1)(1 - t_3 - t_5)$, $c(t_2, t_5) = (1 - t_2)(1 - t_5)$.

The components of $V_1^1(\mathcal{A})$ are

$$t_1 = t_5 = t_2 t_3 t_4 = 1$$

$$t_2 = t_4 = t_1 t_3 t_5 = 1$$

4. Example: deconing the deleted B_3 arrangement



$$t_1 - t_4 = 0,\ t_2 + t_4t_7 = 0,\ t_3 + t_4t_7 = 0,\ t_4^2 - t_5 = 0,\ t_5t_7 - 1 = 0,\ t_6 + 1 = 0$$
$$t_1t_6t_7 - 1 = 0,\ t_2 - 1 = 0,\ t_3 - t_7 = 0,\ t_4 - t_6 = 0,\ t_5 - 1 = 0$$
$$t_1 - t_6 = 0,\ t_2 - t_7 = 0,\ t_3 - 1 = 0,\ t_4t_6t_7 - 1 = 0,\ t_5 - 1 = 0$$
$$t_1t_4t_7 - 1 = 0,\ t_2 - 1 = 0,\ t_3 - 1 = 0,\ t_5 - 1 = 0,\ t_6 - 1 = 0$$
$$t_1t_3t_6 - 1 = 0,\ t_2 - 1 = 0,\ t_4 - 1 = 0,\ t_5 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - t_4 = 0,\ t_2 - t_3 = 0,\ t_3t_4t_6 - 1 = 0,\ t_5 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - t_5 = 0,\ t_2 - t_6 = 0,\ t_3t_5t_6 - 1 = 0,\ t_4 - 1 = 0,\ t_7 - 1 = 0$$
$$t_3 - 1 = 0,\ t_4 - 1 = 0,\ t_5 - 1 = 0,\ t_6 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - 1 = 0,\ t_2t_5t_6 - 1 = 0,\ t_3 - t_6 = 0,\ t_4 - t_5 = 0,\ t_7 - 1 = 0$$
$$t_1 - 1 = 0,\ t_2t_3t_5 - 1 = 0,\ t_4 - 1 = 0,\ t_6 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - 1 = 0,\ t_2t_4t_6 - 1 = 0,\ t_3 - 1 = 0,\ t_5 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - 1 = 0,\ t_2 - 1 = 0,\ t_5 - 1 = 0,\ t_6 - 1 = 0,\ t_7 - 1 = 0$$
$$t_1 - 1 = 0,\ t_2 - 1 = 0,\ t_3 - 1 = 0,\ t_4 - 1 = 0$$

Specializing to the case with $t_1 = t_2 = \cdots = t_n = t$ we obtain that the kernel of ∂_2 is isomorphic to $\left(\mathbb{Z}[t^{\pm 1}]\right)^4$ and the image of ∂_2 can be identified with the submodule

$$\{(t-1)(p_1, p_2, p_3, p_4, 0) \mid p_i \in \mathbb{Z}[t^{\pm 1}]\}$$

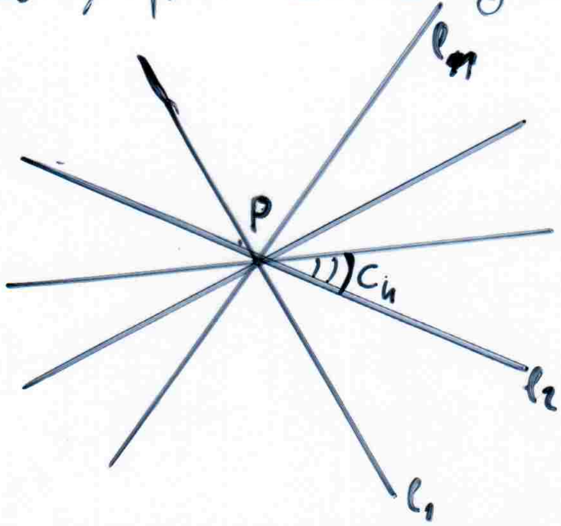
of $\left(\mathbb{Z}[t^{\pm 1}]\right)^5$. We have therefore:

$$H_2(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \left(\mathbb{Z}[t^{\pm 1}]\right)^4$$

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \left(\frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)}\right)^4 \cong \mathbb{Z}^4$$

$$H_0(\mathcal{M}(\mathcal{A}), \mathbb{Z}[t^{\pm 1}]) \cong \frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)} \cong \mathbb{Z}$$

Proof is obtained by first considering the central case:



<u>Thm</u>

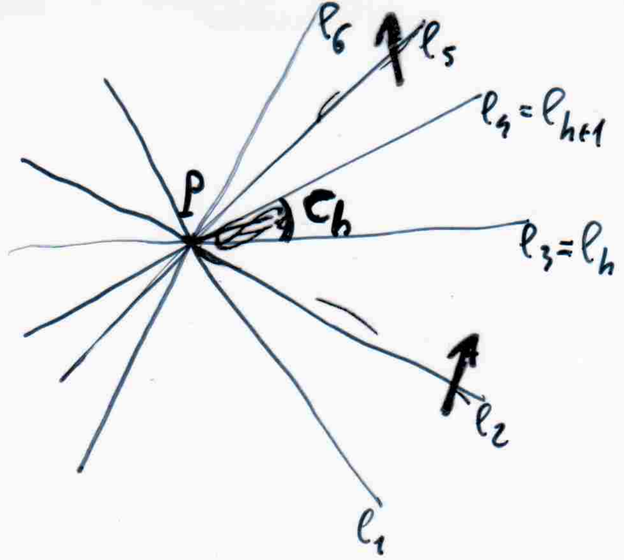
$$\partial_2\left(\ell \cdot e_{[c_h<P]}\right) = \sum_{j=1}^{n}\left[\prod_{\ell_i \in [c_h \to |F_j|]} t_i - \prod_{1 \leq i \leq j-1} t_i\right](\ell) \cdot e_{[c_{j-1}<P]}$$

which is obtained through semplification over the admissible paths;

second, one uses induction over the number of points P_i.

[redacted]

Proof (central case)



Take into account all admissible sequences from $[C_h < P]$ to $[C_{j-1} < F_j]$

I) $j \leq h$. Then $\forall$ any adm. seq. $s = (F_{j_1}, \dots, F_j)$ one has $P \triangleleft F_{j_1}$ and $|F_{j_1}| = \ell_i$ with $i \geq h+1$.

Now fix the last element of s which is bigger than P. The point is that the contribution is the same, up to sign which is ± 1 according whether # facets of s is odd or even.

Then the sum vanishes exept when the last element bigger than P lies in $\boxed{\ell_n}$ or in $\boxed{\ell_h}$

First case $\Rightarrow$ $\exists$ unique adm. seq. which gives $-\prod_{i=1}^{j-1} t_i$

Second case $\Rightarrow$ partitioning according to the first element F_{j_1} it remains only the sequence (F'_{h+1}, F'_h, F_j) which gives

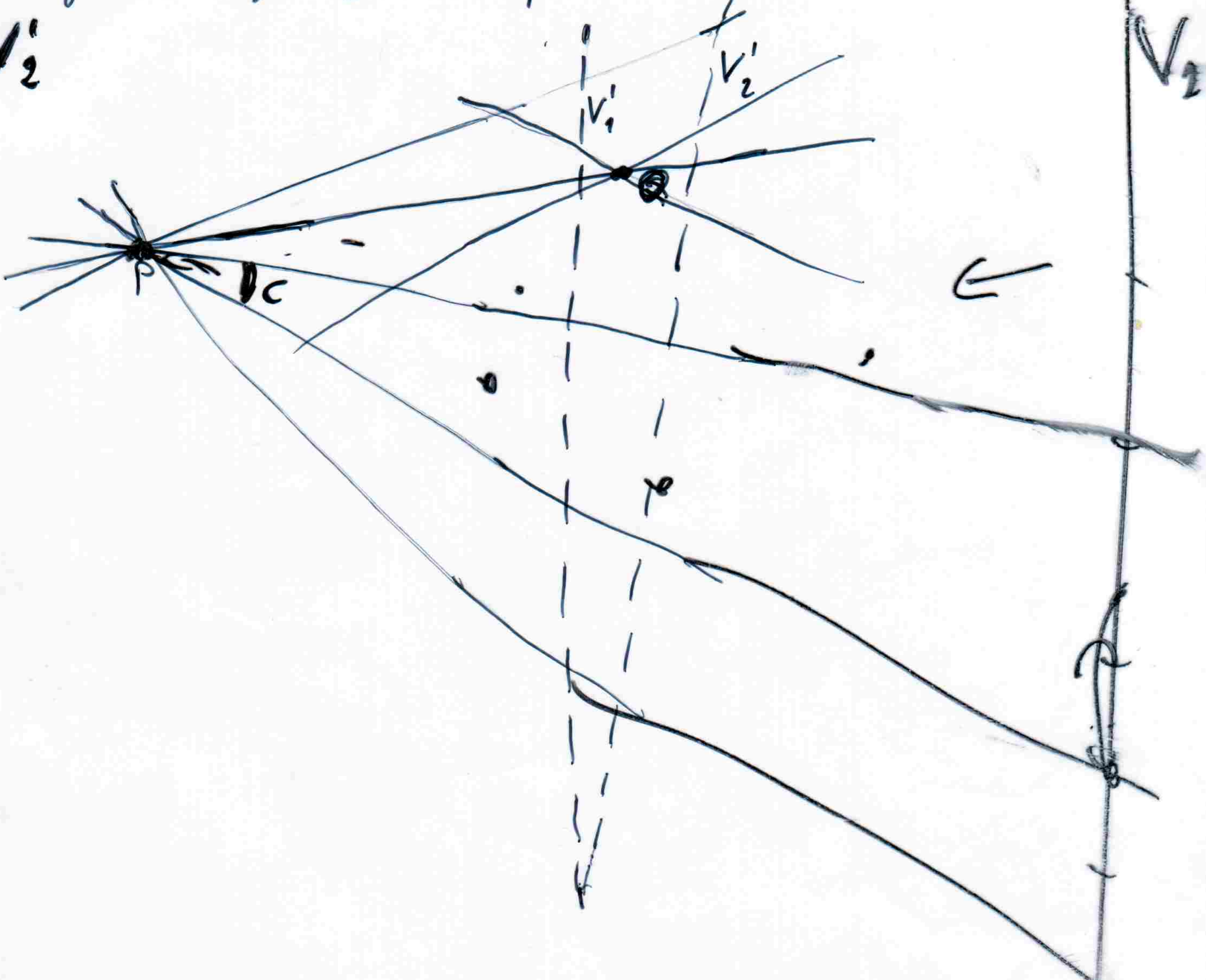
$$\prod_{i=h+1}^{n} t_i \prod_{i=1}^{j-1} t_i$$

so the formula follows.

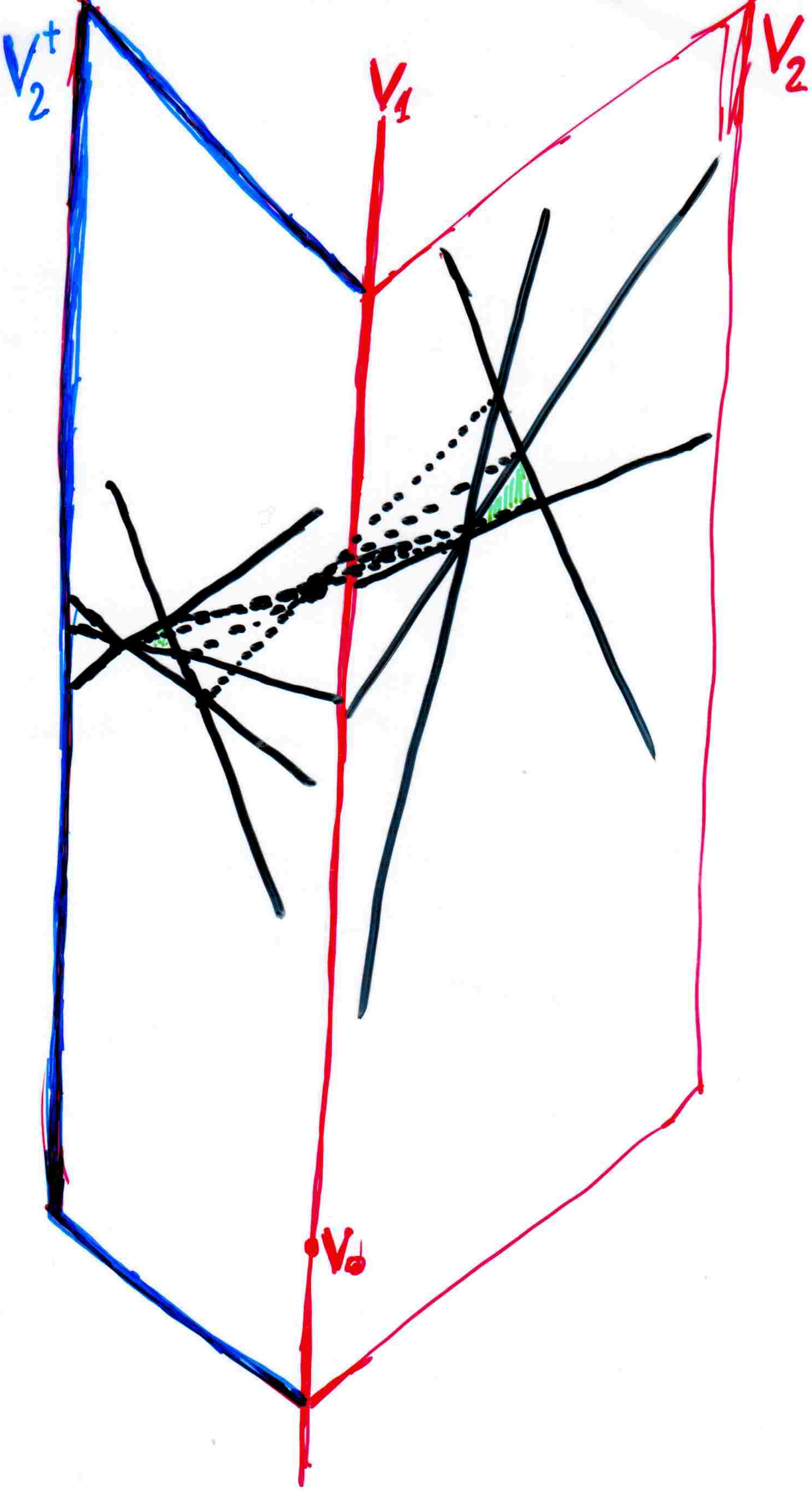
[Here $|F'_i| = \ell_i$, $P \triangleleft F'_i$]

<u>Affine case</u>: by induction on the number of vertices of the arrangement $\mathfrak{a}$.

One assumes the formula for the "truncated" arrangement by V_1' and prove it for the truncated by V_2'

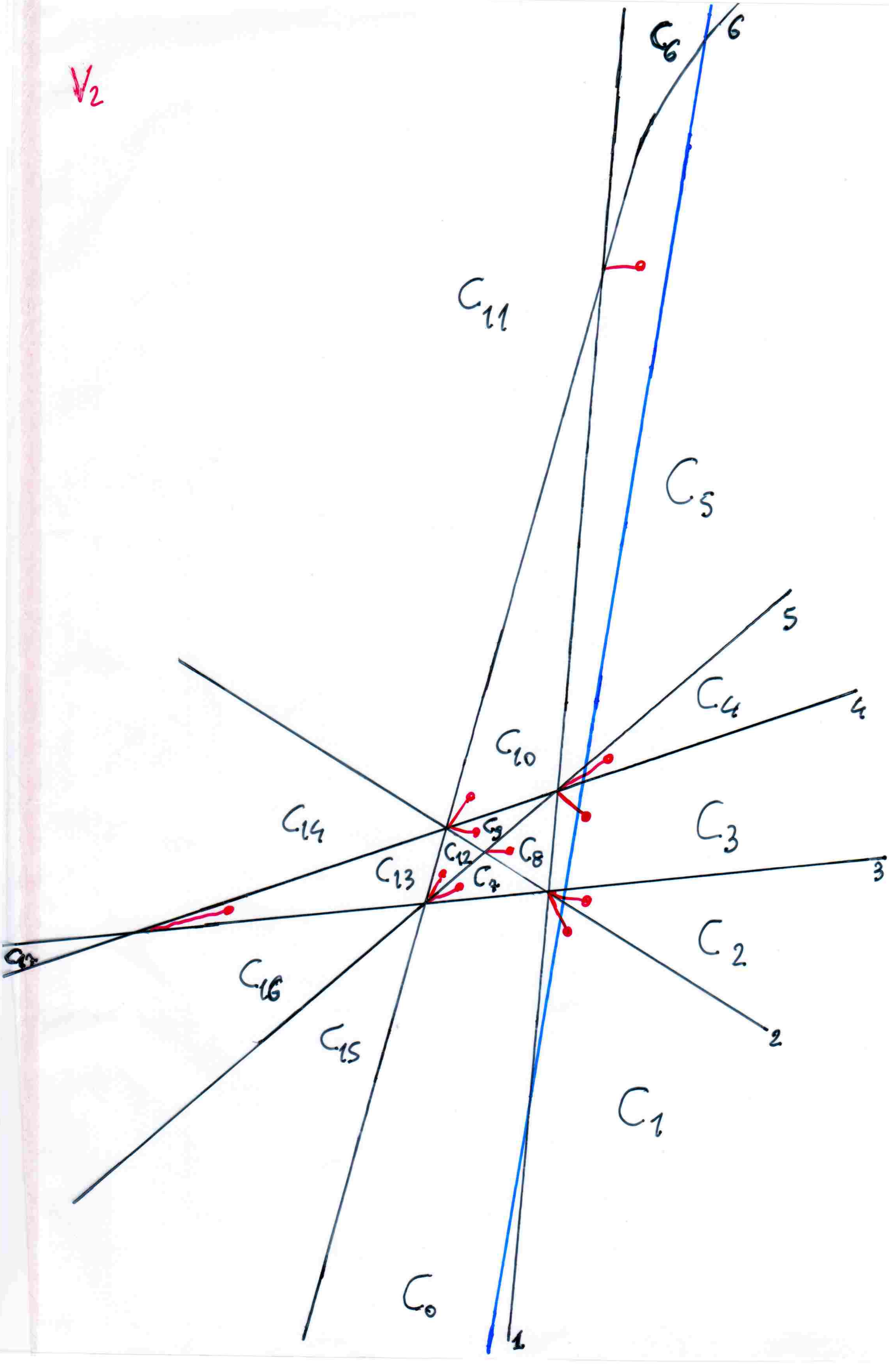


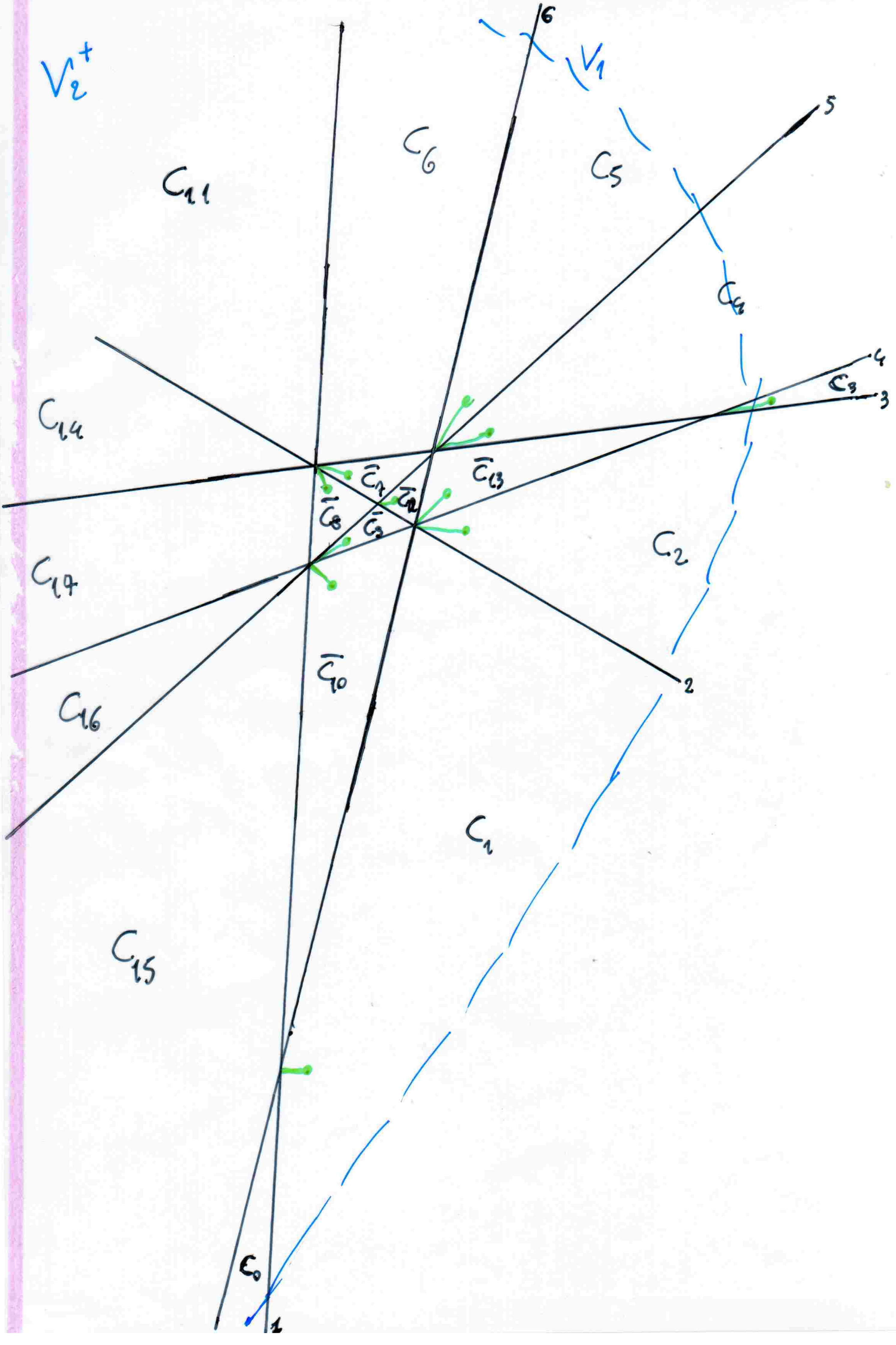
compose with "local" endomorphisms and conclude.



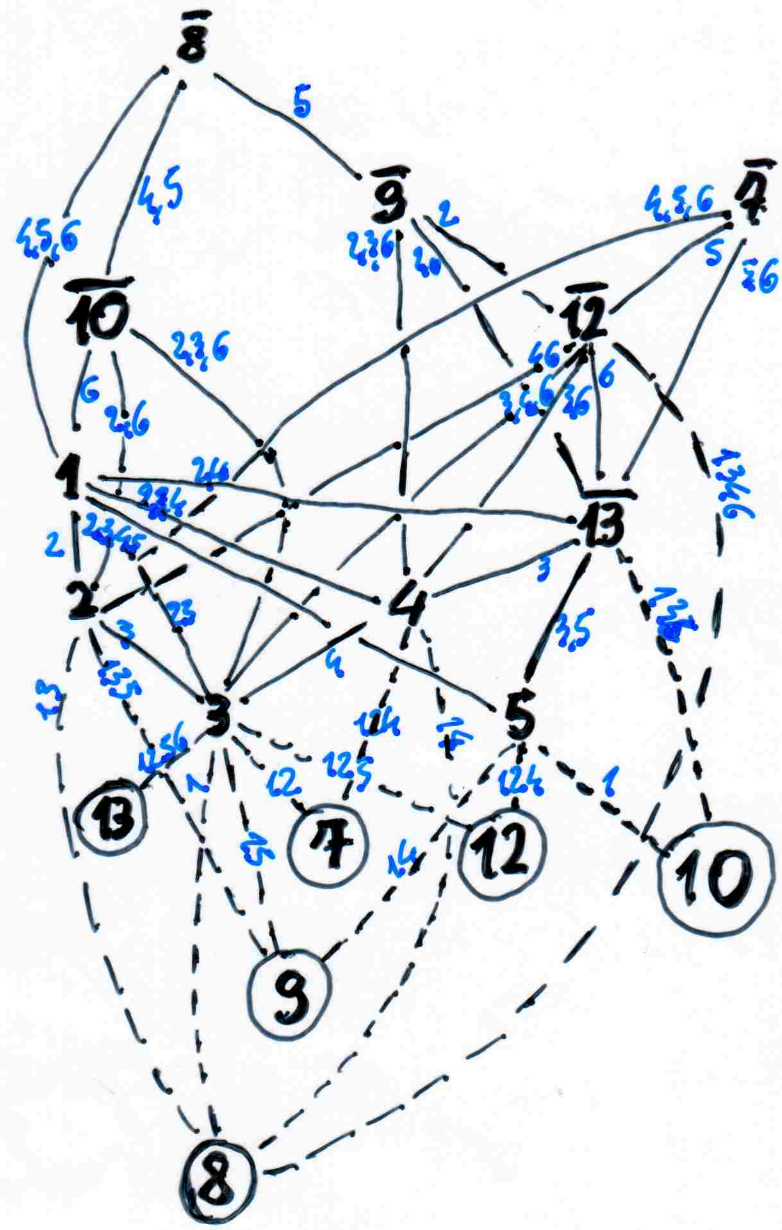
V_2^+
V_1
V_2
V_0

V_2





V2+
V1
C11
C6
C5
C4
C14
C3
1
2
3
4
5
6
C13
C7
C12
C8
C9
C2
C17
C10
C16
C1
C15
C0


8
9
4
10
12
1
13
2
4
3
5
13
7
12
10
9
8

Thm (the top-boundary)

The boundary of each top-base element is given by a tree: for C bounded, $C \cap V^{n-1} \neq \phi$, one has

$$\partial [C < O] = \sum_{[C' < F^{n-1}] \in \mathrm{Sing}^{n-1}} \ \sum_{\substack{D \in \Gamma(C) \\ \mathcal{M}_{F^{n-1}} \text{ does not} \\ \text{separate } C', D}} c(\Gamma(C), C', D) \cdot [C' < F^{n-1}]$$

where:

- $\Gamma(C)$ is a tree connecting C and its opposite chamber $\overline{C}$ constructed explicitely.

- the coefficient c in the formula is also explicit, described only in terms of "hyperplane separations".

example A_3

$\partial[C_7 < 0] =$ $\qquad \Gamma(C_7) = 7 \to 3 \to 2 \to \bar{7}$

$$= t_1 t_2 (1 - t_4 t_5 t_6) [C_2 < P_{C_2}] - t_1 t_2 (1 - t_3) [C_3 < P_{C_3}]$$

$$- t_1 t_2 t_4 t_5 (1 - t_3) [C_5 < P_{C_5}] - t_1 t_2 (1 - t_3) [C_8 < P_{C_8}]$$

$$- t_1 t_2 t_5 (1 - t_3) [C_9 < P_{C_9}] + t_5 t_6 (1 - t_1 t_2) [C_{13} < P_{C_{13}}]$$

$$+ (1 - t_1 t_2) [C_7 < P_{C_7}]$$

Happy Birthday,

Anatoly !