Solving Polynomial Systems With Tropical Methods

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THESIS

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To my wife and family

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$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$
$f^{-1}(0)$	Solution set of f

SUMMARY

In this thesis, we develop a new polyhedral method to solve systems of polynomials. We are primarily interested in obtaining the Puiseux series representations of positive dimensional solutions sets for square polynomial systems and systems, which consists of more equations than variables. While we are primarily interested in positive dimensional solution sets, our polyhedral method can also be used to find isolated solutions of a system of polynomials.

Our polyhedral method has its origins in the work of Newton and Puiseux (105), which lead to the Newton-Puiseux method for plane curves (141) (34). A strong influence on the development of our polyhedral method can be found in the work of Bernshtein (14) (see also (63)), specifically in Bernshtein's Theorem A and Theorem B, and in the work of Maurer (91), which generalized the Newton-Puiseux method to space curves. For us, and this is reflected in the development of our polyhedral method, the work of Bernshtein (14) and Maurer (91) form a basis for our understanding of the field of tropical algebraic geometry ((125), Chapter 9) (107) (68) (70) (19) (72) (71) (100) (77) (20).

From our perspective, the work of Bernshtein (14) and the work of Maurer (91) are seen as precursors of tropical algebraic geometry ((125), Chapter 9) (107) (68) (70) (19) (72) (71) (100) (77) (20), which consolidated these previous results in form of a theorem, called the fundamental theorem of tropical algebraic geometry (73). In particular, we see the the fundamental theorem of tropical algebraic geometry (73) as a generalization of Bernshtein's Theorem B (14).

With the development of our polyhedral method, we aim to generalize polyhedral homotopies (88) (62) (139) (137) (136) (64) (118) (134). Given that we see Bernshtein's work (14), as well as the work of Maurer (91), as precursors of tropical algebraic geometry, the content of this thesis and our polyhedral method can be seen as the symbolic-numeric version of the fundamental theorem of tropical algebraic geometry (73).

In Chapter 1, we introduce the background material, related work, main concepts and formally state the problem. The rest of the chapters cover the different stages in the development of our polyhedral method.

The Chapter 2 covers the construction of a symbolic-numeric algorithm to find the common factor of two bivariate polynomials with approximate complex coefficients. We first express the two bivariate polynomials as a system of polynomials and then proceeded to determine whether or not their solution set is a plane curve. We then show how tropical methods lead to the Puiseux series of the common factor when it exists or efficiently exclude it when the common factor does not exist. By covering the development of curves in the plane, this algorithm represents the first stage in the development of our polyhedral method to develop general algebraic sets. The most significant results of this chapter are expressed by the Algorithm 2.35, Algorithm 2.36 and the Proposition 2.32 and its proof. The Proposition 2.32 is very important because it provides a condition for the existence of the second term in the Puiseux series of the common factor. Our approach is related to methods in numerical algebraic geometry, given in (134).

In Chapter 3, we develop a new polyhedral method to solve systems of polynomials, which have space curves as their solution sets. Here, we extend the ideas we developed in Chapter 2.

Even though our polyhedral method is geared towards the development of positive dimensional algebraic sets, in this chapter we also show how this polyhedral method can be used to find the isolated solutions of a system of polynomials. The main contribution of this chapter has been the Proposition 3.19 and its proof (3).

An important aspect of this chapter is the exploitation of symmetry. We use cyclic permutation and apply it to tropisms and the solutions of the initial form systems to reduce the computational time. In particular, we show that once a tropism and its corresponding initial form systems solutions have been obtained, we can use cyclic permutation to obtain the other solution sets as well as the (total) degrees of the solution sets. The approach of this chapter is related to the methods in numerical algebraic geometry, given in (134).

In Chapter 4, we cover the extension of ideas developed in Chapter 2 and Chapter 3. The main focus is on extending our polyhedral method to develop general algebraic sets. Whereas in the previous chapters we dealt with single tropisms, in this chapter we show how cones of tropisms arise when we consider general algebraic sets and how they lead to multivariate Puiseux series representations of such sets.

In this chapter, we initially focus on solving of binomial systems. For binomial systems, our approach is an algorithm. We next apply our polyhedral method to polynomial systems to obtain general algebraic sets. The most significant aspects of this chapter is the construction of unimodular coordinate transformations, using the Smith and Hermite normal forms, the Proposition 4.16, Proposition 4.31, Proposition 4.32 and their proofs, and the Algorithm Outline 4.12 and Algorithm Outline 4.19.

SUMMARY (Continued)

Application of our polyhedral method to the cyclic n-roots problem leads to two interesting results, when $n = m^2$. We conclude the chapter by giving an exact representation of a solution set for the cyclic m^2-roots problem and the degree of that solution set.

CHAPTER 1

INTRODUCTION

The content of this thesis is based on previously published material (2) (4) (3) (5). All algorithms, algorithm outlines, computational results, concepts, definitions (and variations), examples, figures, ideas, illustrations, pictures, propositions and their proofs, tables, etc. are taken from (4) (3) (5), unless otherwise indicated. We cite (2) for completion, since earlier versions of (4) are based on material from (2). This thesis is a summary of (4) (3) (5). Furthermore, the work of (134) is a major influence on the development of this thesis, which is reflected in our common terminology, definitions, choice of examples, illustrations, etc.

A polyhedral method, within the context of algebraic geometry, is a combination of analytic and discrete procedures, designed to exploit the polyhedral structure of the defining equations of an algebraic set. This thesis covers the development of such a polyhedral method and its application within the context of polynomial system solving. In particular, our polyhedral method is primarily concerned with the term by term development of a (multivariate) Puiseux series expansions of positive dimensional algebraic sets. As such, it can be seen as the generalization of the Newton-Puiseux method (105) (141) for plane curves to a polyhedral method for general algebraic sets.

1.1 Problem Statement

We consider the class of cyclic n-roots polynomial systems (Equation 1.1) as a source of running examples throughout this thesis. Cyclic n-roots polynomial systems arose in the studies of Fourier transforms (11) (15) (57). They quickly rose to prominence as systems, which present extreme challenges for polynomial system solvers when $n \geq 8$. Since their formulation, the cyclic n-roots polynomial systems have been considered academic benchmark problems (43) (134) in the field of computer algebra, see (16) (33) (42) (84) (43) (127) (115) (109). The description (Equation 1.1) of the cyclic n-roots polynomial system $C_n(\mathbf{x})$ is taken from (43) (5).

$$F(\mathbf{x}) = C_n(\mathbf{x}) = \begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ x_0x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \ mod \ n} = 0 \\ x_0x_1x_2 \cdots x_{n-1} - 1 = 0. \end{cases} \tag{1.1}$$

Polynomial system (Equation 1.1) $C_n(\mathbf{x})$ (57) (43) (5) is a square system, that is, the number of equations equals the number of variables. Generally, when solving such a system of polynomials, one expects the solution set to consists only of isolated solutions. However, that is not always the case. Depending on the polyhedral structure and the coefficients of the polynomials in the system, the solution set of $C_n(\mathbf{x})$ may be of any dimension d, for $d < n$. The following Lemma 1.2, due to Backelin (11), captures the challenging nature of the cyclic n-roots polynomial systems. The Lemma 1.2 is taken from (43).

Lemma 1.2 (Backelin)**.** *If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m - 1$.*

The cyclic n-roots polynomial systems posses a characteristic format, which reveals the most significant aspect of their innate structure: the permutation symmetry. As it can be easily observed by looking at the indices of the variables, the cyclic n-roots polynomial systems remain unchanged under the *cyclic permutation* of the variables. This invariance will resonate throughout the application of the polyhedral method, which is being developed in this thesis, to the cyclic n-roots polynomial systems. At the same time, it will illustrate how our polyhedral method takes advantage of symmetry when it is present in a system of polynomials.

The polyhedral method, which is being developed as part of this thesis, is not limited to the square polynomial system. Polynomial systems, which consists of more equations than variables, fall well within the scope of this polyhedral method. Furthermore, no prior knowledge of any information about the solution set is required.

The polyhedral method we are developing is geared towards sparse polynomial systems, which are understood to be systems, having a fewer number of monomials in comparison to the degree (135) (62) (65) (139) (134). Our polyhedral method detects when there exists a positive dimensional solution set to a system of polynomials, like the cyclic n-roots polynomial system or a system with more equations than variables, and it develops its Puiseux series expansion. Otherwise, it returns that all solutions to the system are isolated.

There are the following assumptions on the algebraic sets that this polyhedral method is designed to develop. The algebraic set is in general position with respect to the first d

coordinates, where d is the dimension of the solution set. In practice, this means that the solution set is not contained in a hyperplane, parallel to the first d coordinates. Furthermore, the solution set is assumed to be reduced, i.e. free of multiplicities.

1.3 Background

The approach this thesis takes in the development of the polyhedral method has its origins in the work of Newton and Puiseux (105), which lead to the Newton-Puiseux method for plane curves (141) (34). Further influences can be found in the work of Bernshtein (14) (see also (63)), specifically in the two theorems A and B of Bernshtein, and in the work of Maurer (91), which generalized the Newton-Puiseux method to space curves. All of these results can be seen as a precursors of tropical algebraic geometry ((125), Chapter 9) (107) (68) (70) (19) (72) (71) (100) (77), which consolidated these previous results in form of a theorem, called the fundamental theorem of tropical algebraic geometry (73).

In order to introduce the background material, consider a polynomial system $F(\mathbf{x})$, defined along the lines of the system in (Equation 1.1). The Definition 1.4 is taken from (35) (145) (56) (134) (82) (80) (135) (123) (64) (125) (56) (39) and given here in an adjusted form.

Definition 1.4 (Support, Newton Polytope, Normal Fan)**.** Given a polynomial $f(\mathbf{x}) \in F(\mathbf{x})$ in n variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \neq 0, \; x^{\mathbf{a}} = x_0^{\pm a_0} x_1^{\pm a_1} \cdots x_{n-1}^{\pm a_{n-1}}, \tag{1.2}$$

the *support* A of f consists only of those exponent vectors, whose coefficients $c_{\mathbf{a}}$ in f are non-zero. The *Newton polytope* of a polynomial $f(\mathbf{x}) \in F(\mathbf{x})$ is denoted P and it is the convex hull of the support A. The set of normal vectors $\mathbf{v}$ to the faces of P forms the *normal fan* of P.

As stated in (145) (56), *every* polytope in $\mathbb{R}^n$ can be defined in two equivalent ways:

(1) the convex hull of a finite set of points

(2) an intersection of finitely many closed half-spaces

As an illustration of Definition 1.4 for a bivariate polynomial, consider Example 1.5. It depicts a two-dimensional Newton polytope, called Newton polygon, and its edge normals. In Example 1.5, edge normals form the *normal fan* of the Newton polygon (145) (56). The Example 1.5 and Figure 1 are taken form (4).

Example 1.5.

$$f(x, y) = x^3y + x^2y^3 + x^5y^3 + x^4y^5 + x^2y^7 + x^3y^7 \tag{1.3}$$

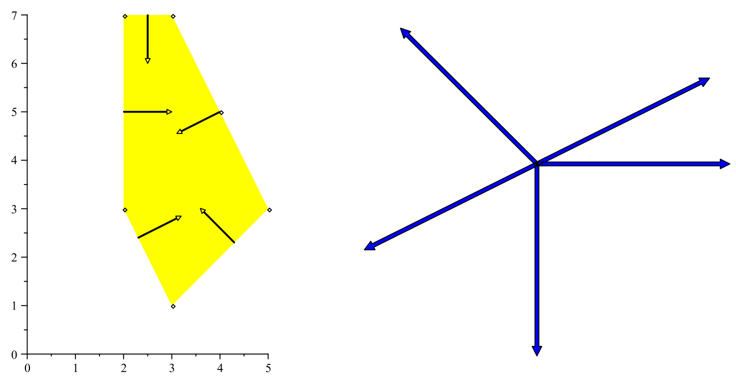


Figure 1. The Newton polygon and its edge normals, forming a normal fan.

The Example 1.5 will be revisited in latter chapters. The Definition 1.4, applied to a bivariate polynomial equation $f(x, y) = 0$ in the Example 1.5, provide enough information to introduce the first major influence in the development of this thesis.

1.5.1 Newton-Puiseux Method

The Newton-Puiseux method for plane curves (105) (141) (34) is quite well known and it does not need a long introduction. The Newton-Puiseux method develops a fractional power series of a plane curve, defined by $f(x, y) = 0$. The method works by associating with the polynomial $f(x, y) = 0$ a polyhedral object, which was introduced above as the Newton polygon. The resulting fractional power series, now called the Puiseux series, can be obtained by manipulating the polynomial $f(x, y) = 0$, using the edge normals of the lower hull of its

Newton polygon. The algorithm to manipulate the polynomial $f(x, y)$ and develop its Puiseux series is given in (141) and (34).

The significance of the Newton-Puiseux method lies in establishing that, in the plane, polynomials and their solution sets can be analyzed via polyhedral objects (i.e. Newton polygons) and that there is a precise way (Definition 1.4) to associate such polyhedral objects with the polynomials. While in the development of the Puiseux series of $f(x, y) = 0$ all monomials are necessary to find the coefficients in the series, only monomials on the lower convex hull lead to the exponents in the Puiseux series of the different branches (105) (141) (34). As given in (141) (34), the exponents in the Puiseux series generated by the Newton-Puiseux method are identified with the edge normals.

1.5.2 Polyhedral Homotopies

The Newton-Puiseux method influenced the work of Bernshtein (14), which in turn lead to polyhedral homotopies (88)(62)(139)(137) (64) (118). Polyhedral homotopies are used to find *isolated* solutions of systems of polynomials over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Consider two simple bivariate polynomials $f_0(x_0, x_1)$, $f_1(x_0, x_1) \in F(x_0, x_1)$.

$$\begin{cases} f_0(x_0, x_1) = x_0x_1 + x_0^2x_1 + x_0^2x_1^3 + x_0x_1^3 & = 0 \\ f_1(x_0, x_1) = x_0 + x_0^2x_1^2 + x_0x_1^3 + x_1^2 & = 0 \end{cases} \tag{1.4}$$

Their Newton polygons and their normal fans can be shown in the same manner as in Example 1.5. Instead of a graphical description, their normal fans N_{f_0} and N_{f_1} are usually given analytically, in terms of vectors anchored at the origin.

$$N_{f_0} = [(1,0),(0,1),(-1,0),(0,-1)]$$
$$N_{f_1} = [(2,1),(-2,1),(-1,-1),(1,-1)] \tag{1.5}$$

Comparing the normal fans N_{f_0} and N_{f_1} with each other, it can be easily seen that they do not have edge normals, which point in the same direction. A polynomial system, whose polynomials lead to such a configuration of normal fans is said to be in *generic* position (14) (62) (139) (137) (118). The fact that the polynomial system is in generic position has far reaching implications. One of these implications is directly related to the construction of polyhedral homotopies.

A construction of a polyhedral homotopy begins by first randomly lifting monomials in the polynomial system $F(\mathbf{x})$, using a new variable $\mathbf{t}$ and a random lifting function $\mathbf{w}$ (134) (14) (62) (136) (64) (139) (137) (118). The main idea behind such a lift is to overcome the situation where polynomials in the system are in generic position with respect to each other. In particular, the goal is to use the new variable $\mathbf{t}^{w_i}$, $w_i \in \mathbf{w}$ and multiply the monomials in polynomial system $F(\mathbf{x})$, such that the monomials of the different polynomials in $F(\mathbf{x})$ share a common facet normal. The introduction of the lift changes the normal fans. The dimension of the normal fan is now higher by one but, more importantly, at least one facet normals is contained in all other normal fans of the polynomials in the system. The lifting is depicted in Example 1.6.

One such common facet normal, here denoted $\mathbf{v} = (v_0, v_1, \cdots, v_{n-1}, v_n)$, alongside other possible ones, leads to what is called *mixed volumes* and bounds on the number of isolated solutions for systems of polynomials (14) (80) (62) (139) (137) (136) (64) (118) (29) (134).

The lifting of polynomials is given in Example 1.6. The lifting is described in (134) (14) (62) (139) (136) (137) (118). The Example 1.6 is taken from (134) (14) (62) (139) (137) (118).

Example 1.6 (Lifting)**.**

$$F(\mathbf{x}) = \sum_{a \in A} c_a \mathbf{x}^a \longrightarrow \widehat{F}(\mathbf{x}, t) = \sum_{a \in A} c_a \mathbf{x}^a t^w \qquad c_a \in \mathbb{C} \tag{1.6}$$

Definition 1.7 is taken from (91) (123) (78) (21) (125) (108) (107) (130) (19) (134) and given here in an adjusted form.

Definition 1.7 (Initial Form, Initial Form System)**.** Let $\mathbf{v} \in \mathbb{Q}^n \setminus \{0\}$, $f(\mathbf{x}) \in F(\mathbf{x})$ and A be given as in Definition 1.4. Let $\langle \cdot, \cdot \rangle$ denote the inner product. The *initial form* of f with respect to $\mathbf{v}$ is

$$in_{\mathbf{v}} f(\mathbf{x}) = \sum_{a \in A} c_a \mathbf{x}^a, \quad m = min\{\langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A\} \textit{ achieved at least two times.} \tag{1.7}$$

The *initial form system* $in_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$ is defined as $in_{\mathbf{v}} F(\mathbf{x}) = (in_{\mathbf{v}} f_0, in_{\mathbf{v}} f_1, \ldots, in_{\mathbf{v}} f_{n-1})$.

For terminology of initial forms, in connection to the Gröbner bases, see (123). In (108), $in_{\mathbf{v}}(f)$ is referred to as an initial term polynomial. For initial ideals, see (130). In (21) and

(78), initial form systems are referred to as truncated systems. We treat initial forms in this thesis as they are treated in (134). For initial forms, see also (125) (107) (19).

The notion of the initial form system can be applied to the lifted polynomial $\widehat{F}(\mathbf{x}, t)$, where $\mathbf{v}$ is one facet normal, which is contained in all other normal fans in the system of polynomials. As described in (14) (62) (139) (137) (118) (134), the coefficients in the polynomial system $F(\mathbf{x})$ and, consequently, in the lifted polynomial $\widehat{F}(\mathbf{x}, t)$, have been replaced with generic complex coefficients.

The initial form system of $\widehat{F}(\mathbf{x}, t)$ with respect to $\mathbf{v}$, denoted $in_{\mathbf{v}}(\widehat{F}(\mathbf{x}, t)) = 0$ is a much sparser polynomial system (134) (14) (62) (139) (137) and it is, therefore, easier to solve. The solution of the initial form $in_{\mathbf{v}}(\widehat{F}(\mathbf{x}, t)) = 0$, denoted $r_0, r_1, \cdots, r_{n-1}$, does not necessarily satisfy the original system $F(\mathbf{x})$. However, it provides a starting point for the development of the solution of $F(\mathbf{x})$ (14) (62) (64) (139) (137) (118) (134). In particular, it allows for the construction of a start system for a polyhedral homotopy $\widehat{F}^{\mathbf{v}}(\mathbf{x}, t)$, as described in (14) (62) (139) (137) (64) (118) (134), which can be numerically deformed into the solution of the target system F($\mathbf{x}$) via homotopy continuation methods (14) (62) (136) (131) (64) (139) (137) (118) (134).

The construction of a start system for a polyhedral homotopy and identification of mixed cells (14) (62) (64) (139) (137) (118) is directly related to the two theorems of Bernshtein (14). In order to obtain the facet normals to identify mixed subdivisions and formally make a connection with the two theorems of Bernshtein, consider Theorem 1.7.1 of Minkowski (41)

(49) (35) and, more applicable to this thesis, a construction called the *Cayley trick* (50) (122) (61) (35).

Theorem 1.7.1 (Minkowski, Mixed Volume)**.** *Let $P_1, P_2, \ldots, P_k$ be convex bodies in $\mathbb{R}^n$, and $\lambda_i \geq 0$, $i = 1, \ldots, k$. Then, $V(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_k$,*

$$V(\lambda_1 P_1 + \cdots + \lambda_k P_k) = \sum_{j_1,\ldots,j_n=1}^{k} V(P_{j_1}, \ldots, P_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}$$

The Theorem 1.7.1 of Minkowski is taken from (41). The volume, given by $V(P_{j_1}, \ldots, P_{j_n})$ is the *mixed volume* of $P_{j_1}, \ldots, P_{j_n}$ (41) (80) (64) (134) (136) (49) (29).

The Cayley trick formulates a resultant as a discriminant (61), as given in [(50) Proposition 1.7, page 274]. For the purpose of this thesis, the geometric variant of the Cayley trick is considered, as given in (122) [(35) §9.2.] (61).

The Definition 1.8 is taken from [(35) §9.2.] (50) (122) (61) (35) (122) (61) (136) (139) (3), and it is given here in an adjusted form.

Definition 1.8 (Cayley Embedding)**.** Let A_k be the support of $f_k \in F(\mathbf{x})$ and $\mathbf{e}_k$ the k-th $(n-1)$-dimensional standard unit vector. Then

$$C_E = (A_0 \times \{0\}) \cup (A_1 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_{n-1} \times \{\mathbf{e}_{n-1}\}) \tag{1.8}$$

is called the *Cayley embedding*.

In (139), the convex hull of the Cayley embedding is refered to as the *Cayley polytope.* The Cayley trick makes it possible to combine all the individual Newton polytopes of a system of polynomials $\widehat{F}(\mathbf{x}, t)$ into one Cayley polytope. As stated in (139), the [Lemma 5.2 in (122)] establishes that the cells in the mixed subdivisions of $\widehat{F}(\mathbf{x}, t)$ are directly related to the triangulation of the Cayley polytope (139), which allow for identification of facet normals used in the construction of start systems for polyhedral homotopies (14) (62) (64) (139) (137) (118) (134).

This section concludes with the two theorems of Bernshtein (14), the two essential results in the development of polyhedral homotopies and the foundation for the polyhedral method, which is being developed in this thesis. The two theorems of Bernshtein (14), Theorem 1.8.1 and Theorem 1.8.2, are taken from (134).

Theorem 1.8.1 (Bernshtein, Theorem A). *The number of roots of a generic system equals the mixed volume of its Newton polytopes. For any system, the mixed volume bounds the number of isolated solutions in* $(\mathbb{C}^*)^n$*,* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Theorem 1.8.2 (Bernshtein, Theorem B). *Consider* $F(\boldsymbol{x}) = 0$*,* $F = (f_0, f_1, \ldots, f_{n-1})$*,* $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$*. If for all* $\boldsymbol{v} \neq 0$*:* $in_{\boldsymbol{v}} F(x) = 0$ *has no solutions in* $(\mathbb{C}^*)^n$*, then* $F(\boldsymbol{x}) = 0$ *has exactly as many isolated solutions in* $(\mathbb{C}^*)^n$ *as the mixed volume.*

1.8.1 Tropisms and Space Curves

The generalization of the Newton-Puiseux method for plane curves to space curves was first completed by Maurer (91). Maurer's approach (91) involved taking a point on the space curve, defined by a system of polynomials, and performing a change of coordinates, such that the point selected becomes the origin (134). The next step in Maurer's approach was to introduce the notion of *tropism* (91). However, before defining a tropism, it is more convenient to define a *pretropism* first.

The word pretropism, we take from (134). It is, of course, related to Maurer's notion of *tropism* (91). However, it is more closely related to the notion of *tropical prevariety* (19) (107).

Definition 1.9 is taken from (107) (19) (134). We give it here in an adjusted form.

Definition 1.9 (Pretropism)**.** A pretropism is a vector $\mathbf{v}$, which leads to an initial form system.

While there may be many edge, face or facet normals, not all of them necessarily lead to a Puiseux series expansion of a solution set. In our approach, the Puiseux series of a space curve has the form given in Example 1.10. The Example 1.10 is taken from (3) and (134), where it is given in compact form, using slightly different notation.

Example 1.10.

$$\begin{aligned} x_0 &= t^{v_0} \\ x_1 &= t^{v_1}(r_1 + c_1 t^{w_1} + \dots) \\ &\vdots \\ x_{n-1} &= t^{v_{n-1}}(r_{n-1} + c_{n-1} t^{w_{n-1}} + \dots) \end{aligned} \qquad c_1, \dots, c_{n-1} \in \mathbb{C}. \tag{1.9}$$

where $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ is a tropism and $r_1, \dots, r_{n-1}$ solution of the initial form system.

Definition 1.11 is based on Maurer's notion of tropism (91) (86), with some important differences, described below. The Definition 1.11 is taken from (91) (86) (134). We give it here in an adjusted form.

Definition 1.11 (Tropism)**.** A tropism is a vector $\mathbf{v}$, perpendicular to at least one edge of each Newton polytope P_i of the polynomial $f_i \in F(\mathbf{x})$, which leads to a Puiseux series approximation of a solution set, for small values of the free parameter t.

Note that every tropism is a pretropism (107) (19).

In the development of the polyhedral method in this thesis, the notion of tropism is used as in (134) and similarly how it was used by Maurer in (91). However, there is an important difference between the approach of this thesis and the approach developed by Maurer. In our approach, we do not assume that there is a tropism to begin with or that there is a curve as a solution set at all. Maurer, however, does. Also, while Maurer picks a random point on the space curve and changes the coordinates, such that the selected point is the origin, the approach of this thesis starts the space curve development at the solutions of the initial form system (134), as defined in Definition 1.7. Our solutions of the initial form systems are denoted as $r_1, \ldots, r_{n-1}$. They are the leading coefficients of the Puiseux series expansion of a space curve, as shown in Example 1.10. Tropisms can also be found in the singularity theory (86).

A polynomial system, whose solution set consists of isolated solutions, will have its polynomials in generic position with respect to each other. This is a consequence of Bernshtein's Theorem 1.8.2 (Theorem B). The lifting (134) (14) (62) (139) (137) (136) (118) of (Equation 1.6) overcomes such an obstacle by introducing an additional parameter $\mathbf{t}$. A system of polynomials, which vanishes on a space curve, can be seen as already lifted. In many ways, the development of the first terms of the space curve, as given in Example 1.10, and the basic ideas involved in polyhedral homotopies, are analogous. Both developments start at the solutions of the initial form systems, which have been identified by a facet normal. In the case of the space curve such a facet normal is called a tropism. In the latter chapters of this thesis, the notion of tropism will be extended to cover general algebraic sets in the same manner.

1.11.1 Tropical Algebraic Geometry

In comparison with the development of the Newton-Puiseux method for plane curves (105) (141) (34), Bernshtein's work (14) and Maurer's work to develop space curves (91), the field of tropical algebraic geometry, in its standardized format, is quite new. From the modern point of view and from the perspective of this thesis, the field of tropical algebraic geometry is super set, whose main contribution generalizes ideas developed by Newton, Puiseux, Bernshtein and Maurer. Our understanding of tropical algebraic geometry comes from understanding the work of Bernshtein (14), the work of Maurer (91), specifically Maurer's notion of tropism, and the Newton-Puiseux method for plane curves (105) (141) (34).

The beginnings of the field of tropical algebraic geometry can be seen in the work of Puiseux (105), Ostrowski (98) and in the definition of an amoeba in (50). The tropical algebraic geometry generalizes the notions of an edge normal, facet normal, pretropism and tropism and combines them into one concept, the *tropical variety* (70) (19) (72) (71) (100) (77). The major result of tropical algebraic geometry is the fundamental theorem of tropical algebraic geometry, due to Jensen, Markwig, Markwig (73). Theorem 1.11.1, the fundamental theorem of tropical algebraic geometry, is taken from (73).

Theorem 1.11.1 (Fundamental Theorem of Tropical Algebraic Geometry)**.** *If K is algebraically closed of characteristic zero and $J \trianglelefteq K\{\{t\}\}[\underline{x}]$ is an ideal then*

$$\omega \in Trop(J) \cap \mathbb{Q}^n \iff \exists\, p \in V(J) : -val(p) = \omega \in \mathbb{Q}^n \tag{1.10}$$

where val is the coordinate-wise valuation.

The fundamental theorem of tropical algebraic geometry establishes a relationship between a tropical variety and an algebraic variety (an algebraic set), via the Puiseux series. For the purpose of this thesis, the fundamental theorem of tropical algebraic geometry can be seen in the light of the two theorems of Bernshtein (Theorem 1.8.1 and Theorem 1.8.2), specifically Theorem B.

The fundamental theorem of tropical algebraic geometry is a theoretical result, which essentially generalizes Bernshtein's Theorem B. The polyhedral method, which is being developed as part of this thesis, aims to generalize polyhedral methods from methods for isolated solutions to a method to develop, general, positive dimensional algebraic sets. As a result of that, the polyhedral method of this thesis can be seen as the symbolic-numeric version of the fundamental theorem of tropical algebraic geometry.

For other related background material, see (83) (137) (62) (63) (124) (88) (118).

1.12 Related Work

An additional extension of the Newton-Puiseux method to space curves is presented in (7) and it is implemented in CoCoA. For more on computations of Puiseux series, see also CASA (59) and [(111), Appendix A]. For symbolic algorithms, see (74) (102). For symbolic-numeric algorithms, see (101) (103) (104). For complexity aspects of the computation of the Puiseux series of a plane curve, see (142). For Newton-Puiseux expansions and multivariate polynomials, see (13). Solutions in terms of a general fractional power series for systems with more variables than equations are given in (93). For more on Puiseux series as solutions to systems of polynomials and related material, see (6) (8) (92) (93) (9) (81) (10) (39).

For introductory material on tropical algebraic geometry, see ((125), Chapter 9) (107) (68) (77). For tropical varieties and their computations, see (70) (19) (72) (71) (129) (20). For proof of the fundamental theorem of tropical algebraic geometry, see (73).

The software, which played an important role in the development of this thesis are the *PHCpack* - a general purpose solver for polynomial systems (132), *cddlib* - C implementation of the double description method (45), *Gfan* - software for the computation of Gröbner fans and tropical varieties (71), and *Sage Mathematics Software* (121).

We emphasize (134) as one of the first attempts to compute the Puiseux series of algebraic sets in the manner that we accomplish in this thesis.

1.13 Thesis Contribution

The polyhedral method, which is being developed as part of this thesis, aims to generalize polyhedral homotopies to develop positive dimensional solution sets for systems of polynomials over $(\mathbb{C}^*)^n$. In particular, what is considered a start system for a polyhedral homotopy to find isolated solutions, in this thesis generalizes to a start system for the development of a (multidimensional) Puiseux series of a positive dimensional algebraic sets. In effect, this approach generalizes the Newton-Puiseux method (141) (34) and it can be seen as the symbolic-numeric version of the fundamental theorem of tropical algebraic geometry (73).

We want to emphasize that our understanding of tropical algebraic geometry comes from understanding polyhedral homotopies (88) (62) (139) (137) (118) (134), the work of Bernshtein (14), and the work of Maurer (91). Furthermore, the notion of a tropical method in this thesis is based on Maurer's notion of tropism (91). We see the fundamental theorem of tropical algebraic geometry (73) as an important theoretical result. For us, however, it is the two theorems of Bernshtein (14), Theorem A and Theorem B, and Maurer's notion of tropism (91) that are forming the basis for our polyhedral method and the way we solve polynomial systems in this thesis.

This thesis is divided into four chapters. Following this introductory chapter, Chapter 2 introduces our polyhedral method and addresses its application to the problem of finding a common factor of two bivariate polynomials in form of a Puiseux series. Chapter 3 addresses the development of a Puiseux series for one dimensional algebraic sets, via the notion of (pre)tropism, with an emphasis on exploitation of symmetry. We exploit symmetry by applying the various permutations to the pretropism, resulting initial form systems and their solutions. In this manner, we can obtain the Puiseux series representation for other solution sets, which also satisfy the system of polynomials. Chapter 4 addresses the development of a multidimensional Puiseux series for general algebraic sets. The second, third and the fourth chapters should be seen as generalizations of each other, having an origin in polyhedral homotopies for zero dimensional solution sets.

CHAPTER 2

PLANE CURVES

2.1 Introduction

The content of this Chapter 2 is based on previously published material (2) and (4). All algorithms, algorithm outlines, computational results, concepts, definitions (and variations), examples, figures, ideas, illustrations, pictures, propositions and their proofs, tables, etc. are taken from (4), unless otherwise indicated. We cite (2) for completion, since earlier versions of (4) are based on material from (2). This Chapter 2 is a summary of (4).

We begin the development of our polyhedral method by addressing curves in the plane, with a specific application in mind: detection and development of a common factor of two sparse bivariate polynomials with approximate complex coefficients. We use the treatment of the curves in the plane to introduce our polyhedral method and to show how the notion of tropism (91) is connected with polyhedral homotopies and the two theorems of Bernshtein (14), which form a foundation in the development of our polyhedral method.

For introductory material on tropical algebraic geometry, see ((125), Chapter 9) (107) (68) (100). For tropical varieties and their computations, see (70) (19) (72) (71) (129) (20). For implementations in terms of libraries, software packages and software systems, see Gfan (70; 71) (121), SINGULAR library (72) and TrIm (126).

With the proof of the fundamental theorem of tropical algebraic geometry (73) and the development of software around it, as listed in the paragraph above, tropical algebraic geometry has acquired a certain norm and a standard. While the field is relatively new, many of the aspects of tropical algebraic geometry can be seen in other areas of mathematics, predating the today's standard by centuries. We can find it in the work of Newton and Puiseux (105) (141) (34) and Ostrowski (98).

From the point of view of this thesis and the polyhedral method we are developing, tropical algebraic geometry is seen within the context of polyhedral homotopies (62) (64) (88) (137), the two theorems of Bernshtein (14), and Maurer's notion of tropism (91). Therefore, in this thesis and our polyhedral method, tropical algebraic geometry is seen and understood as a way to solve systems polynomials, by building on ideas described in the work of Puiseux (105) (141) (34), Bernshtein (14) and the work of Maurer (91).

There are other areas, which relate to tropical algebraic geometry, such as idempotent analysis, presented in (90). We also refer to (106), which uses aspects of tropical algebraic geometry in relationship to boundary value problems in differential equations, along with a Maple implementation.

In this chapter, we are considering the problem of finding a common factor of two sparse bivariate polynomials with approximate complex coefficients and the development of its Puiseux series. The problem we are addressing is related to the problem of using the Newton polytopes to factor sparse polynomials (1) (40) (47). For work on approximate factorization, see (28) (27) (46) (76) (117). For polynomials with approximate coefficients and their GCD, see (144). For the absolute factorization of polynomials, see (24) and (25). For criteria for irreducibility of polynomials, based on polytopes, see (98).

In this chapter we are working in the plane, hence, our objects of study are not polytopes but polygons. For theoretical aspects of polyhedral geometry, the relationships between polygons and polytopes, we rely on (145) (56).

A system of polynomials, consisting of two bivariate polynomials with approximate complex coefficients is a system that may have an origin in a specific application. Such systems rarely contain a common factor as they are more likely to vanish only on isolated solutions. From a point of view of a computer algebra system, detecting and immediately excluding cases where the common factor does not exist is more preferable than carrying out expansive computations only to determine at some later point that the common factor is not there.

In this chapter, we present a polyhedral (preprocessing) method to develop a Puiseux series expansion of the common factor. The development of the Puiseux series starts with the solution of the initial form system, which we will refer to as the common solution at infinity. In order to determine if a solution at infinity is isolated or not, we will extend the Newton-Puiseux method (105)(141)(34) along the lines of Maurer (91) and his method for space curves.

Related material to this aspect of our polyhedral method can be found in (7) with a CoCoA implementation. For computation of the Puiseux series over $\mathbb{Q}$, see CASA in (59) and additionally see ((111), Appendix A). For general Puiseux series solutions, see (93). For symbolic algorithms, see (73) (74) (102). For symbolic-numeric algorithms, see (101) (103) (104).

For complexity aspects of the computation of a Puiseux series in the plane, see (142). For application of Hansel series, see (110) and (67).

Deciding whether there is a common factor for a system of two bivariate polynomials with approximate complex coefficients can be reduced to finding roots of univariate polynomials. Using a pretropism to obtain an initial form system results in a much smaller system, where all the monomials lie on an edge of the Newton polygons, which share the same edge normal. A particular kind of coordinate transformation, called the unimodular coordinate transformation, can be used to eliminate one of the two variables in the initial form system, resulting in a univariate polynomial system. If the transformed initial form system (now univariate) does not have a solution, then there is no root at infinity and, hence, no starting point for the development of the Puiseux series. If that is the case for all pretropisms, then there is no common factor.

If there is a non-zero solution at infinity, then we can start building the Puiseux series term by term. The entire process is summarized in Figure 2, taken from (4).

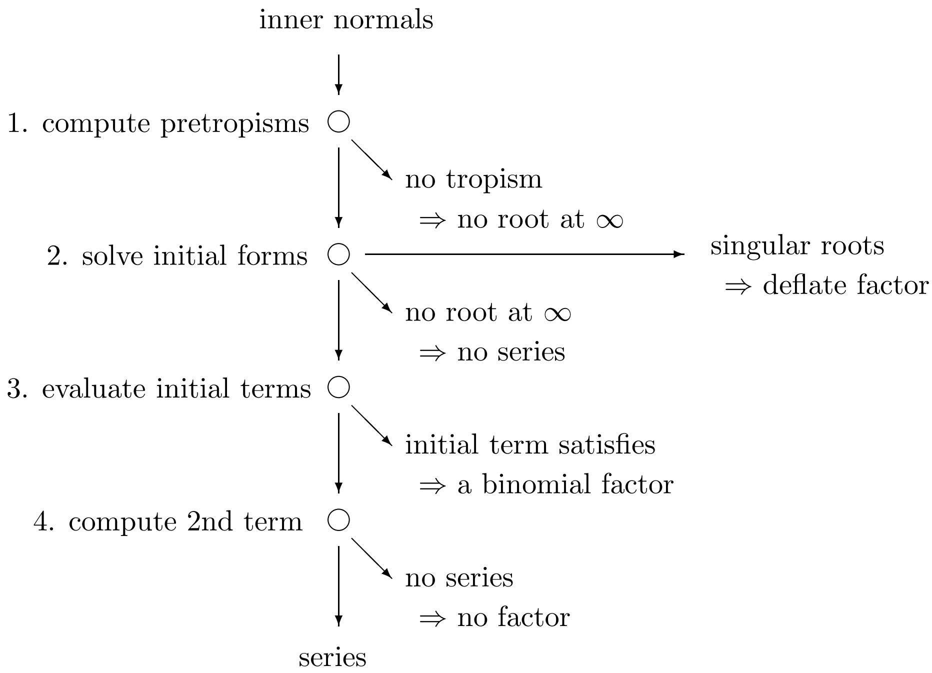


Figure 2. A staggered approach for a regular common factor of two polynomials in two variables.

A variant of Figure 2 is also given in (134).

The tropisms are the leading exponents of the Puiseux series expansion of the common factor, while the solutions of initial forms systems (solutions at infinity) are the leading coefficients in the Puiseux series. Should more terms in the Puiseux series of the common factor be needed, we refer to sparse interpolation techniques (32) (51) (75) (85) as a way to obtain them.

2.2 Amoeba

So far in our polyhedral method, we have been relying on the Newton polygon/polytope (145) (56) as a way to identify monomials, which lead to the solutions at infinity and the leading coefficients in the Puiseux series. We now formally give a justification for the connection between algebraic sets and Newton polygons/polytopes.

2.2.1 Logarithm of Varieties

As we are considering bivariate polynomials in this chapter, we define the *variety* to be the solution set of our bivariate system of polynomials. While varieties are a major object of study in algebraic geometry, we focus on the approach of Bergman (12), who investigated the logarithms of varieties - now a major object of study in the field of tropical algebraic geometry. The following illustration is taken from (12) (4).

$$\begin{array}{rcccc} \log & : & \mathbb{C}^* \times \mathbb{C}^* & \rightarrow & \mathbb{R} \times \mathbb{R} \\ & & (x, y) & \mapsto & (\log(|x|), \log(|y|)) \end{array} \tag{2.1}$$

When we are developing the Puiseux series of the common factor, we start with a *non-zero* solution at infinity, i.e. we are solving the initial form system by looking for solutions in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This is a consequence of the fact that the logarithm is not defined at zero.

The Definition 2.3 is taken from (50) and (140).

Definition 2.3 (Gel'fand, Kapranov, and Zelevinsky 1994)**.** The *amoeba* of a variety is its image under the log map.

In order to give a visual description of an amoeba, consider Example 2.4 and Figure 3, both taken from (4)

Example 2.4.

$$f := \frac{1}{2}x + \frac{1}{5}y - 1 = 0 \qquad A := \left[\ln\left(\left|re^{I\theta}\right|\right), \ln\left(\left|\frac{5}{2}re^{I\theta} - 5\right|\right)\right]. \tag{2.2}$$

Figure 3. The amoeba of a linear polynomial.

While the plot Figure 3 of Example 2.4 appears rather simple, in general, plotting amoebas is very difficult. In (128), homotopy continuation methods (132) are suggested. See also (87).

2.4.1 The Newton Polytope and the Amoeba

As stated in the main introduction, using (145) (56), every polytope in $\mathbb{R}^n$ can be defined in two equivalent ways:

(1) the convex hull of a finite set of points

(2) an intersection of finitely many closed half-spaces

We can use the alternative definition of a Newton polygon, as an intersection of finitely many closed half-spaces, to enclose the amoeba by placing the half-spaces perpendicular to the *tentacles* of the amoeba. This process compactifies the amoeba of the solution set $f^{-1}(0)$ and it defines the Newton polygon of f. The map, given in (120), maps all the points in the variety to the interior of the Newton polygon of f.

As an illustration, we give Example 2.5 and Figure 4, both taken from (4).

Example 2.5 (Example 2.4 continued)**.** Note the direction of the amoeba tentacles with respect to the edges of the Newton polytope.

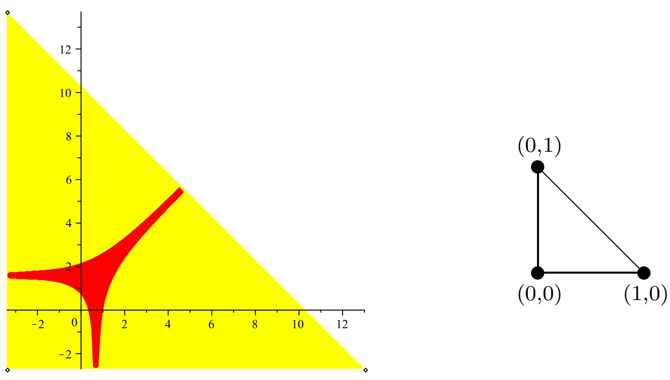


Figure 4. The compactification of the amoeba: the edges of the Newton polygon (displayed at the right) are perpendicular to the tentacles of the amoeba.

For more information on the amoebas, see (140) (94) (95) (68) (96) (99). In the next several subsection, we will see precisely how the relationship between amoebas, the direction of the amoeba tentacles, Newton polygons, and pretropisms leads to initial form systems and the Puiseux series expansions of the common factor.

2.6 Tentacles of the Amoeba

Given the relationship between the compactified amoeba and the Newton polygon, we can represent the tentacles of the amoeba by the edge normals of the Newton polygons. For the purpose of answering the question whether two bivariate polynomials have a common factor, only the directions of the amoeba tentacles, as they stretches out to infinity, play the significant role.

2.6.1 Tentacles Point Towards Infinity

We now consider the problem of the existence of a common factor of two bivariate polynomials from the perspective of tropical algebraic geometry. Now, the emphasis is on the shape of the amoeba of the common factor. As given in (140), the tentacles are the thinning ends of the amoeba. Formally, along ((94), Remark 9), we take into consideration the closure $\bar{A}$ of the amoeba in the toric variety (30) (31) (29), associated with the Newton polygon of the defining polynomial of the amoeba. The tentacles of the amoeba correspond then to the intersections of $\bar{A}$ with the edges of the Newton polygon (94) (140). For plane curves, these intersections consists of isolated points. Again, for more on the amoeba and the toric varieties, see (140) (128) (94) (95) (68) (96) (99) (87) and (30) (31) (29) (120).

From the perspective of tropical algebraic geometry, the problem of finding the common factor of two bivariate polynomials can first be addressed at infinity. Moreover, the question, in the negative, can be decided immediately. The ability to exclude the existence of a common factor provides us with an effective preprocessing criterion. We capture this paragraph in form of Proposition 2.7. We give here the Proposition 2.7 and its proof exactly as we gave it in (4).

Proposition 2.7. *Let f and g be two polynomials. If the amoebas of f and g have no tentacle stretching out to infinity in the same direction, then f and g have no common factor.*

Proof. We proceed by contraposition, assuming f and g have a common factor, say r, and we write $f = rf_1$ and $g = rg_1$. The tentacles of the amoebas of f and g will contain the tentacles of the common factor r because of $f^{-1}(0) = r^{-1}(0) \cup f_1^{-1}(0)$ and therefore $A_f = A_r \cup A_{f_1}$, where the amoebas of f, r, and f_1 are denoted respectively by A_f, A_r, and A_{f_1}. Similarly, for g: $A_g = A_r \cup A_{g_1}$. So A_f and A_g contain both A_r and the same intersection points with the edges of the Newton polygons and therefore tentacles stretching out in the same directions. $\square$

As an illustration of Proposition 2.7, we give the Figure 5, taken from (4).

Considering the fact that amoebas are computationally difficult objects to capture, the Proposition 2.7 may seem as not particularly practical. Given, however, the relationship between the Newton polygons and the amoeba, we can associate the tentacles of the amoeba with the edge normals of the Newton polygons. As such, the directions of the amoeba tentacles correspond to our tropisms, given in Definition 1.11.

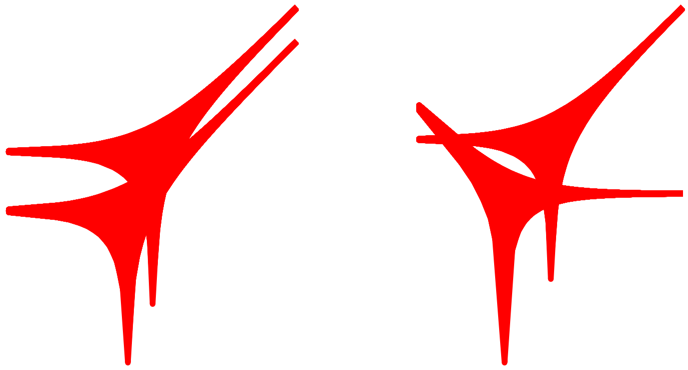


Figure 5. The amoebas of $(\frac{1}{2}x + \frac{1}{5}y + 1)(x + y + 1)$ and $(\frac{1}{2}x + \frac{1}{5}y + 1)(xy + y + \frac{1}{2})$, respectively at the left and right. The amoeba of a product is the union of the amoebas of the factors. Observe the directions of the tentacles.

2.7.1 A Tropicalization and the Normal Fan

The term *a tropicalization*, within our context, is closely related to the common refinement of the normal fans (145) of Newton polygons. Our tropicalization is more limited than those in ((125), §9.4) (99), for reasons explained below. The Definition 2.8 is taken from ((125), §9.4) (99) (145) (56) (4) and given here in an adjusted form.

Definition 2.8 (A Tropicalization)**.** A tropicalization is the set of inward pointing vectors of the form (u, v), normal to the Newton polygon, with gcd(u,v)=1. We denote a tropicalization of a polynomial f as $Trop(f)$.

In order to illustrate the Definition 2.8, we give Example 2.9 and Figure 6, taken from (4).

Example 2.9 (Example 1.5 revisited)**.**

$$f(x,y) = x^3y + x^2y^3 + x^5y^3 + x^4y^5 + x^2y^7 + x^3y^7 \tag{2.3}$$

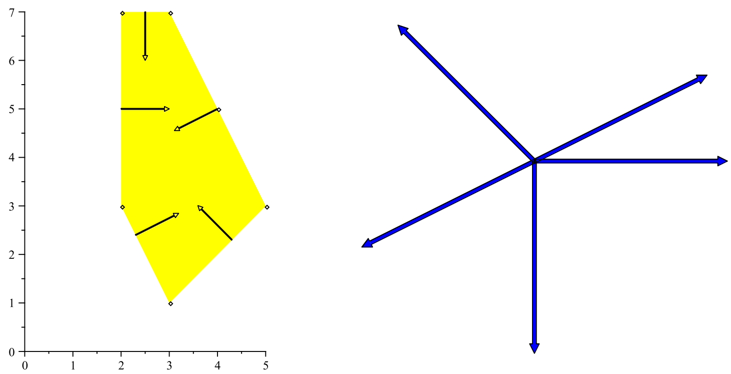
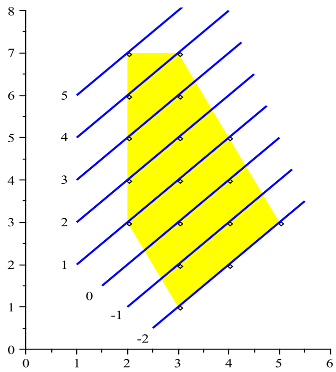


Figure 6. The Newton polygon and its edge normals, forming a normal fan and a tropicalization.

Using the inner product, we continue with Example 1.5 and illustrate how the direction of the amoeba tentacles (i.e. edge normals) are grading all the vertices in the support set. The following Example 2.10 and Figure 7 are taken from (4).

Example 2.10 (Example 1.5 continued)**.** In Figure 7 we use the inner product to grade the points in the support set, with respect to the inner normal $(-1, +1)$.



$$
\begin{aligned}
-1 \times i + (+1) \times j &= +5 \\
-1 \times i + (+1) \times j &= +4 \\
-1 \times i + (+1) \times j &= +3 \\
-1 \times i + (+1) \times j &= +2 \\
-1 \times i + (+1) \times j &= +1 \\
-1 \times i + (+1) \times j &= 0 \\
-1 \times i + (+1) \times j &= -1 \\
-1 \times i + (+1) \times j &= -2
\end{aligned}
$$

Figure 7. Grading the points in the support along $(-1, +1)$.

The grading of vertices (i.e. monomials) in this manner gives rise to homogeneous coordinates, see (30) (29) (31) (133), in the study of toric varieties. For usage of Newton polytopes in connection with Gröbner bases, see (123).

In the general construction of tropical variety of an ideal ((125), §9.4), one frequently introduces the auxiliary variable t. As t does not occur in our setup, our tropicalizations are more limited. For that reason, we refer to our tropicalization as *a tropicalization*, instead of *the tropicalization.* Specifically, in (99), the tropicalization f^τ of a Laurent polynomial f with support A is defined as

$$f^\tau(\mathbf{x}) = \max_{\mathbf{a}\in A}\{\ \log|c_\mathbf{a}| + \langle \mathbf{a}, \mathbf{x}\rangle\ \} \quad \text{for} \quad f(\mathbf{x}) = \sum_{\mathbf{a}\in A} c_\mathbf{a}\mathbf{x}^\mathbf{a}, c_\mathbf{a} \in \mathbb{C}^*. \tag{2.4}$$

In our setup, we chose the *min* convention, since we consider the Puiseux series expansion near zero. Not taking the coefficient size $O(c_\mathbf{a}) = 1$ into consideration and excluding $\log|c_\mathbf{a}|$ from the definition of f^τ, the tropical variety $(f^\tau)^{-1}(0)$ is comprised of those points $\mathbf{v} \in \mathbb{Q}$, where at least two of the monomials have the extremal (i.e. *minimal*) value $\langle \mathbf{a}, \mathbf{v}\rangle$.

2.11 Tropisms in the Plane

In this section we will establish why pretropisms, which may eventually become tropisms, are an effective concept in determining when two bivariate polynomials have a common factor. In particular, we will show how pretropisms give us conditions, which lead to an effective preprocessing stage to decide when a common factor can occur.

2.11.1 The Particular Direction of Varieties

As stated in the main introduction chapter, the existence of a positive dimensional solution set for a system of polynomials is directly related to the position of the associated Newton polytopes with respect to each other. We now translate the meaning of this statement into the language of tropicalizations and pretropisms.

We begin by showing when the common factor cannot exist and we show that, that is an immediate consequence of the Theorem 1.8.2, the Theorem B of Bernshtein (14). We state

Bernshtein's Theorem B for Newton polygons in the language of this chapter and give the proof for this specific case next, in form of Theorem 2.11.1. A version of Theorem 2.11.1 is also given in (134), based on an earlier version of (4). We give Theorem 2.11.1 and its proof here exactly as we gave it in (4).

Theorem 2.11.1. *Let f and g be two polynomials in x and y. If $Trop(f) \cap Trop(g) = \emptyset$ then the system $f(x, y) = 0 = g(x, y)$ has no solutions at infinity.*

Proof. Rephrasing part (a) of (14, Theorem B), using our notations and restricting to two polynomials f and g in x and y: If the system defined by the equations $in_{\mathbf{v}}(f)(x, y) = 0$ and $in_{\mathbf{v}}(g)(x, y) = 0$ does not have any roots in $(\mathbb{C}^*)^2$ for any $\mathbf{v} \neq (0, 0)$, then all roots of the system defined by $f(x, y) = 0$ and $g(x, y) = 0$ are isolated and their number equals the mixed volume of the polygons spanned by the supports of f and g. The condition $Trop(f) \cap Trop(g) = \emptyset$ implies there is no $\mathbf{v}$ so that $in_{\mathbf{v}}(f)$ and $in_{\mathbf{v}}(g)$ have each at least two monomials. Equivalently, for all $\mathbf{v} \neq (0, 0)$, $in_{\mathbf{v}}(f)$ or $in_{\mathbf{v}}(g)$ (possibly both for general $\mathbf{v}$, but at least one of them for particular choices of $\mathbf{v}$) consist only of one monomial. Therefore the system defined by the equations $in_{\mathbf{v}}(f)(x, y) = 0$ and $in_{\mathbf{v}}(g)(x, y) = 0$ does not have any roots in $(\mathbb{C}^*)^2$. Hence, the system defined by $f(x, y) = 0$ and $g(x, y) = 0$ has no roots at infinity. $\square$

Next, we show when there cannot be a common factor, using Proposition 2.7. We give the Proposition 2.12 and its proof here exactly as we gave it in (4).

Proposition 2.12. *If for two polynomials f and g: $Trop(f) \cap Trop(g) = \emptyset$, then f and g have no common factor.*

Proof. By Theorem 2.11.1, $Trop(f) \cap Trop(g) = \emptyset$ implies there is no common root at infinity. However, if f and g had a common factor, they would have a common root at infinity as well. This common root would then correspond to one of the ends of the tentacles of the amoeba of the common factor as in Proposition 2.7. $\square$

As an illustration of when two polynomials do not have a common factor, we give the following Figure 8, which shows the configuration of normal fans for such a case. The Figure 8 is taken form (4).

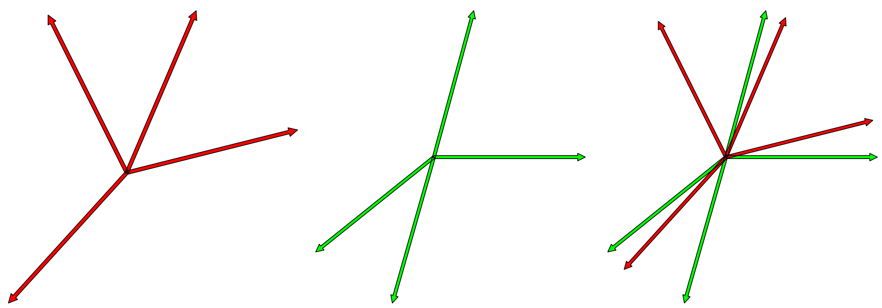


Figure 8. The first two pictures from the left represent the normal fans of two polynomials. By superposition of the fans at the far right we see there are no common directions. Therefore, for all nonzero coefficients, the polynomials can have no common factor.

2.12.1 A Certificate for a Numerical Computation

As we have seen, the pretropisms can tell us when the common factor does not exist. That property classifies the pretropism as a certificate, which can be used in a computation. The Example 2.13 and Figure 9 illustrate the configuration of pretropisms (normal fans) when there exists a common factor. We call every pretropisms, which leads to the Puiseux series expansion of the common factor, a tropism (91). The Example 2.13 and Figure 9 are taken from (4).

Example 2.13.

$$F(\mathbf{x},\mathbf{y}) = \begin{cases} r = 2xy + x^2y + 9xy^2 + 7x^3y + x^4y + 9x^3y^2 \\ f = r\left(6x^{10} + 6x^6y^3 + 5x^4y + 3x^3y^5 + 5y^4 + 5y^5\right) = 0 \\ g = r\left(2x^{13} + 5x^9 + x^6y^3 + 8x^6y^8 + 6x^2 + 5y^5\right) = 0 \end{cases} \tag{2.5}$$

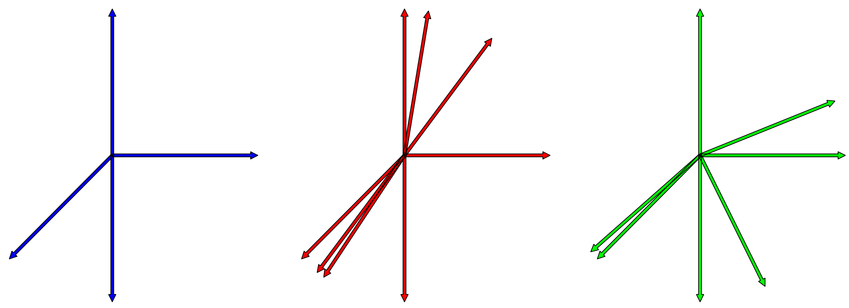
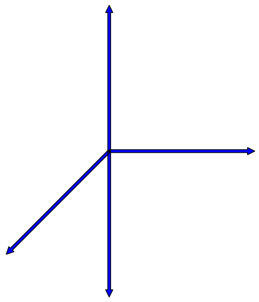


Figure 9. The normal fan at the left is the normal fan of the factor common to two polynomials f and g. The normal fans of f and g are displayed in the middle and at the right. We recognize the fan at the left as a part of the other fans.

The trivial factor xy is excluded, as we do not consider the coordinate axes. We now consider the initial form system of polynomial system in (Equation 2.5) in one of the directions given by the edge normals. We take the inner product between the edge normal and the monomial exponents and keep only those monomials where the minimal value of the inner product has been achieved at least twice. The Example 2.14 and Figure 10 are taken from (4).

Example 2.14 (Example 2.13 continued)**.** The initial forms:



Take $(1, 0)$ as one of the 4 directions:

$in_{(1,0)}(r) = 2xy + 9xy^2$

Initial forms of f and g:

$in_{(1,0)}(f) = 55xy^6 + 10xy^5 + 45xy^7 = in_{(1,0)}(r)(5y^4 + 5y^5)$

$in_{(1,0)}(g) = 10xy^6 + 45xy^7 = in_{(1,0)}(r)(5y^5)$

Figure 10. The normal fan of the common factor and the initial form systems corresponding to the direction (1,0).

In Example 2.15, we give the factored initial form system of Example 2.14, to make the computation of the common root at infinity more clear. Example 2.15 is taken from (4).

Example 2.15 (Example 2.14 continued)**.**

$$\begin{cases} in_{(1,0)}(f) = x\left(5y^5(y+1)(2+9y)\right) = 0 \\ in_{(1,0)}(g) = x\left(5y^5(2+9y)\right) = 0 \end{cases} \tag{2.6}$$

The initial form (Equation 2.6) has only one non-trivial solution, $y = \frac{-2}{9}$. We can set $x = t$, and let it be the free parameter. The beginning of Puiseux series expansion in the direction of tropism $(1, 0)$ is given in Example 2.16. Example 2.16 is taken from (4).

Example 2.16 (Example 2.14 continued)**.**

$$\begin{aligned} x &= t^1 \\ y &= t^0\ (\tfrac{-2}{9} + \dots) \end{aligned} \tag{2.7}$$

Note that when $t = 0$, we are at infinity.

As we are allowing polynomials to have approximate coefficients, the solution of the initial form system, the root at infinity, will also be an approximate solution. For a procedure on how to bound the radius of convergence for the Newton's method, we refer to α-theory in (18). See also (112) for inclusion of approximate functions in this manner. For an altogether different approach, see (60).

In order to work with tropisms, which are not standard basis vectors, we need a special kind of coordinate transformation, called unimodular coordinate transformation. In (21), the unimodular coordinate transformations are called power transformations. Unimodular coordinate transformations are described in (17) (131) (41) (133) (36) (134). The Definition 2.17 is taken from (131) (21) (138) (17) (136) (132) (133) (139) (36) (4) and given here in an adjusted form.

Definition 2.17. For a pretropism (u, v), normalized so the greatest common divisor $gcd(u, v) = 1$, the unimodular matrix M

$$M = \begin{bmatrix} u & v \\ -l & k \end{bmatrix}, \quad gcd(u, v) = 1 = ku + lv = \det(M) \tag{2.8}$$

defines the unimodular coordinate transformation $x = X^u Y^{-l}$ and $y = X^v Y^k$.

Continuing with the Example 2.13, we investigate Puiseux series expansions of system in (Equation 2.5), using a pretropism $(-1,-1)$, which is not a standard basis vector. The Example 2.18 is taken from (4).

Example 2.18 (Example 2.13 continued)**.** Initial form system in direction of $(-1,-1)$:

$$\begin{cases} in_{(-1,-1)}(f) &= 54x^{13}y^2 + 6x^{14}y = \left(x^4y + 9x^3y^2\right) 6x^{10} \\ \\ in_{(-1,-1)}(g) &= 72x^9y^{10} + 8x^{10}y^9 = \left(x^4y + 9x^3y^2\right) 8x^6y^8 \end{cases} \tag{2.9}$$

In Example 2.19, we show how we use the unimodular matrix M to obtain new coordinates, denoted X and Y, and transform the initial form system. The Example 2.19 is taken from (4).

Example 2.19 (Example 2.18 continued)**.** The unimodular coordinate transformation matrix $M = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$, $gcd(-1,-1) = (-1)(-1)+0(-1) = 1$, leads to new coordinates $x = X^{-1}Y^0$, $y = X^{-1}Y^{-1}$, which are used to transform the initial form system (Equation 2.9), resulting in

$$\begin{cases} in_{(-1,-1)}(f)(x = X^{-1}Y^0, y = X^{-1}Y^{-1}) &= (54Y + 6)/(X^{15}Y^2) &= 0 \\ \\ in_{(-1,-1)}(g)(x = X^{-1}Y^0, y = X^{-1}Y^{-1}) &= (72 + 8Y)/(X^{19}Y^{10}) &= 0 \end{cases} \tag{2.10}$$

We want to draw the attention to the effects of the unimodular coordinate transformations on the initial form systems. In particular, notice how the monomials in the X variable have the equal minimal degree in each equation, which allows us to factor them out entirely. As a justification, consider the following Example 2.20, given for any monomial of the form $x^a y^b$. Example 2.20 is taken from (4).

Example 2.20.

$$(X^u Y^{-l})^a (X^v Y^k)^b = X^{au+bv} Y^{-la+kb} = X^{\langle (a,b),(u,v)\rangle} Y^{-la+kb} \tag{2.11}$$

As all monomials of the form $x^a y^b$ lie on the edge, perpendicular to the edge normal (u, v), we can turn any initial form system in two variables into a univariate system of equations.

The initial form system (Equation 2.10) has a solution $X = t$, $Y = \frac{-1}{9}$, where $X = t$ is again the free parameter. We can return this common solution at infinity back to the original (x, y) coordinates, using the same coordinate transformation. We illustrate this with Example 2.21, taken from (4).

Example 2.21 (Example 2.18 continued)**.**

$$\begin{cases} X = t \\ Y = -1/9 \end{cases} \quad \begin{pmatrix} x = X^{-1}Y^0 \\ y = X^{-1}Y^{-1} \end{pmatrix} \quad \Rightarrow \quad \begin{cases} x = t^{-1} \\ y = t^{-1}\,(-9 + \dots). \end{cases} \tag{2.12}$$

We remarked earlier that we are expanding our Puiseux series for $t \approx 0$. As $t \to 0$, we are approaching the location of the common root at infinity. This is consistent with stretching of the amoeba tentacles towards the infinity (140) and the location of the common solution.

The tropicalization $Trop(r)$ of the common factor in (Equation 2.5) of Example 2.13, consisted of four pretropisms. All four pretropisms are given in Example 2.22, taken from (4).

Example 2.22 (Example 2.13 continued)**.**

$$Trop(r) = \{ \ (1,0),(0,1),(-1,-1),(0,-1) \ \} \,. \tag{2.13}$$

Adding up the pretropisms (edge normals) in $Trop(r)$, will equal to zero. This is known as the balancing condition (107). In (54), it is used to factor tropical polynomials.

2.22.1 The Cost of the Preprocessing Algorithm

In this section we revisit Figure 2 in form of Figure 11 and offer the Figure 11 as an illustration of our algorithm. The Figure 11 is taken from (4).

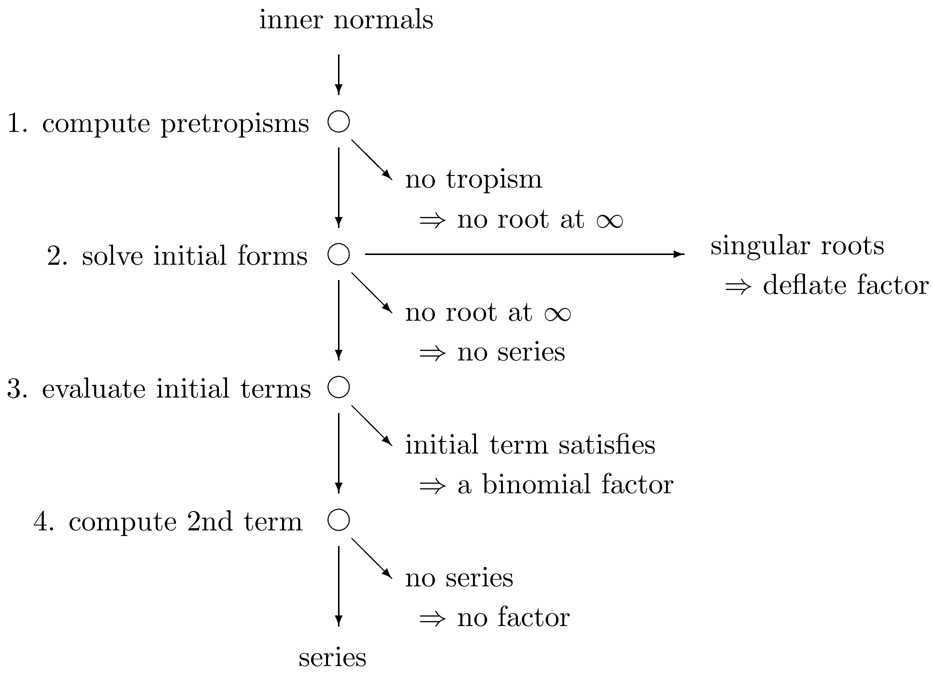


Figure 11. A staggered approach for a regular common factor of two polynomials in two variables.

Again, a variant of Figure 11 is also given in (134).

We give the cost of the pretropisms computation for a system of two bivariate polynomials in form of a Proposition 2.23. We give Proposition 2.23 and its proof here exactly as we gave it in (4).

Proposition 2.23. *Let f and g be two polynomials given by respectively n and m monomials. The cost of computing pretropisms $Trop(f) \cap Trop(g)$ is $O(n\log(n)) + O(m\log(m))$.*

Proof. It takes $O(n \log(n))$ operations for computing a tropicalization $Trop(f)$ because computing the convex hull of a set of n points amounts to sorting the points in the support. Likewise, computing $Trop(g)$ takes $O(m \log(n))$ operations. Merging sorted lists of normals to find the pretropisms in $Trop(f) \cap Trop(g)$ takes linear time in the length of the lists. □

While coefficients in the system of polynomials always play a role in the process of solving the system, in the case of the non-existence of pretropisms, coefficients play no role. For bivariate polynomial systems, the non-existence of pretropisms is an exact certificate that the common factor does not exist.

When there are pretropisms, they lead to initial form systems, which we must solve in order to obtain the common root at infinity. We now give the cost of solving an initial form system in terms of a Proposition 2.24. We give Proposition 2.24 and its proof here exactly as we gave it in (4).

Proposition 2.24. *Let f and g be two polynomials given by respectively n and m monomials. For every pretropism $t \in Trop(f) \cap Trop(g)$ it takes at most $O((n+m)^3)$ operations to find a common solution in $(\mathbb{C}^*)^2$ to the initial form system defined by $\boldsymbol{v}$.*

Proof. For a pretropism $\mathbf{v}$, we solve the initial form system. In particular, an initial root $\mathbf{z}$ satisfies

$$\begin{cases} in_{\mathbf{v}}(f)(\mathbf{z}) = 0 \\ in_{\mathbf{v}}(g)(\mathbf{z}) = 0 \end{cases} \quad \mathbf{z} \in (\mathbb{C}^*)^2. \tag{2.14}$$

We perform a unimodular transformation so the pretropism we consider is a unit vector, (1,0) or (0,1). This implies that the two equations in the initial form system are defined by two polynomials in one variable. To decide whether two polynomials in one variable admit a common solution we determine the rank of the Sylvester matrix. Using singular value decomposition, the cost of this rank determination is cubic in the size of the matrix. For rank deficient matrices, the singular vectors give the coefficients of the common factor. The roots of this common factor are the eigenvalues of a companion matrix. The cost of methods to compute eigenvalues is also cubic in the dimension of the matrix. □

In the proof of the Proposition 2.24, we used the singular value decomposition. However, the rank revealing methods (89) are more efficient in practice. It may appear hard, in general, to find the common root at infinity, should the root turn out to be a root with multiplicity. After the unimodular coordinate transformation, however, our initial form systems are univariate. We can then use methods of (143) to find the solutions.

2.25 Germs

Having obtained an initial form system via a pretropism, which has a solution in $\mathbb{C}^*$, we can start building the Puiseux series of the common factor term by term. The solution at infinity is the leading coefficient of such a Puiseux series expansion. In this way, the pretropism becomes a tropism. For information on *germs*, see (34) (44) (79).

2.25.1 How Amoebas Grow Their Tentacles Starting at Infinity

Before we show how the amoeba grows its tentacles, we must ensure that the solution at infinity is an isolated solution. Consider the following form of the Puiseux series of the common factor, which we want to build term by term. We express this in form of Definition 2.26, which is related to the notion of *germ* (34) (44) (79). The Definition 2.26 is taken from (4).

Definition 2.26. The common factor is given by the equation $r(x,y) = 0$ and it has no multiple factors, with the exception of a monomial factor, which is excluded here as we do not consider coordinate axes. In canonical form for the tropism $(1,0)$, a *Puiseux series for* $(1,0)$ has the form

$$\begin{cases} X = t \\ Y = c_0 + c_1 t^w (1 + O(t)), \quad c_0, c_1 \in \mathbb{C}^*, w \in \mathbb{N}, w > 0. \end{cases} \tag{2.15}$$

For a general tropism $(u,v) \in \mathbb{Z}^2$, with $gcd(u,v) = ku + lv = 1$, a *Puiseux series for* (u,v) has the form

$$\begin{cases} x = t^u \left(c_0 + c_1 t^w (1 + O(t))\right)^{-l} & c_0, c_1 \in \mathbb{C}^* & x = X^u Y^{-l} \\ y = t^v \left(c_0 + c_1 t^w (1 + O(t))\right)^{k} & w \in \mathbb{N}, w > 0 & y = X^v Y^k \end{cases} \tag{2.16}$$

Following the setup described in Definition 2.26, we use unimodular coordinate transformations to transform initial form systems to obtain the first coefficient in the Puiseux series expansion. Example 2.27 is taken from (4).

Example 2.27.

$$\begin{cases} in_{(1,0)}(f)(t, c_0) = 0 \\ in_{(1,0)}(g)(t, c_0) = 0 \end{cases} \tag{2.17}$$

Eventually, we transform the original system as well in order to obtain the second term in the Puiseux series expansion. Should the tropism be a standard basis vector, then it is not necessary to perform unimodular coordinate transformation.

We continue with system of Example 2.13 (Equation 2.5), and illustrate the computation of the second term in direction of tropism $(1,0)$. The Example 2.28 is taken from (4).

Example 2.28 (Example 2.13 continued)**.**

$$\begin{cases} f = 10xy^5 + 45xy^7 + 55xy^6 + x^2(\text{ 30 other terms }) \\ \\ g = 45xy^7 + 10xy^6 + x^2(\text{ 34 other terms }) \end{cases} \tag{2.18}$$

As this polynomial system is rather large, we show only monomials, which form the initial form system. Again, because tropism $(1,0)$ is a standard basis vector, we do not perform change of coordinates.

Next, we let $f_1 = f/x$ and $g_1 = g/x$. Solving yields the solution at infinity $z = \frac{-2}{9}$, as given before in (Equation 2.6).

Using the solution at infinity, the Puiseux series expansion of the common factor takes the form, given in Example 2.29. Example 2.29 is taken from (4).

Example 2.29 (Example 2.28 continued)**.**

$$\begin{cases} x = t^1 \\ y = -\frac{2}{9}t^0 + Ct(1 + O(t)), \quad C \in \mathbb{C}^*. \end{cases} \tag{2.19}$$

Substituting this form into the system of polynomials Equation 2.18, leads to a linear and overdetermined system. Example 2.30 is taken from (4).

Example 2.30 (Example 2.29 continued)**.**

$$\begin{cases} -\frac{1120}{531441} - \frac{1120}{59049}C = 0 \\ \\ -\frac{320}{59049} - \frac{320}{531441}C = 0 \end{cases} \tag{2.20}$$

The solution $C = \frac{-1}{9}$ satisfies this linear system and provides us with the second term in the Puiseux series expansion of the common factor, as given in Example 2.31, taken from (4).

Example 2.31 (Example 2.29 continued)**.**

$$\begin{cases} x = t \\ y = -\frac{2}{9} - \frac{1}{9}t(1 + O(t)). \end{cases} \tag{2.21}$$

The second term in the Puiseux series of the common factor happens to have a degree equal to one. It may happen that the second term does not exist at all or, if it does, it may have a degree greater than one. We now give an explicit condition on the second term exponent in the Puiseux series in form of Proposition 2.32. We give the Proposition 2.32 and its proof here exactly as we gave it in (4).

Proposition 2.32. *Given are two polynomials f and g in X and Y, after a unimodular co-ordinate transformation and a multiplication or division by a monomial so f and g have the form*

$$\begin{cases} f(X,Y) = p(Y) + P(X,Y), & p(Y) = in_{(1,0)}(f)(X,Y), \\ g(X,Y) = q(Y) + Q(X,Y), & q(Y) = in_{(1,0)}(g)(X,Y). \end{cases} \tag{2.22}$$

By the given form of f and g, the initial forms p and q are polynomials in Y with nonzero constant term. Moreover, all terms in the remainder polynomials P and Q have a positive power in X. Let $c_0 \neq 0$:

$$\begin{cases} p(c_0) = 0, \quad p'(c_0) \neq 0, \quad f(t,c_0) \neq 0 \\ q(c_0) = 0, \quad q'(c_0) \neq 0, \quad g(t,c_0) \neq 0 \end{cases} \qquad p' = \frac{\partial p}{\partial Y}, q' = \frac{\partial q}{\partial Y}. \tag{2.23}$$

Let $P_k \in \mathbb{C} \setminus \{0\}$: $P(X,c_0) = P_k X^k(1+O(X))$ and $Q_l \in \mathbb{C} \setminus \{0\}$: $Q(X,c_0) = Q_l X^l(1+O(X))$. If $k = l$ and $Q_k p'(c_0) - P_k q'(c_0) = 0$, then $c_1 = -P_k/p'(c_0) = -Q_k/q'(c_0)$ is the coefficient of the second term in $(X = t, Y = c_0 + c_1 t^k)$, the leading part of a Puiseux series expansion of a regular common factor of f and g. If $k \neq l$ or $Q_k p'(c_0) - P_k q'(c_0) \neq 0$, then f and g have no common factor with expansion starting at $(X = t, Y = c_0)$.

Proof. Let us consider the effect of substituting $X = t, Y = c_0 + c_1 t^w$ into f and g, using the value for the initial root c_0 and treating the second coefficient c_1 and the exponent w as unknowns. We may write $p(Y)$ as

$$p(Y) = \alpha_1(Y - c_0)(Y - \alpha_2)(Y - \alpha_3)\cdots(Y - \alpha_d), \quad d = \deg(p), \alpha_i \in \mathbb{C}, \tag{2.24}$$

$i = 1, 2, 3, \ldots, d$. Because c_0 is a regular root of the initial forms: $p'(c_0) \neq 0$ and $c_0 \neq \alpha_i$, $i = 2, 3, \ldots, d$. Then:

$$\begin{aligned} p(Y = c_0 + c_1 t^w) &= \alpha_1 (c_1 t^w)(c_0 + c_1 t^w - \alpha_2)(c_0 + c_1 t^w - \alpha_3) \cdots && (2.25) \\ & \quad \cdots (c_0 + c_1 t^w - \alpha_d) && (2.26) \\ &= c_1 t^w \alpha_1 (c_0 - \alpha_2)(c_0 - \alpha_3) \cdots (c_0 - \alpha_d)(1 + O(t^w)) && (2.27) \\ &= c_1 t^w p'(c_0)(1 + O(t^w)). && (2.28) \end{aligned}$$

Similarly: $q(Y = c_0 + c_1 t^w) = c_1 t^w q'(c_0)(1 + O(t^w))$.

Substitution of $X = t$ and $Y = c_0 + c_1 t^w$ into $P(X, Y)$ leads to $P_k t^k (1 + O(t))$ for a nonzero constant P_k. Observe that the lowest power of t does not involve c_1, but only depends on c_0. If the constant P_k were zero, then this would imply $P(t, c_0) = 0$ for all t and also $f(t, c_0) = 0$, contradicting the assumption $f(t, c_0) \neq 0$. Note that $f(t, c_0) = 0$ occurs in case the common factor is binomial, i.e.: consists only of two monomials with nonzero coefficients.

The result of substituting $X = t, Y = c_0 + c_1 t^w$ into f and g is then

$$\begin{cases} f(X = t, Y = c_0 + c_1 t^w) = c_1 t^w p'(c_0)(1 + O(t^w)) + P_k t^k (1 + O(t)) = 0 \\ g(X = t, Y = c_0 + c_1 t^w) = c_1 t^w q'(c_0)(1 + O(t^w)) + Q_l t^l (1 + O(t)) = 0. \end{cases} \qquad (2.29)$$

For the dominant terms to vanish, we must have $w = k = l$ and solve

$$\begin{bmatrix} p'(c_0) & P_k \\ q'(c_0) & Q_k \end{bmatrix} \begin{bmatrix} c_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.30}$$

For this linear system in c_1 to have the nonzero solution $c_1 = -P_k/p'(c_0) = -Q_k/q'(c_0)$ the determinant $Q_k p'(c_0) - P_k q'(c_0)$ must equal zero.

To prove the second if statement of the proposition, we first observe that if $Q_k p'(c_0) - P_k q'(c_0) \neq 0$, the linear system in c_1 has no solution and hence there is no Puiseux series expansion of a common factor starting at $(X = t, Y = c_0)$. Now we consider the case $k \neq l$. If $k < l$, then the determinant of the linear system in c_1 equals $-P_k q'(c_0) \neq 0$. Otherwise, for $k > l$, the determinant is $Q_l p'(c_0) \neq 0$. Therefore if $k \neq l$, no solution for c_1 exists and there is also no Puiseux series expansion. □

The Proposition 2.32 allows us to obtain the exponent and the coefficient of the second term in a concrete manner. While the exponent w has an exact value, the coefficient c_1 consists of an approximate solution of the linear system we had to solve. Nevertheless, the exponent, and the coefficient up to a certain predetermined tolerance, are the final certificates that there exists a common factor for the two bivariate polynomials in the system.

2.32.1 Convergence of Puiseux Series in Their Canonical Form

We consider the Puiseux series to be in their canonical form if their first terms are given with respect to the tropism $(1, 0)$. In particular, the Puiseux series in canonical form is given by: $(X = t,\ Y = c_0 + c_1 t^w (1 + O(t)))$, for $c_0, c_1 \in \mathbb{C}^*$ and $w \in \mathbb{N}, w > 0$. For any tropism

(u, v), the Puiseux series can be brought into this form. This is the effect of the unimodular coordinate transformation, where X and Y denote the new coordinates.

In order to verify whether the newly obtained term $c_1 t^w$ is a valid second term of the series, we substitute the two terms of the series ($X = t,\ Y = c_0 + c_1 t^w$) into the original polynomial system, whose coordinates have also been transformed using the same unimodular coordinate transformation.

We analyze the effects of ($X = t,\ Y = c_0 + c_1 t^w$) substitution by disregarding the terms of order greater or equal to $O(t^{w+1})$ and we compare anything strictly smaller than $O(t^{w+1})$ with the lowest power of t, obtained from substitution of ($X = t,\ Y = c_0$) into the transformed polynomial system. As we are dealing with approximate values for c_0 and c_1, we take a certain predetermined tolerance into consideration while comparing the values.

We formally express this verification procedure in Example 2.33, using the notation as it was used in Proposition 2.32. Example 2.33 is taken from (4).

Example 2.33.

$$\begin{cases} f(X = t, Y = c_0) = O(t^{m_1}), \quad m_1 > 0, \\ g(X = t, Y = c_0) = O(t^{m_2}), \quad m_2 > 0, \end{cases} \tag{2.31}$$

and

$$\begin{cases} f(X = t, Y = c_0 + c_1 t^w) = O(t^{m_1 + k_1}), \quad k_1 > 0, \\ g(X = t, Y = c_0 + c_1 t^w) = O(t^{m_2 + k_2}), \quad k_2 > 0. \end{cases} \tag{2.32}$$

The expressions in (Equation 2.31) and (Equation 2.32) of Example 2.33 describe the extent to which the first two terms of the Puiseux series correspond to the points on the curve, defined by the system of two bivariate polynomials with approximate coefficients. The power of t consists of exact data. It gives the beginnings of an algebraic description of the common factor. However, the coefficient of t is not. As the coefficients are numerical approximations, the immediate vicinity of the point $(0, c_0)$ in $\mathbb{C}^2$, where we start the development of the Puiseux series of the common factor, is treated as a disk with a sufficiently small radius, where we may continue to build the Puiseux series of the common factor, term by term.

In this subsection, we used the unimodular coordinate transformations to bring the Puiseux series expansion in its canonical form. The unimodular coordinate transformations are also used in accurate evaluation of polynomials (36). For weighted projective transformations, used in rescaling of high degree polynomials, see (133).

We conclude the section on *germs* by saying that in our setup, the tropisms lead to solutions at infinity, which we use to build the Puiseux series of the common factor. This is an analog of the relationship between a tentacle of an amoeba and a germ. For more information on *germs*, see (34) (44) (79).

2.34 The Algorithm

In this section we give the formal description of the the algorithms, which we covered in this chapter. Even though, through out Chapter 2 we used variables x and y, formally, a polynomial f (similarly for polynomial g) has the form

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad c_{\mathbf{a}} \in \mathbb{C}^*, \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \tag{2.33}$$

In order to represent the support set A, we use a list. For the coefficients, we use a lookup table, which we denote $C[A]$ and we let the exponent vectors $\mathbf{a} \in A$ denote the indices of the lookup table $C[A]$. The variables X and Y denote new coordinates, after the unimodular coordinate transformation, as given in Definition 2.17. We let Mz denote the value of Y.

For the two polynomials f and g, we denote their support sets as A_f and A_g. Our algorithms take the two polynomials f and g as input in form of tuples $(A_f, C[A_f])$ and $(A_g, C[A_g])$. The Algorithm 2.35 is taken from (4).

Algorithm 2.35. *Tropisms and Initial Roots*

Input: $(A_f, C[A_f])$ *and* $(A_g, C[A_g])$.

Output: $T = \{\ (u,v) \in \mathbb{Z}^2 \setminus (0,0) \mid (u,v) \ \ \textit{is tropism}\ \}$,

$$R[T] = \{\ \{\ z \in \mathbb{C}^* \mid in_{(u,v)}(f)(Mz) = 0, in_{(u,v)}(g)(Mz) = 0\ \} \mid (u,v) \in T\ \}.$$

For the cost of Algorithm 2.35, see Proposition 2.23 and Proposition 2.24. Next, we give the Algorithm 2.36, which covers the second stage in our polyhedral method for plane curves.

The Algorithm 2.36 is taken from (4).

Algorithm 2.36. *Second Term of Puiseux Expansion*

Input: $(A_f, C[A_f])$, $(A_g, C[A_g])$, T, *and* $R[T]$.

Output: $W[R[T]] = \{\ (c,w) \in \mathbb{C}^* \times \mathbb{N}^+ \mid z \in Z \in R[T]\ \}$.

The second term of the Puiseux series is given by the elements of the set $W[R[T]]$. For every $(c,w) \in W[R[T]]$, the first two terms of the Puiseux series, in coordinates after the unimodular transformation, have the form as given in Example 2.37, taken from (4).

Example 2.37.

$$\begin{cases} X = t^1 \\ Y = z + ct^w \end{cases} \tag{2.34}$$

For the conditions on the existence of the exponent w, see Proposition 2.32. We can use the unimodular coordinate transformation to return the series expansion in (Equation 2.34) back to the original coordinates.

2.38 Summary of the Chapter

In this chapter we covered the construction of a symbolic-numeric algorithm to find the common factor of two bivariate polynomials with approximate complex coefficients. We first expressed the two bivariate polynomials as a system of polynomials and then proceeded to determine whether or not their solution set is a plane curve. We then showed how tropical methods lead to the Puiseux series of the common factor when it exists or efficiently exclude it when the common factor does not exist. By covering the development of curves in the plane, this algorithm represents the first stage in the development of our polyhedral method to develop general algebraic sets. The most significant results of this chapter are expressed by the Algorithm 2.35, Algorithm 2.36 and the Proposition 2.32 and its proof. The Proposition 2.32 is very important because it provides a condition for the existence of the second term in the Puiseux series of the common factor. Our approach is related to methods in numerical algebraic geometry, given in (134).

CHAPTER 3

SPACE CURVES

3.1 Introduction

The content of this Chapter 3 is based on previously published material (3). All algorithms, algorithm outlines, computational results, concepts, definitions (and variations), examples, figures, ideas, illustrations, pictures, propositions and their proofs, tables, etc. are taken from (3), unless otherwise indicated. This Chapter 3 is a summary of (3).

In this chapter we extend the tropical methods of the previous chapter to develop space curves, defined by a more general system of polynomials. In particular, we aim at polynomial systems defined along the lines of the system Equation 1.1. Similar to the approach we applied to plane curves, we will start the development of the space curve at infinity. Using a single tropism, we will obtain an initial form system. The obtained initial form system will be transformed, via the tropism based unimodular coordinate transformation, and solved. The isolated solutions of the transformed initial form system are the solutions at infinity. As before, they will be the leading coefficients of the Puiseux series expansion of the space curve and the tropism will be its leading exponents.

In our setup, the tropisms determine where the isolated solutions at infinity can be obtained. We perform a unimodular coordinate transformation to isolate these solutions. The process of using the tropism and changing the coordinates is an alternative to intersecting the space curve at infinity with hyperplanes and isolate the solutions in that manner. For more on unimodular transformations, facet normals and the introduction of projective coordinates, see (131) (21) (138) (17) (133).

A space curve, which is a solution set to a system of polynomials, like (Equation 1.1), stretches from the torus $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to infinity. We use this property to start the development of the Puiseux series of the space curve. Our approach can be seen as a symbolic-numeric variant of the proof of the fundamental theorem of tropical algebraic geometry in (73). While we restrict ourselves to space curves in this chapter, we point out that the first use of this approach is to be found in (134), which aimed at a generalization of this approach to any dimension. In this Chapter 3 and the next Chapter 4, we generalized the approach further than what was accomplished in (134). As a result, in Chapter 3 (and Chapter 4), we build on and expand the results of (134). For us, the work of (134) is a major influence, which is reflected in our common terminology, definitions, choice of examples, illustrations, etc.

In our approach, the Puiseux series of a space curve, which satisfies a system of polynomials, has the form given in Example 3.2. The Example 3.2 is taken from (3) and (134), where it is given in compact form, using a slightly different notation.

Example 3.2.

$$\left.\begin{aligned} x_0 &= t^{v_0} \\ x_1 &= t^{v_1}(r_1 + c_1 t^{w_1} + \dots) \\ &\vdots \\ x_{n-1} &= t^{v_{n-1}}(r_{n-1} + c_{n-1} t^{w_{n-1}} + \dots) \end{aligned}\right. \qquad c_1, \dots, c_{n-1} \in \mathbb{C}. \tag{3.1}$$

where $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ is a tropism and $r_1, \dots, r_{n-1}$ solution of the initial form system.

Note the form of the Puiseux series and the particular role, which is played by the first coordinate. By default, our application of the unimodular coordinate transformation factors out the first variable in order to find the isolated solutions of the initial form system. The factorization of the first variable frees the first coordinates, which is then used as a free parameter, here denoted as t.

The development of the Puiseux series of a space curve can be split into three parts. In each part, we compute the following in the given order:

1. pretropism i.e. the leading powers candidates for the Puiseux series (105) (141) (91) (107).
2. leading coefficients
3. second coefficient in the Puiseux series

The method, which is used to compute the second coefficient in the Puiseux series, can be adapted to compute more terms in the series. Adaptation consists of a a simple back substitution and solving of a linear system. For us, however, the existence of the second term is a certificate that there exists a space curve as a solution. After we have computed the second term, we do not proceed with the computation of more terms.

With this chapter, we will start to address another issue, which was not addressed in the previous chapter: the exploitation of symmetry when it present in a system of polynomials. Recognizing and taking advantage of symmetry reduces the computational time of our polyhedral method. For example, when a system of polynomials is invariant under the some form of permutation of the variables, we focus here on cyclic permutation, then it is possible to reduce the computational aspects of our polyhedral method by considering just the generators of the orbits. The generator may be a solution set of an initial form system or a tropism. Applying the permutation to such a generator, we can obtain the other elements, i.e. valid solutions or valid tropisms.

In addition to exploiting the permutation symmetry, we will also address other types of symmetry, not necessarily related to the permutation of variables. Among them we include *extensions of tropisms*. Under extension of tropisms we specifically mean the instances among the cyclic n-roots polynomial systems, where an extended version of the tropisms occurs also in other cyclic n-roots polynomial systems. For example, the tropisms, which leads to the Puiseux series expansion of the space curve, defined by the cyclic 4-roots polynomial system, can be extended to lead to the Puiseux series of a space curve for the cyclic 8, 12, $\ldots$ - roots polynomial systems.

In this Chapter we will, exclusively, focus on the cyclic n-roots polynomial system. By applying our polyhedral method to the cyclic 4-roots and 8-roots polynomial system, we illustrate our method for space curves in a concrete manner. Application of our polyhedral method to the cyclic 5-roots serves as an illustration of how we exploit symmetry. At the end of this chapter we will tackle the cyclic 12-roots polynomial system to show that our polyhedral method can also be applied to solve large and difficult problems. Specifically, we will find the exact representations of the space curves, defined by cyclic 12-roots polynomial system, that have been previously obtained in (109).

3.3 An Example: Cyclic 4-Roots

We consider the cyclic 4-roots polynomial system, given in (Equation 3.2). The cyclic 4-roots problem has also been treated in (134), in the same way that we treat it here in Example 3.4 and in (3). Hence, our Example 3.4 is taken from (3) and (134). We give Example 3.4 here in more detail than we gave it in (3) or that is given in (134), in order to give a better illustration of our polyhedral method.

Example 3.4.

$$C_4(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{cases} \tag{3.2}$$

The cyclic 4-roots polynomial system has one pretropism $\mathbf{v} = (1, -1, 1, -1)$, see (3) (134). We investigate the system (Equation 3.2) in the direction of pretropism $\mathbf{v} = (1, -1, 1, -1)$. Our polyhedral method returns the initial form system, given in Example 3.5, taken from (3). See also (134).

Example 3.5 (Example 3.4 continued)**.**

$$in_{\mathbf{v}}(C_4)(\mathbf{x}) = \begin{cases} x_1 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_1x_2x_3 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{cases} \tag{3.3}$$

Using the pretropism $\mathbf{v} = (1, -1, 1, -1)$, we generate the unimodular coordinate transformation via the matrix M, with $det(M) = \pm 1$. The unimodular matrix M and the coordinate transformation are given in Example 3.6, taken from (3). See also (134).

Example 3.6 (Example 3.4 continued)**.**

$$M = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_0 = z_0, \quad x_1 = \frac{z_1}{z_0}, \quad x_2 = z_0z_2, \quad x_3 = \frac{z_3}{z_0} \tag{3.4}$$

Using the change of coordinates in Equation 3.4, we transform the initial form system Equation 3.3 and obtain a new system. Note how the z_0 variable can be completely eliminated from the system. This is an effect of the unimodular coordinate transformation on the monomials, which share a common facet normal. We illustrate this in Example 3.7, taken from (3). See also (134).

Example 3.7 (Example 3.4 continued)**.**

$$in_{\mathbf{v}}(C_4)(\mathbf{z}) = \begin{cases} \frac{z_1}{z_0} + \frac{z_3}{z_0} = 0 \\ \\ z_0\frac{z_1}{z_0} + \frac{z_1}{z_0}z_0z_2 + z_0z_2\frac{z_3}{z_0} + \frac{z_3}{z_0}z_0 = 0 \\ \\ \frac{z_1}{z_0}z_0z_2\frac{z_3}{z_0} + z_0\frac{z_1}{z_0}\frac{z_3}{z_0} = 0 \\ \\ z_0\frac{z_1}{z_0}z_0z_2\frac{z_3}{z_0} - 1 = 0 \end{cases} \tag{3.5}$$

We clear the z_0 variable from the system in Equation 3.5 and obtain a transformed initial form system, which, when solved, leads to isolated solutions. We give the initial form system and its solutions in Example 3.8, taken from (3). See also (134)

Example 3.8 (Example 3.4 continued).

$$in_{\mathbf{v}}(C_4)(\mathbf{z}) = \begin{cases} z_1 + z_3 = 0 \\ z_1 + z_1 z_2 + z_2 z_3 + z_3 = 0 \\ z_1 z_2 z_3 + z_1 z_3 = 0 \\ z_1 z_2 z_3 - 1 = 0 \end{cases} \tag{3.6}$$

$$(z_0 = t, \;\; z_1 = 1, \;\; z_2 = -1, \;\; z_3 = -1)$$

$$(z_0 = t, \;\; z_1 = -1, \;\; z_2 = -1, \;\; z_3 = 1)$$

These are the solutions at infinity, the leading coefficients in the Puiseux series of the space curve, defined by the cyclic 4-roots polynomial system. It turns out that in this case there is no second term in the series. The first terms satisfy the entire system. We, hence, have an exact representation of two curves, which satisfy the cyclic 4-roots polynomial system. Using the coordinate transformation given in Equation 3.4, we return the two solution curves back to the original coordinates, as illustrated in Example 3.9, taken from (3). See also (134).

Example 3.9 (Example 3.4 continued).

$$\begin{array}{ll} x_0 = t^1 & x_0 = t^1 \\ x_1 = t^{-1} & x_1 = -t^{-1} \\ x_2 = -t^1 & x_2 = -t^1 \\ x_3 = -t^{-1} & x_3 = t^{-1} \end{array} \tag{3.7}$$

In order to emphasize the role the tropisms play in our polyhedral method again, note the powers in the parametrization of the two space curves. They correspond to the tropism $\mathbf{v} = (1, -1, 1, -1)$ and differ only in coefficients, the solutions of the initial form system.

3.9.1 Cyclic 4-Roots: Degree of the Solution Curves

Given the exact representation of the two solution curves, given in Equation 3.7 of Example 3.9, we can compute their degrees in the following way. For $c_0, c_1, c_2, c_3, c_4 \in \mathbb{C}^4$, we construct random hyperplanes of the form $c_0 + c_1x_0 + c_2x_1 + c_3x_2 + c_4x_3 = 0$ and substitute the solutions, given in Equation 3.7, into the hyperplanes. This kind of computation of the degree is described in (134). We give an illustration for the first solution curve of the cyclic 4-roots problem in Example 3.10, taken form (3) and given here in more detail.

Example 3.10.

$$c_0 + c_1x_0 + c_2x_1 + c_3x_2 + c_4x_3 = 0 \tag{3.8}$$

$$c_0 + c_1t^1 + c_2\frac{1}{t^1} + c_3t^1 + c_4\frac{1}{t^1} = 0 \tag{3.9}$$

Clearing the denominator, we obtain an univariate quadratic polynomial

$$c_0t_1 + c_1t^2 + c_2 + c_3t^2 + c_4 = 0 \tag{3.10}$$

or simplified

$$(c_1 + c_3)t^2 + c_0t + (c_2 + c_4) = 0 \tag{3.11}$$

From (Equation 3.11) of Example 3.10, we see that the degree for the first solution curve is 2. We can repeat this process for the second solution curve and obtain its degree, which is also 2. Having computed the degrees, we conclude that the cyclic 4-roots polynomial system vanishes on two quadratic curves.

3.11 Using Symmetry

In this section, we illustrate how the permutation symmetry can be exploited. For that, we focus on the cyclic 5-roots polynomial system. An attempt to compute the pretropisms for the cyclic 5-roots system will yield the empty set. The Newton polytopes, corresponding to the polynomials in the cyclic 5-roots system, are all in generic position with respect to each other. In other words, they do not have any facet normals in common. Consequently, all solutions of the cyclic 5-roots system will be isolated and their number is given by the mixed volume. For mixed volume, see (14) (62) (139) (137) (134) (118) (29). Because the Newton polytopes are in generic position with respect to each other, the mixed volume will be sharp (134). Using PHCpack (132), we computed the mixed volume for the cyclic 5-roots system and obtained a value of 70. Hence, the number of isolated solutions is exactly 70. We will now use our polyhedral method to exploit the symmetry and obtain all the 70 isolated solutions.

By using the cyclic 5-roots polynomial system, we want to emphasize that our polyhedral method can also be applied to exploit symmetry and find isolated solutions as well as positive dimensional solutions sets. In later sections, we will illustrate how the exploitation of symmetry is used when the solution sets are positive dimensional.

We present here an alternative approach to (135), where polyhedral homotopies were adjusted to exploit the permutation symmetry of the cyclic 5-roots polynomial system.

The application of our polyhedral method to the cyclic 5-roots problem, as described above, will be given in form of Example 3.12, taken from (3).

Example 3.12. The cyclic 5-roots polynomial system

$$C_5(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 = 0 \\ x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 = 0 \\ x_0x_1x_2x_3x_4 - 1 = 0 \end{cases} \tag{3.12}$$

As stated before, all the solutions of the system are isolated. Instead of considering the entire cyclic 5-roots system, we can remove the last equation temporarily and consider a modified system, which consists of more variables than equations. This is illustrated in Example 3.13, taken from (3).

Example 3.13 (Example 3.12 continued)**.**

$$\begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 = 0 \\ x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 = 0 \end{cases} \tag{3.13}$$

Because monomials in each individual equations of (Equation 3.13) have the same degree, the minimal inner product value of the pretropism $\mathbf{v} = (1, 1, 1, 1, 1)$ with the monomial exponents will be the same. Hence, all monomials of (Equation 3.13) form the initial form system.

Using the pretropism $\mathbf{v} = (1, 1, 1, 1, 1)$, we generate the unimodular matrix M and obtain the coordinate transformation, as given in Example 3.14, taken from (3).

Example 3.14 (Example 3.12 continued)**.**

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

$$x_0 = z_0,\ x_1 = z_0 z_1,\ x_2 = z_0 z_2,\ x_3 = z_0 z_3,\ x_4 = z_0 z_4 \tag{3.15}$$

We apply the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$, as given in (Equation 3.15), and obtain the transformed initial form system, given in Example 3.15, taken from (3).

Example 3.15 (Example 3.12 continued).

$$in_{\mathbf{v}}(C_5)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_2 + z_3 + z_4 + 1 = 0 \\ z_1z_2 + z_2z_3 + z_3z_4 + z_1 + z_4 = 0 \\ z_1z_2z_3 + z_2z_3z_4 + z_1z_2 + z_1z_4 + z_3z_4 = 0 \\ z_1z_2z_3z_4 + z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = 0 \end{cases} \tag{3.16}$$

Note that, again, the variable z_0 is entirely absent form the system as a result of the unimodular coordinate transformation.

Solving the system (Equation 3.16) in Example 3.15, we obtain 14 isolated solutions. These solution have the general form, given in (Equation 3.17) of Example 3.16. Since z_0 was entirely absent form the system in (Equation 3.16), we let it be the free parameter $z_0 = t$. Using the coordinate transformation given in (Equation 3.15), we return the isolated solutions to the original coordinates and obtain solutions, which generically represents the 14 solution lines of the form (Equation 3.18), given in Example 3.16, taken from (3).

Example 3.16 (Example 3.12 continued).

$$z_1 = c_1,\ z_2 = c_2,\ z_3 = c_3,\ z_4 = c_4 \tag{3.17}$$

$$x_0 = t,\ x_1 = tc_1,\ x_2 = tc_2,\ x_3 = tc_3,\ x_4 = tc_4 \tag{3.18}$$

We now proceed to find all the isolated solutions of the original cyclic 5-roots system by considering the initially omitted equation $x_0x_1x_2x_3x_4 - 1 = 0$. Substitution of the solution lines, given in Equation 3.18, into the equation $x_0x_1x_2x_3x_4 - 1 = 0$, results in an univariate polynomial $kt^5 - 1$, with some constant $k \in \mathbb{C}$.

On substitution of the 14 solutions, given generically in Equation 3.18, into the equation $x_0x_1x_2x_3x_4 - 1 = 0$, 10 have the form $t^5 - 1 = 0$, contributing $10 \times 5 = 50$ solutions. Two of the solutions have the form $(-122.99186938124345)t^5 - 1 = 0$, contributing $2 \times 5 = 10$ solutions. The last two solutions have the form $(-0.0081306187557833118)t^5 - 1 = 0$, contributing $2 \times 5 = 10$ solutions. In total, we have $50+10+10 = 70$ solutions of the cyclic 5-roots polynomial system.

More symmetry can be observed in the last two polynomials, which contribute 10 solutions each. Their coefficients are inverses of each other: $\frac{1}{(-122.99186938124345)} \approx -0.0081306187557833118$.

We have just presented a very special case among the cyclic n-roots polynomial systems, where it happens that the first $n-1$ equations lead to an explicit representation of the solution lines. A more likely scenario is the case where we have a Puiseux series approximation of the curve. In such a case, we can use the first terms (usually more than just one or two) of the Puiseux series to compute the witness sets (118) (134) of the curve, which is defined as in our example above, by the first $n - 1$ equations. In this thesis, we do not deal with witness sets computation. We just want to remark that witness sets also offer an avenue to obtain a representation of the curve. In conclusion of this paragraph, we emphasize that the initial terms of the Puiseux series and witness sets allow for the use of the diagonal homotopy (116) to intersect the curve with the omitted n^{th} equation.

In the next several sections, we will consider cyclic n-roots polynomial systems, which vanish on positive dimensional solution sets. Specifically, we will consider the cases when $n = 8$ and

$n = 12$. Backelin's lemma (11) guarantees that there are positive dimensional solution sets when $n = 8$ and $n = 12$. At this point, we will postpone the case when $n = 9$. We will cover the case when $n = 9$ in Chapter 4, where we cover general algebraic sets.

3.17 Computation of Pretropisms

In our computation of pretropisms, we use the Cayley trick (50, Proposition 1.7, page 274) (122) (35, §9.2) (61). We also refer to the computation of tropical varieties in (19), with an implementation in Gfan (71).

In order to compute the H-representation (45), also called H-polytope in (145), of the Cayley polytope (50, Proposition 1.7, page 274) (122) (35, §9.2) (61), we use *cddlib* (45).

The computation of the H-representation of the Cayley polytope, for the cyclic 8-roots polynomial system, took *cddlib* (45) less than a second. The computation was performed on one core of a machine, running GNU/Linux, with a 3.07 GHz processor and 4GB of RAM. The computation produced 94 pretropisms for the cyclic 8-roots. Using permutation symmetry, we reduced that number to 11 generators, which are displayed in Table I. See also (3).

(1, -1, 0, 1, 0, -1, 1, -1)	(1, -1, 0, 1, 0, 0, -1, 0)	(1, -1, 1, -1, 1, -1, 1, -1)
(1, -1, -1, 1, 1, -1, -1, 1)	(1, -1, 0, 0, 0, 1, 0, -1)	(1, -1, 0, 0, 1, 0, 0, -1)
(1, -1, 1, -1, 1, 0, -1, 0)	(1, -1, 1, 0, -1, 0, 0, 0)	(1, -1, 1, 0, 0, -1, 0, 0)
(3, -1, -1, -1, 3, -1, -1, -1)	(1, -3, 1, 1, 1, -3, 1, 1)	

TABLE I

PRETROPISM GENERATORS OF THE CYCLIC 8-ROOT PROBLEM

While the computation of the pretropisms for the cyclic 8-roots system was very fast, the pretropism computation for the cyclic 12-roots system was not. Using the same machine and method that was used for the pretropism computation of the cyclic 8-roots system, the pretropism computation for the cyclic 12-roots system took cddlib (45) about a week. The computation produced 907923 facet normals, out of which only a small portion were pretropisms.

3.18 Computation of the Second Term in the Puiseux Series

When we considered the cyclic 4-roots system and the modified cyclic 5-system earlier, we obtained the exact representation of their solution sets. Generally, that will not be the case. In most cases, in order to obtain a better approximation of the solution set, we will have to compute more terms of its Puiseux series expansion.

The solutions at infinity provide us with an initial term in the Puiseux series. However, if they do not satisfy the entire system, as it was the case with the cyclic 4-roots, then they may just be isolated solutions at infinity. In order to ensure that the solution set is a curve, we need to compute at least the second term in its Puiseux series. For us, the second term in the series is a certificate that the solution set is a curve. In this manner, the pretropism becomes a tropism.

We use PHCpack (132) to solve the transformed initial form systems and obtain the solutions at infinity. For symbolic manipulation involving pretropisms and the computation of terms in the Puiseux series, we use Sage (121).

In the computation of the second term in the Puiseux series of the space curve, we do not guess the exponent of the second term in the Puiseux series, which we then try to either confirm or reject. Such a process would be computationally expensive, especially since we are dealing with sparse polynomial systems. The method we use to obtain the exponent of the second term we give in form of a proposition. We give the Proposition 3.19 and its proof here exactly as we gave it in (3), adjusting the notation to match the convention used in this thesis.

Proposition 3.19. *If the initial root does not satisfy the entire transformed polynomial system, then there must be at least one nonzero constant exponent a_i forming monomial $c_i t^{a_i}$.*

Proof. Let $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1})$ and $\bar{\mathbf{z}} = (z_1, z_2, \ldots, z_{n-1})$ denote variables after the unimodular transformation. Let $(z_0 = t, z_1 = r_1, \ldots, z_{n-1} = r_{n-1})$ be a regular solution at infinity and t the free variable.

The ith equation of the original system after the unimodular coordinate transformation has the form

$$f_i = z_0^{m_i}(P_i(\bar{\mathbf{z}}) + O(z_0)Q_i(\mathbf{z})), \quad i = 0, 1, \ldots, n-1 \tag{3.19}$$

where the polynomial $P_i(\bar{\mathbf{z}})$ consists of all monomials which form the initial form component of f_i and $Q_i(\mathbf{z})$ is a polynomial consisting of all remaining monomials of f_i.

For the second term in the series expansions, we denote

$$\begin{aligned} z_0 &= t \\ z_i &= r_i + k_i t^w, \quad i = 1, 2, \ldots, n-1. \end{aligned} \tag{3.20}$$

We first show that polynomial $z_0^{m_i} P_i(\bar{\mathbf{z}})$ cannot contain a monomial of the form $c_i t^{a_i}$ on substitution of (Equation 3.20). The polynomial $z_0^{m_i} P_i(\bar{\mathbf{z}})$ is the initial form of f_i, hence solution at infinity ($z_0 = t, z_1 = r_1, z_2 = r_2, \ldots, z_{n-1} = r_{n-1}$) satisfies $z_0^{m_i} P_i(\bar{\mathbf{z}})$ entirely. Substituting (Equation 3.20) into $z_0^{m_i} P_i(\bar{\mathbf{z}})$ eliminates all constants in $t^{m_i} P_i(\bar{\mathbf{z}})$. Hence, the polynomial $P_i(t)$ $= R_i(t^w)$ and, therefore, $t^{m_i} P_i(t) = R_i(t^{w+m_i})$.

We next show that polynomial $Q_i(\mathbf{z})$ contains a monomial $c_i t^{a_i}$. The polynomial $Q_i(\mathbf{z})$ can be rewritten as

$$z_0^{a_i} Q_i(\bar{\mathbf{z}}) = z_0^{a_i} T_{i,0}(\bar{\mathbf{z}}) + z_0^{a_i+1} T_{i,1}(\bar{\mathbf{z}}) + \cdots + z_0^{a_i+(n-1)} T_{i,(n-1)}(\bar{\mathbf{z}}). \tag{3.21}$$

The polynomial $Q_i(\mathbf{z}) = z_0^{a_i} Q_i(\bar{\mathbf{z}})$ consists of monomials which are not part of the initial form of f_i. Hence, on substitution of solution at infinity (Equation 3.20), $z_0^{a_i} Q_i(\bar{\mathbf{z}}) = t^{a_i} Q_i(t)$ does not vanish entirely and it must contain monomials which are constants. Since $Q_i(t)$ contains monomials which are constants, $t^{a_i} Q_i(t)$ must contain a monomial of the form $c_{ij} t^{a_i}$. $\square$

We may have a second term in the Puiseux series expansion of a space curve when the solutions of the initial form systems, the solutions at infinity, do not satisfy the original system completely. In such a case, the first two terms of the series have the form as given in Example 3.20, taken from (3).

Example 3.20.

$$\begin{cases} z_0 = t \\ z_i = r_i + k_i t^w, \quad i = 1, 2, \ldots, n-1 \end{cases} \tag{3.22}$$

where $r_i \in \mathbb{C}^*$ are the solutions at infinity and k_i is the yet undetermined coefficient of the second term t^w, with $w > 0$.

The goal is to find the smallest value for w, such that the linear system in the k_i's has at least one non-zero solution. Substitution of (Equation 3.22) results in equations of the form as given in Example 3.21, taken from (3). Here we have adjusted the notation slightly to match the convention, used in this thesis.

Example 3.21.

$$r_i t^{a_i}(1 + O(t)) + t^{w+b_i} \sum_{j=0}^{n-1} \gamma_{ij} k_j (1 + O(t)) = 0, \quad i = 0, 1, \ldots, n-1 \tag{3.23}$$

where a_i, b_i are constant exponents and r_i and γ_{ij} are constant coefficients.

The resulting equations, given in (Equation 3.23), contain more terms than it is needed to obtain the second term. For that reason, we cut off the $O(t)$ terms and keep only those equations, which contain the smallest exponents a_i. The main reason for doing this is the following: on back substitution of the first two terms of the series into the original system, we want the lowest powers of t to disappear. This setup allows us to have a condition on the value the unknown exponent w of the second term can have.

Should we not be able to match t^{a_i} with t^{w+b_i}, in all the equations where the same lowest value for a_i is found, then there is no w. As a result, there is no second term and, consequentially, no space curve as a solution set.

On the other hand, if we can match t^{a_i} with t^{w+b_i}, in all the equations where the same lowest value for a_i is found, then there will be a linear system in the unknowns k_i, which needs to be solved to obtain the coefficients of the second term. The non-existence of the solution for this linear system would mean that there is, again, no space curve as a solution set. However, if a non-zero solution for the linear system exists, then there exists a space curve as a solution set of the original polynomial system.

3.22 Polyhedral Method For Space Curves

3.22.1 Puiseux Series of the Cyclic 8-Roots Polynomial System

We denote the cyclic 8-roots polynomial system by $C_8(\mathbf{x}) = \mathbf{0}$ and investigate the system in the direction of pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$. In (134), the cyclic 8-roots problem is treated in a nearly identical way to what we give in Example 3.23, using a different pretropism. For our pretropism case, the initial form system is given in Example 3.23, taken from (3).

Example 3.23.

$$in_{\mathbf{v}}(C_8)(\mathbf{x}) = \begin{cases} x_1 + x_6 = 0 \\ x_1x_2 + x_5x_6 + x_6x_7 = 0 \\ x_4x_5x_6 + x_5x_6x_7 = 0 \\ x_0x_1x_6x_7 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_6x_7 + x_0x_1x_5x_6x_7 = 0 \\ x_0x_1x_2x_5x_6x_7 + x_0x_1x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases} \tag{3.24}$$

As the next step, we use the tropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ to generate an unimodular matrix $M \in \mathbb{Z}^{n \times n}$ with $det(M) = \pm 1$ and transform the initial form system $in_{\mathbf{v}}(C_8)(\mathbf{x}) = 0$.

After the unimodular coordinate transformation, the initial form system (Equation 3.24) will have one variable less than before.

Placing the pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ on the first row, the unimodular matrix M and the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ are given in Example 3.24, taken from (3).

Example 3.24 (Example 3.23 continued)**.**

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.25}$$

$$x_0 = z_0,\ x_1 = z_1/z_0,\ x_2 = z_2,\ x_3 = z_0 z_3,\ x_4 = z_4,\ x_5 = z_5,\ x_6 = z_6/z_0,\ x_7 = z_7 \tag{3.26}$$

Using the change of coordinates, given in (Equation 3.26) of Example 3.24, we transform the initial form system (Equation 3.24) and obtain the new initial form system, given in Example 3.25, taken from (3).

Example 3.25 (Example 3.23 continued)**.**

$$in_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_6 = 0 \\ z_1 z_2 + z_5 z_6 + z_6 z_7 = 0 \\ z_4 z_5 z_6 + z_5 z_6 z_7 = 0 \\ z_4 z_5 z_6 z_7 + z_1 z_6 z_7 = 0 \\ z_1 z_2 z_6 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases} \tag{3.27}$$

Note that, after the unimodular coordinate transformation, the first variable z_0 is no longer present in the transformed initial form system (Equation 3.27). Solving the system (Equation 3.27), we obtain a regular solution, given in Example 3.26, taken from (3).

Example 3.26 (Example 3.23 continued)**.**

$$z_0 = t,\ z_1 = -I,\ z_2 = \frac{-1}{2} - \frac{I}{2},\ z_3 = -1,\ z_4 = 1 + I,\ z_5 = \frac{1}{2} + \frac{I}{2},\ z_6 = I,\ z_7 = -1 - I \tag{3.28}$$

with $I = \sqrt{-1}$ and $z_0 = t$ as the free parameter in the Puiseux series.

With the solutions of the initial form system, we have obtained the solutions at infinity and the leading coefficients in the Puiseux series of the space curve. Next, we proceed as in Proposition 3.19. We do not show the entire computational process here but we remark that the exponent of the second term, which is initially assumed to have the form $k_i t^w$, can be matched and has a solution $w = 1$. Hence, the second term in the Puiseux series may be of the form as given in Example 3.27, taken from (3).

Example 3.27 (Example 3.23 continued)**.**

$$\begin{cases} z_0 = t \\ z_1 = -I + k_1 t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + k_2 t \\ z_3 = -1 + k_3 t \\ z_4 = 1 + I + k_4 t \\ z_5 = \frac{1}{2} + \frac{I}{2} + k_5 t \\ z_6 = I + k_6 t \\ z_7 = (-1 - I) + k_7 t \end{cases} \tag{3.29}$$

The second term in the Puiseux series may be of the form as given in Example 3.27, provided that there exists a non-zero solution for the resulting linear system in $k_1, k_2, \ldots, k_7$.

In order to find the solutions $k_1, k_2, \ldots, k_7$, we construct the linear system in the following manner. Using the unimodular coordinate transformation (Equation 3.26), we transform the original cyclic 8-roots system $C_8(\mathbf{x}) = \mathbf{0}$. The system $C_8(\mathbf{x}) = \mathbf{0}$ becomes, hence, $C_8(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$. Next, we substitute the form (Equation 3.29) into $C_8(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ and collect all the coefficients in $k_1, k_2, \ldots, k_7$ of the term t^1. The resulting linear system has a non-zero solution, providing us with the second term in the Puiseux series expansion of the space curve. The two terms in the Puiseux series are given in Example 3.28, taken from (3).

Example 3.28 (Example 3.23 continued)**.**

$$\begin{cases} z_0 = t \\ z_1 = -I + (-1 - I)t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t \\ z_3 = -1 \\ z_4 = 1 + I - t \\ z_5 = \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t \\ z_6 = I + (1 + I)t \\ z_7 = (-1 - I) + t \end{cases} \tag{3.30}$$

The solutions at infinity were regular. Furthermore, the solutions of the linear system, which provided us with the second coefficients in the series, were also regular solutions. Consequentially, we have a symbolic-numeric certificate that there is a space curve, which satisfies the cyclic 8-roots polynomial system. At this point, we start to refer to the pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ as a tropism.

Here, we used the first variable as the free parameter. However, we can also select other variables as free parameters by placing the pretropism in a different row of the unimodular matrix M, before generating the rest of the entries, such that $det(M) = \pm 1$.

The selection of a different variable as a free parameter may actually be necessary. In particular, it may happen that the solution curve is not in general position with respect to the variable (i.e. plane $x_0 = 0$) that is selected as a free parameter. In that case, a different free variable must be chosen in order to obtain the Puiseux series of the curve.

When we are dealing with polynomial systems, which have symmetry, we exploit that symmetry by applying the various permutations to the pretropisms, resulting initial form systems and their solutions. In this manner, we can obtain the Puiseux series representation for other solution curves, which also satisfy the system of polynomials.

We have already encountered a special situation, with the cyclic 4-roots system, where the first term in the series ends up satisfying the entire system. Such an occurrence is frequent among the cyclic n-roots polynomial system. Here, we present it for the cyclic 8-roots system. For this type of pattern, see also (134).

We investigate the $C_8(\mathbf{x}) = \mathbf{0}$ system in the direction of tropism $\mathbf{v} = (1, -1, 1, -1, 1, -1, 1, -1)$. Again, for the cyclic 8-roots problem, see (134). The initial form system in direction of pretropism $\mathbf{v} = (1, -1, 1, -1, 1, -1, 1, -1)$ is given in Example 3.29, taken from (3).

Example 3.29.

$$in_{\mathbf{v}}(C_8)(\mathbf{x}) = \begin{cases} x_1 + x_3 + x_5 + x_7 = 0 \\ x_0x_1 + x_0x_7 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 = 0 \\ x_0x_1x_7 + x_1x_2x_3 + x_3x_4x_5 + x_5x_6x_7 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_7 + x_0x_1x_6x_7 + x_0x_5x_6x_7 \\ +\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_7 + x_0x_1x_5x_6x_7 + x_1x_2x_3x_4x_5 + x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_7 + x_0x_1x_2x_3x_6x_7 + x_0x_1x_2x_5x_6x_7 \\ +\ x_0x_1x_4x_5x_6x_7 + x_0x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_7 + x_0x_1x_2x_3x_5x_6x_7 + x_0x_1x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases} \tag{3.31}$$

Using the tropism $\mathbf{v} = (1, -1, 1, -1, 1, -1, 1, -1)$, we generate the unimodular matrix M and the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ in Example 3.30, taken from (3).

Example 3.30 (Example 3.29 continued).

$$M = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_0 = z_0,\ x_1 = z_1/z_0,\ x_2 = z_0 z_2,\ x_3 = z_3/z_0,\ x_4 = z_0 z_4,\ x_5 = z_5/z_0,\ x_6 = z_0 z_6,\ x_7 = z_7/z_0.$$

Using the new coordinates, we transform the initial form system (Equation 3.31) of Example 3.29 and obtain the transformed initial form system, given in Example 3.31, taken from (3).

Example 3.31 (Example 3.29 continued)**.**

$$in_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_3 + z_5 + z_7 = 0 \\ z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_6 + z_6z_7 + z_1 + z_7 = 0 \\ z_1z_2z_3 + z_3z_4z_5 + z_5z_6z_7 + z_1z_7 = 0 \\ z_1z_2z_3z_4 + z_2z_3z_4z_5 + z_3z_4z_5z_6 + z_4z_5z_6z_7 + z_1z_2z_3 \\ + \ z_1z_2z_7 + z_1z_6z_7 + z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5 + z_3z_4z_5z_6z_7 + z_1z_2z_3z_7 + z_1z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6 + z_2z_3z_4z_5z_6z_7 + z_1z_2z_3z_4z_5 + z_1z_2z_3z_4z_7 \\ + \ z_1z_2z_3z_6z_7 + z_1z_2z_5z_6z_7 + z_1z_4z_5z_6z_7 + z_3z_4z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6z_7 + z_1z_2z_3z_4z_5z_7 + z_1z_2z_3z_5z_6z_7 + z_1z_3z_4z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6z_7 - 1 = 0 \end{cases} \tag{3.32}$$

The solution set of the initial form system of Example 3.31 consists of 72 solutions. Out of the 72, we select a particular solution and return it to the original coordinates, via the unimodular coordinate transformation. This is illustrated in Example 3.32, taken from (3).

Example 3.32 (Example 3.29 continued)**.**

$$z_0 = t,\ z_1 = -1,\ z_2 = I,\ z_3 = -I,\ z_4 = -1,\ z_5 = 1,\ z_6 = -I,\ z_7 = I, \tag{3.33}$$

$$x_0 = t,\ x_1 = -1/t,\ x_2 = It,\ x_3 = -I/t,\ x_4 = -t,\ x_5 = 1/t,\ x_6 = -It,\ x_7 = I/t \tag{3.34}$$

This particular solution satisfies the cyclic 8-roots polynomial system $C_8(\mathbf{x}) = \mathbf{0}$ completely. Furthermore, by cyclically permuting this solutions set, we find the other 7 solution sets, which satisfy the $C_8(\mathbf{x}) = \mathbf{0}$ system completely as well.

The number of solutions of an initial form system and the tropism, which lead to that initial form system, are used in (134) to compute the branch degree of a curve. The Definition 3.33 is taken from (134) and we use it here for the computation of the degree of a branch.

Definition 3.33 (Branch Degree)**.** Let $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ be a tropism and let R be the set of initial roots of the initial form system $in_{\mathbf{v}}(f)(\mathbf{z})$. Then the degree of the branch is

$$\#R \times \left| \max_{i=0}^{n-1} v_i - \min_{i=0}^{n-1} v_i \right|. \tag{3.35}$$

We use Definition 3.33 (134) and the data we computed to obtain the degrees of the cyclic 8-roots solution curve. These results are listed in Table II. The Table II is taken from (3).

$(1,-1,1,-1,1,-1,1,-1)$	$8 \times 2 = 16$
$(1,-1,0,1,0,0,-1,0) \rightarrow (1,0,0,-1,0,1,-1,0)$	$8 \times 2 + 8 \times 2 = 32$
$(1,0,-1,0,0,1,0,-1) \rightarrow (1,0,-1,1,0,-1,0,0)$	$8 \times 2 + 8 \times 2 = 32$
$(1,0,-1,1,0,-1,0,0) \rightarrow (1,0,-1,0,0,1,0,-1)$	$8 \times 2 + 8 \times 2 = 32$
$(1,0,0,-1,0,1,-1,0) \rightarrow (1,-1,0,1,0,0,-1,0)$	$8 \times 2 + 8 \times 2 = 32$
	TOTAL $= 144$

TABLE II

TROPISMS, CYCLIC PERMUTATIONS, AND DEGREES FOR THE CYCLIC 8 SOLUTION CURVE.

We conclude our treatment of the cyclic 8-roots polynomial system by remarking that our polyhedral method can also be used to obtain the isolated solutions of the cyclic 8-roots polynomial system, in the same way it was done for the cyclic 5-roots polynomial system, given in Example 3.12.

3.33.1 The Cyclic 12-Roots Polynomial System

We investigate the cyclic 12-roots polynomial system in the direction of pretropism $\mathbf{v}$=$(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)$. This exact problem has been already treated in (134). Hence, our Example 3.34 is taken from (134) and (3). In this thesis, as well as in (3), we give, however, a more general result regarding the cyclic 12-roots problem in the direction of pretropism $\mathbf{v}$=$(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)$ than what is given in (134).

In Table III, we give the solutions of the transformed initial form system in the direction of pretropisms $\mathbf{v}$. The Table III is taken from (3). Any solution in Table III can be returned to the original coordinates. This process is given in Example 3.34, taken from (3). See also (134).

Example 3.34.

$$\begin{aligned} &z_0 = t,\ z_1 = r_1,\ z_2 = r_2,\ z_3 = r_3,\ z_4 = r_4,\ z_5 = r_5, \\ &z_6 = r_6,\ z_7 = r_7,\ z_8 = r_8,\ z_9 = r_9,\ z_{10} = r_{10},\ z_{11} = r_{11} \end{aligned} \tag{3.36}$$

$$\begin{aligned} &x_0 = t,\ x_1 = r_1/t,\ x_2 = r_2 t,\ x_3 = r_3/t,\ x_4 = r_4 t,\ x_5 = r_5/t \\ &x_6 = r_6 t,\ x_7 = r_7/t,\ x_8 = r_8 t,\ x_9 = r_9/t,\ x_{10} = r_{10} t,\ x_{11} = r_{11}/t \end{aligned} \tag{3.37}$$

All the solutions are quadratic space curves. This can be easily established by using the Definition 3.33, see also (134). This result has been obtained in (109), by using factorization methods. We obtained this result by a straightforward application of our polyhedral method.

z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}$
$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1
1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1
$-\frac{1}{2}$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1
$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
-1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	-1	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$

TABLE III

GENERATORS OF THE ROOTS OF THE INITIAL FORM SYSTEM $IN_{\mathbf{V}}(C_{12})(\mathbf{X}) = \mathbf{0}$ WITH THE TROPISM $\mathbf{V}$ IN THE TRANSFORMED $\mathbf{Z}$ COORDINATES. EVERY SOLUTION DEFINES A SOLUTION CURVE OF THE CYCLIC 12-ROOTS SYSTEM.

3.35 Summary of the Chapter

In his chapter we developed a new polyhedral method to solve systems of polynomials, which have space curves as their solution sets. Here, we extended the ideas we developed in Chapter 2. Even though our polyhedral method is geared towards the development of positive dimensional algebraic sets, in this chapter we also showed how this polyhedral method can be used to find the isolated solutions of a system of polynomials. The main contribution of this chapter has been the Proposition 3.19 and its proof (3).

An important aspect of this chapter has been the exploitation of symmetry. We used cyclic permutation and applied it to tropisms and the solutions of the initial form systems to reduce the computational time. In particular, we showed that once a tropism and its corresponding initial form systems solutions have been obtained, we can use cyclic permutation to obtain the other solution sets as well as the (total) degrees of the solution sets. The approach of this chapter is related to the methods in numerical algebraic geometry, given in (134).

CHAPTER 4

GENERAL ALGEBRAIC SETS

4.1 Introduction

The content of this Chapter 4 is based on previously published material (5). All algorithms, algorithm outlines, computational results, concepts, definitions (and variations), examples, figures, ideas, illustrations, pictures, propositions and their proofs, tables, etc. are taken from (5), unless otherwise indicated. This Chapter 4 is a summary of (5).

In this chapter, we continue the generalization of the methods, which were covered in Chapter 2 and Chapter 3. Specifically, we focus on extending our polyhedral method to develop general algebraic sets and express them in form of a multivariate Puiseux series.

In Chapter 3, we saw how a tropism can be used to obtain the Puiseux series of a space curve, which satisfies a system of polynomials. As a parametrization of a space curve requires only one free parameter, we only had to use one tropism to obtain the parametrization. However, when the solution set of a system of polynomials is a d-dimensional surface, then one free parameter is no longer enough to parametrize such a surface.

A parametric representation of a d-dimensional surface requires d free parameters. In our polyhedral method, each tropism leads to a free parameter in the Puiseux series. In order to obtain a parametrization of a d-dimensional surface, we need to use d tropisms. As tropisms are facet normals, i.e. vectors, we see the d tropisms as vectors, which form a d-dimensional cone (145) (56).

Our polyhedral method always starts with the computation of facet normals. In this case, the polyhedral method starts with the computation of cones of facet normals, which may lead to cones of pretropisms. The cones of pretropisms lead to initial form systems, which are transformed using the unimodular matrices generated by cones of pretropisms. If the transformed initial form systems lead to solutions, which are isolated and regular, then we may have the leading coefficients of the Puiseux series of the d-dimensional surface.

If the solutions of the initial form systems can be used to obtain more terms in the Puiseux series of the d-dimensional surface or if they satisfy the original system of polynomials when they are returned to the original coordinates, then the cone of pretropisms becomes a cone of tropisms. Otherwise, the solutions are just isolated solutions at infinity and there is no Puiseux series expansion of the d-dimensional surface for that particular cone of pretropisms.

The approach of our polyhedral method stems from ideas, which were covered in (134). In our computations, we used Sage (121) for the symbolic manipulation of the polynomial systems and the numerical solver PHCpack (132) to solve the transformed initial form systems and obtain the solutions at infinity. In both cases, other software can be used to obtain the same results.

As in Chapter 3, we devote a considerable amount of attention to the exploitation of symmetry when developing Puiseux series for higher-dimensional solution sets. In (118), the higher-dimensional solution sets are approached by the introduction of hyperplanes to intersect the higher-dimensional surface and, in that manner, decrease the dimension of the solution sets down to the curve case. However, the introduction of hyperplanes in general position does not take advantage of symmetry. Hence, what we will present in this chapter can be seen as an alternative approach to (118), which takes symmetry into account.

In order to illustrate our polyhedral method, we will consider binomial systems as well continue to investigate the benchmark cyclic n-roots polynomial systems (Equation 1.1). We refer to (43) for additional information on cyclic n-roots polynomial systems, especially the cyclic 9-roots system. See (127) for recent work on the classification of complex Hadamard matrices. In (115), the cyclic 9-roots solution set, which is a two dimensional surface, has been numerically factored into its irreducible components. The components consist of six cubics (115). For recent work on the cyclic 12-roots, see (109).

Our polyhedral method builds the Puiseux series of the positive dimensional solution sets term by term. However, as we saw in Chapter 3, it sometimes happens that the first term computed already satisfies the entire system. While such cases are exceptional, we find such cases again when we consider cyclic n-roots systems, whose solution sets are higher-dimensional surfaces. The results produced by our polyhedral method correspond to what is already known about the cyclic n-roots polynomial systems, see (11) and (43).

Related material to our approach can be found in (53), which covers the geometric resolution of a polynomial system for Newton's iterator (22). Our approach is rooted in the work and results of (14, Theorem B) and (91). We are impressed by the tropical results of (19) (100) (77) and the proof of the fundamental theorem of tropical algebraic geometry (73). As Puiseux series are often used when resolving singularities, we refer to (10), which addresses an extension of Newton's method via the tropical variety. For software in connection with (73), we refer to Gfan (71) and the Singular library `tropical.lib` (72).

For finiteness proofs in celestial mechanics, which use polyhedral and tropical methods, see (58) and (69). For two dimensional varieties and their truncations, see (78). For a relationship with Gröbner bases, we refer to (123). The power transformations, described in (21), are closely related to the unimodular coordinate transformations on which we rely. Unimodular coordinate transformations are described in (131) (41) (134). For related and concurrent material on unimodular matrices to (5), see (66) and (55). For Newton-Puiseux expansions and multivariate polynomials, see (13). For more on Puiseux series as solutions to systems of polynomials and related material, see (6) (8) (92) (93) (9) (81) (10) (39).

4.2 Binomial Systems

The polyhedral method, which is being developed as part of this thesis, is geared towards solving of sparse polynomial systems. Binomial systems, systems consisting of two monomials per equation, are the sparsest systems in that category. We refer to (23) (37) (31) regarding the binomial ideals. In our treatment of binomial systems, we were influenced by the approach, given in (134).

In Chapter 3, we considered polynomial systems, whose solution sets consisted of space curves. In the process of obtaining the Puiseux series of the space curves, we used the first variable as the free parameter. This was a direct consequence of the way we use unimodular coordinate transformations, generated by a single tropism.

In this chapter, we are going to be looking for parametrization of d-dimensional solution sets. We will obtain the parametrization of such solution sets by first obtaining the cones, consisting of d tropisms, which will lead to initial form systems. Generating the unimodular matrices with the cones of tropisms will transform the initial form systems, eliminating d variables, which will then be used as the free parameters. We now illustrate this approach on the binomial systems.

4.2.1 Binomial System Example

Consider the binomial system (Equation 4.1) and the exponents of its non-constant monomial, given in the matrix (Equation 4.2) in the Example 4.3. The Example 4.3 is taken from (5).

Example 4.3.

$$\begin{cases} x_0^2 x_1 x_2^4 x_3^3 - 1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{cases} \tag{4.1}$$

$$A = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.2}$$

In the next step, we need to obtain the the kernel of the matrix A. The system $A\mathbf{x} = \mathbf{0}$ can be satisfied by two linearly independent vectors $\mathbf{u}$=(-3,2,1,0) and $\mathbf{v}$=(-2,1,0,1). We generate the unimodular matrix M by placing the pretropisms $\mathbf{u}$ and $\mathbf{v}$ in the first two rows of M and fill up the rest of the entries with integers, such that $det(M) = \pm 1$. Note that in this thesis, and generally in our polyhedral method, we place our pretropisms in the *rows* of the matrix M. In the binomial system we gave in (5), we made an exception and placed the pretropisms in the *columns* of M. The effect of this change was the lower degree in the transformed binomial system. Computationally, this is an important aspect and we want to make that clear. However, while we use the binomial system, which we gave in (5), here as a running example, we place the pretropisms **u** and **v**, according to our general convention, in the rows of the matrix M. We do this for the sake of consistency. Hence, the resulting transformed binomial system, in this thesis, will have a slightly higher degree in comparison to the transformed binomial system, which we gave in (5).

Constructing the matrix M for pretropisms **u** and **v** is trivial. However, in the latter sections of this chapter, we will show how the construction of the unimodular matrix M can be done for any cone of pretropisms, which we might encounter in the problems that we consider.

The matrix M and the corresponding coordinate transformation are given in Example 4.4, taken from (5), and adjusted according to the description above.

Example 4.4 (Example 4.3 continued)**.**

$$M = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{cases} x_0 = z_0^{-3} z_1^{-2} \\ x_1 = z_0^2 z_1 \\ x_2 = z_0 z_2 \\ x_3 = z_1 z_3 \end{cases} \tag{4.3}$$

As the two linearly independent pretropisms **u** and **v** are in the the kernel of A, the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ (Equation 4.3) will eliminate two variables form the system. Because we have placed the pretropisms on the first two rows of M, the first two variables z_0 and z_1 will be eliminated and they will become the free parameters, describing the solution set.

Substituting the coordinate transformation of (Equation 4.3) into (Equation 4.1), we obtain the transformed and simplified binomial system, given in Example 4.5, taken from (5), and adjusted according to the description above.

Example 4.5 (Example 4.3 continued)**.**

$$\begin{cases} z_0^{-6} z_1^{-4} z_0^2 z_1^1 z_0^4 z_2^4 z_1^3 z_3^3 - 1 = 0 \\ z_0^{-3} z_1^{-2} z_0^2 z_1^1 z_0^1 z_2^1 z_1^1 z_3^1 - 1 = 0 \end{cases} \tag{4.4}$$

$$\begin{cases} z_2^4 z_3^3 - 1 = 0 \\ z_2 z_3 - 1 = 0 \end{cases} \tag{4.5}$$

Solving the system (Equation 4.5) yields isolated solutions for z_2 and z_3. We can return these solutions to the original coordinates via the unimodular coordinate transformation, given in (Equation 4.3), and obtain an exact representation of the solution, in which the previously eliminated variables z_0 and z_1 serve as the free parameters.

4.5.1 Unimodular Coordinate Transformations

The unimodular coordinate transformation has the following form $\mathbf{x} = \mathbf{z}^M$, where M is a square matrix with $det(M) = \pm 1$. We illustrated the construction of such a transformation in (Equation 4.3), of the previous section. In that particular example, the construction of the matrix M was trivial. In this section we address the construction of matrix M in a systematic manner. In our approach to unimodular matrices, we were influenced by (131) (41) (21) (133) (17) (36) (134).

For binomial systems, we begin by obtaining the kernel of the exponents matrix A. The kernel vectors are used to form a matrix B as its row vectors, such that we have $AB^T = \mathbf{0}$. The elements of the matrix B are considered to be pretropisms. As a result, once we have obtained the matrix B, the method for constructing the matrix M for binomial system is the same as for the general polynomial system case. Additionally, all pretropisms are divided by the greatest common divisor of its elements prior to being used to construct the unimodular matrix M.

In the next step, we obtain the Smith Normal Form (97) (26) (135) of the matrix B, which results in three matrices (U, S, V), such that $S = UBV$. For matrices U and V, we have $det(U) = \pm 1$ and $det(V) = \pm 1$. The matrix S is a matrix with nonzero elements on the diagonal and zeros everywhere else.

When the matrix U is an identity matrix, it follows from $S = UBV$ that $B = SV^{-1}$. Hence, the results of $B\mathbf{x}$ and $SV^{-1}\mathbf{x}$ are the same for any $\mathbf{x}$. When, furthermore, the entries of the diagonal matrix S are all $1's$, then the unimodular matrix M is given by V^{-1}.

With the following example, we show the case when the matrix U, $det(U) = \pm 1$, is not the identity matrix and the entries of the diagonal matrix S are all $1's$.

The matrix B consists of the vectors in the kernel of the matrix A, i.e. $AB^T = \mathbf{0}$, given in (Equation 4.2). We gave the matrix B in Example 4.6, taken from (5).

Example 4.6 (Example 4.3 continued)**.**

$$B = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}. \tag{4.6}$$

We used Sage (121) software to compute the Smith Normal Form of the matrix B in Example 4.7. In (5), we used the software GAP (48) to compute the Smith Normal Form of the matrix B. As Sage (121) is the main computational software for this thesis, we give the Sage (121) output for the Smith Normal Form of the matrix B in Example 4.7. The Example 4.7 is taken from (5) but the output matrices look differently from what we gave in (5), as a result of software change.

Example 4.7 (Example 4.3 continued)**.**

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{4.7}$$

$$V = \begin{bmatrix} -2 & 1 & 2 & -1 \\ -3 & 2 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.8}$$

In order to obtain the unimodular matrix M, used for the coordinate transformation $\mathbf{x} = \mathbf{z}^M$, we first need to invert the matrix U. Note that in this particular case $U = U^{-1}$, which in the general case is not to be expected. Then, the matrix U^{-1} needs to be extended with an identity matrix, such that it is a square matrix of dimension equal to that of the matrix V^{-1}. The extended matrix U^{-1} is denoted as E. Then, the matrix M is given by the product EV^{-1}. For our running example, we describe this in Example 4.8. Again, the Example 4.8 is taken from (5) but the output matrices look differently from what we gave in (5) as a result of software change, described above.

Example 4.8 (Example 4.3 continued)**.**

$$EV^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 1 \\ -3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.9}$$

$$M = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.10}$$

Note that the matrices in (Equation 4.10) and (Equation 4.3) have the same tropisms, i.e. vectors of the matrix B, in their first two rows but that the entries of the last two rows are interchanged. This is due to the fact that their determinants are -1 and 1. While the determinant may be either -1 or 1, the tropisms cannot be altered in any way, except being rescaled. This rescaling process will be addressed next.

The rescaling of the pretropisms arises as a necessity. The pretropism matrix B may contain pretropisms for which we cannot construct the unimodular matrix M, such that the matrix M is a square matrix with $det(M) = \pm 1$ and that it contains d pretropisms in their first d rows as the pretropisms appear in the matrix B.

As an example, consider the case in Example 4.9. In Example 4.9, we use Sage (121) to compute the Smith Normal Form (97)(26) of the matrix B, denoted matrix S. The Example 4.9 is taken from (5).

Example 4.9.

$$B = \begin{bmatrix} 2 & 6 & 17 & 9 \\ 4 & 14 & 13 & 3 \end{bmatrix}. \tag{4.11}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}, \tag{4.12}$$

The matrix S does not have $1's$ on the diagonal. Hence, the two methods we used earlier to obtain the matrix M will not work in this case. In order to obtain the matrix M, we can rescale the pretropisms, i.e. row vectors of the matrix B, by using the matrices, which are generated in the process of obtaining the Hermite Normal Form of the matrix B. For information on the Hermite normal form of a matrix, see (52) (26).

The Hermite Normal Form (52) (26) of the matrix B leads to the following identity: $UB = H$. The matrix H is an upper triangular matrix and U is a square matrix with $det(U) = \pm 1$.

We work under the assumption that the matrix B is of full rank and that the diagonal of the matrix H contains only nonzero elements. If the matrix H happens to have zeros on the diagonal, the elements of the matrix B can be permuted in such a way, that the matrix H will contain only the nonzero elements on its diagonal.

Next we construct the matrix D, which will be used to rescale the elements of matrix B. The matrix D is a diagonal matrix, whose diagonal elements consist of the diagonal elements of the matrix H. All other elements of D are zero and $dim(D) = dim(U)$. Having constructed the matrix D, we can obtain the unimodular coordinate transformation matrix M as given in Example 4.10, taken from (5).

Example 4.10. Let I denote the identity matrix, then

$$M = \begin{bmatrix} D^{-1}B \\ \mathbf{0} \quad I \end{bmatrix}. \tag{4.13}$$

We next show that $det(M) = \pm 1$. First, we extend the matrix U as in Example 4.11, taken from (5).

Example 4.11.

$$\widehat{U} = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{4.14}$$

It is clear that $det(\widehat{U}) = \pm 1$, since $det(U) = \pm 1$ and I is an identity matrix. Hence, $det(\widehat{U}M) = \pm det(M)$. Let d be the number of pretropisms in matrix B. The product $\widehat{U}M$ results in a matrix, whose $d \times d$ sub-matrix, located in upper-left corner, now has a determinant equal to ± 1. This is a direct consequence of the multiplication by the rescaling matrix D^{-1}. As $det(\widehat{U}M) = \pm 1$, it follows that $det(M) = \pm 1$.

Obtaining the unimodular matrix M in this manner, results in a fractional parametrization of the solution set. Observe also that, for the d-dimensional solution sets, the fractional exponents occur only for the first d variables, which have been initially eliminated by the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$. These first d variables, upon the return to the original coordinates, become the free parameters.

At this point, we remark that this method presents a general method to obtain the unimodular coordinate transformation matrix M. We, however, prefer the solution sets to have integer exponents, when ever that is possible. Hence, the first two methods that were described earlier are preferred, when ever possible, to this general method.

4.11.1 Method for Solving Binomial Systems

We start the treatment of binomial systems by first putting the system into the following form: $\mathbf{x}^A - \mathbf{c} = \mathbf{0}$. The matrix A, $A \in \mathbb{Z}^{k \times n}$, is the matrix containing the exponents and the vector $\mathbf{c}$=$(c_0, c_1, \ldots, c_{k-1})^T$ contains the constant terms of the binomial system, such that $c_i \neq 0$, $i = 0, 1, \ldots, k-1$.

When $rank(A) = k$, then the codimnesion of the solution set is equal to k. Furthermore, the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ gives the form of the solution set, in which the last $n-k$ variables consist of computed constant values.

We next give an outline of the method to solve binomial systems. It is assumed that the matrix A is of full rank k. We give the the Algorithm Outline 4.12 here exactly as we gave it in (5).

Algorithm Outline 4.12. *(Polyhedral Method For Binomial Systems)*

1. *Compute the null space B of A, $d = n - k$.*
2. *Compute the Smith normal form (U, S, V) of B.*
3. *Depending on U and S do one of the following:*
   - *If U is the identity matrix, then $M = V^{-1}$ and the first d variables have positive denominators in their powers when not all elements on the diagonal of S are equal to one.*
   - *If U is not the identity matrix and if all elements on the diagonal of S are one, then extend U^{-1} with an identity matrix to obtain an n-by-n matrix E that has U^{-1} in its first d rows and columns. Then, $M = EV^{-1}$.*
   - *. In all other cases, define M as in (Equation 4.13).*
4. *After the coordinate transformation $\boldsymbol{x} = \boldsymbol{z}^M$, compute the leading coefficients solving a binomial system of k equations in k unknowns. Return M and the corresponding solutions of the binomial system.*

The output of the method to solve binomial systems consists of a parametrization of a d-dimensional solution set, involving d free parameters. Setting the free parameters equal to *zero*, means that that we are cutting the solution set of the binomial system with d coordinate hyperplanes. By default, in our polyhedral method, the first d variables assume the role of the d free parameters. The points, where the coordinate hyperplanes meet the solution set, are the isolated points, which take the role of the coefficients in the parametrization of the solution set of a binomial system. These are the points, which are obtained by solving the binomial system, after the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$.

When the d free parameters are not set equal to zero, then the exponents of the parameters are given directly by *a* basis of the kernel of the matrix A, which contains the exponents of the non-constant monomials in its rows. We denoted the matrix, which contains the basis of the kernel of the matrix A, as matrix B. Note also that any vectors in the basis of the kernel can be taken to be the exponents of the free d parameters.

4.13 Polyhedral Method for Sparse Systems of Polynomials

With this section, we are moving towards the development of the solution sets for general systems of polynomials. As we do not know beforehand what the dimension of such a solution set is, we begin the search for the d-dimensional solution sets by looking for the d-dimensional cones of pretropisms. Such cones of pretropisms lead to initial form systems, defined in (Definition 1.7). With the cones of pretropisms, we can transform the initial forms systems by using the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$. This procedure is analogues to the way that we used the vectors of the matrix B, when we were considering binomial systems. Specifically, vectors of the matrix B were considered to be cones of pretropisms.

If solving of the transformed initial form systems leads to isolated solutions, then we *might* have a starting point for the development of a multivariate Puiseux series of the d-dimensional solution set. Verifying the existence of the Puiseux series follows the process, described in Chapter 3, where we considered space curves as solution sets. For the remainder of this Chapter, we will describe the d-dimensional solution sets by developing Puiseux series of the space curves on the d-dimensional surface, using each free parameter as a parameter in such a Puiseux series.

In our investigation of the d-dimensional solution sets and their Puiseux series, we relied on insights that were described in (114) and (119). We refer to (114) and (119) for more information about decomposition of solution sets.

4.13.1 Asymptotic View of the Algebraic Surfaces

The system of polynomials $F(\mathbf{x})$, as given in Equation 1.1, may define a d-dimensional algebraic surface. In order to develop and represent that d-dimensional surface using a Puiseux series, we assume that the defining equations of the polynomial system $F(\mathbf{x})$ are in Noether position (38) (119). When the equations of the polynomial system $F(\mathbf{x})$ are in Noether position (38), we may set the first d variables of $F(\mathbf{x})$ to equal random complex numbers (38) (119) (113). Consequentially, solving the system $F(\mathbf{x})$, with the first d variables set to random complex numbers, would now result in a zero dimensional solution set. Furthermore, we make the assumption that when we set the first d variables in the system $F(\mathbf{x})$ to equal zero, that the d-dimensional algebraic surface does not change its dimension. In other words, we assume that the d-dimensional algebraic surface intersects the first d coordinate planes in a generic manner.

Otherwise, a different choice of the d variables would need to be made to accommodate for the situation. Regarding positive dimensional solution sets, see also (83).

We now offer the following illustration of what has been described in the previous paragraph, using (38) (119) (113) (114). Here, we restrict ourselves to the case of a two dimensional surface, i.e. $d = 2$. Consider Example 4.14, containing the general polynomial system (Equation 1.1), amended with two additional parameters t_0 and t_1. The Example 4.14 is taken from (5).

Example 4.14.

$$\begin{cases} F(\mathbf{x}) & = & \mathbf{0} \\ x_0 & = & c_0 t_0 \\ x_1 & = & c_1 t_0^{v_{0,1}} t_1^{v_{1,1}} (c_{1,1} + O(t_0, t_1)), \end{cases} \tag{4.15}$$

where $c_0, c_1, c_{1,1} \in \mathbb{C} \setminus \{0\}$ and $v_{0,1}, v_{1,1} \in \mathbb{Q}$.

This illustration show how we can move on a two dimensional surface by allowing the two parameters t_0 and t_1 to move from 1 to 0. For example, at $t_0 = 1$ and $t_1 = 1$ we are at a random point on the two dimensional surface, given by $x_0 = c_0$ and $x_1 = c_1$.

Here we have allowed the first variable x_0 to be represented by a linear parameter $c_0 t_0$. However, we may not allow a similar linearity to describe the second variable x_1 as it is not flexible enough to describe a general surface. Hence, the second variable has the form $x_1 = c_1 t_0^{v_{0,1}} t_1^{v_{1,1}} (c_{1,1} + O(t_0, t_1))$ and with it a greater flexibility, necessary to describe a more general surface.

By letting t_0 move to zero, x_0 is also moving to zero. Next we illustrate what can happen with x_1 in such a case. We can let the variable x_1 also move to zero, provided that $v_{0,1} > 0$

and that $v_{1,1} > 0$. If $v_{0,1} < 0$ or $v_{1,1} < 0$ then x_1 moves to infinity. Otherwise, if $v_{0,1} = 0$ and $v_{1,1} = 0$, x_1 moves to $c_1 c_{1,1}$.

The statement of (Equation 4.15) gives what we call a multivariate Puiseux series. Furthermore, what was just described for the variable x_1 can be extended to other variables to obtain a general Puiseux series, which describes the algebraic surface. The form of the extension is given in Example 4.15, taken from (5).

Example 4.15 (Example 4.14 continued)**.**

$$x_k = c_k t_0^{v_{0,k}} t_1^{v_{1,k}} (c_{1,k} + O(t_0, t_1)), \tag{4.16}$$

where $c_k, c_{1,k} \in \mathbb{C} \setminus \{0\}$, $v_{0,k}, v_{1,k} \in \mathbb{Q}$, for $k = 2, \ldots, n-1$.

In specific instances, the general, multivariate, Puiseux series reverts to the multivariate Taylor series. In the current example, such an instance will occur if the exponents in the Puiseux series are positive integers and, as the the parameters t_0 and t_1 approach zero, we have a finite number of solutions at *infinity* with multiplicity one.

In the next paragraph, we want to establish a connection between the Puiseux series, which we encountered when we considered algebraic curves and the general, multivariate, Puiseux series, which are necessary to describe d-dimensional solution sets. In particular, we want to show how the notion of *tropism* extends to a notion of *cone of tropisms* when we consider polynomial systems, whose solution sets are d-dimensional.

Consider the content of (Equation 4.15) and (Equation 4.16) in Example 4.14 and Example 4.15. Let $\Phi(t_0, t_1) = \mathbf{0}$ denote the system $F(x_0 = c_0 t_0,\ x_1 = c_1 t_0^{v_{0,1}} t_1^{v_{1,1}}(c_{1,1} + O(t_0, t_1)),\ x_k = c_k t_0^{v_{0,k}} t_1^{v_{1,k}}(c_{1,k} + O(t_0, t_1))) = \mathbf{0}$, for $k = 2, \ldots, n-1$.

Letting the parameters t_0 and t_1 go to zero, the new polynomial system $\Phi(t_0, t_1) = \mathbf{0}$ consists of at least two monomials per equation. This follows from the fact that all the leading coefficients are such that $c_k, c_{1,k} \in \mathbb{C} \setminus \{0\}$, for all $k = 0, 1, \ldots, n-1$. In other words, all the solutions at infinity are non-zero, regular and isolated solutions. The sum of monomials in each equation of the system $F(\mathbf{x}) = \mathbf{0}$, for which t_0 and t_1 in $\Phi(t_0, t_1) = \mathbf{0}$ have the lowest exponents, is referred to as the initial form system of $F(\mathbf{x}) = \mathbf{0}$ with respect to cone of (pre)tropisms $\mathbf{v}_0 = (1, v_{0,1}, v_{0,1}, \ldots, v_{0,n-1})$ and $\mathbf{v}_1 = (0, v_{1,1}, v_{1,2}, \ldots, v_{1,n-1})$. See also (Definition 1.7).

The notion of tropism from Chapter 2 and Chapter 3 has now been extended to a cone of tropisms. By *cone* we mean a polyhedral cone, see (145) (56). As tropisms are linearly independent vectors, each generator of the cone corresponds to a parameter in the multivariate Puiseux series.

Before we give the next result, we want to point out [(58), Proposition 1] as an important and related result. We now give our result, regarding the development of a Puiseux series for d-dimensional solution sets. We give Proposition 4.16 and its proof here exactly as we gave it in (5).

Proposition 4.16. *If $F(\boldsymbol{x}) = \boldsymbol{0}$ is in Noether position and defines a d-dimensional solution set in $\mathbb{C}^n$, intersecting the first d coordinate planes in regular isolated points, then there are d linearly independent tropisms $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots \boldsymbol{v}_{d-1} \in \mathbb{Q}^n$ so that the initial form system*

$in_{\boldsymbol{v}_0}(in_{\boldsymbol{v}_1}(\cdots in_{\boldsymbol{v}_{d-1}}(F)\cdots))(\boldsymbol{x} = \boldsymbol{z}^M) = \boldsymbol{0}$ has a solution $\boldsymbol{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$. This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

$$\begin{aligned}
x_0 &= t_0^{v_{0,0}} \\
x_1 &= t_0^{v_{0,1}} t_1^{v_{1,1}} \\
&\vdots \\
x_{d-1} &= t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} \\
x_d &= c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
x_{d+1} &= c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
&\vdots \\
x_n &= c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots
\end{aligned} \tag{4.17}$$

Proof. Because the set defined by $F(\mathbf{x}) = \mathbf{0}$ is in Noether position, we can let the first d variables go to zero, using for example a multiparameter homotopy as in (Equation 4.15) and still obtain regular isolated solutions, denoted as $(0, 0, \ldots, 0, c_0, c_1, \ldots, c_{n-d-1}) \in \mathbb{C}^n$.

The tropisms $\mathbf{v}_0$, $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_{d-1}$ define the initial form system, i.e.: those monomials in the system $F(\mathbf{x}) = \mathbf{0}$ that become dominant as the parameters t_0, t_1, $\ldots$, t_{d-1} move to zero. In particular: for any vector $\mathbf{v}$ in the cone spanned by the tropisms, we have that every monomial $\mathbf{x}^{\mathbf{a}}$ in the initial form system makes minimal inner product $\langle \mathbf{a}, \mathbf{v} \rangle$, minimal with respect to any other monomial $\mathbf{x}^{\mathbf{b}}$ not in the initial form system, i.e.: $\langle \mathbf{a}, \mathbf{v} \rangle$ ¡ $\langle \mathbf{b}, \mathbf{v} \rangle$.

Because the leading terms of the Puiseux series vanish at the initial form system, the inner product with the monomials and the leading powers must be minimal compared to all other monomials in the system. Hence the shape of the Puiseux series. □

4.16.1 Polyhedral Method for Algebraic Sets

When we apply our polyhedral method to develop Puiseux series representation of a d-dimensional solution set, the Proposition 4.16 plays a major role. In Example 4.17, we give the description of the way we use Proposition 4.16. We give the description in Example 4.17 exactly as we gave it in (5).

Example 4.17. (Proposition 4.16 Applied)

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $F(\mathbf{x}) = \mathbf{0}$ of dimension d, then the system $F(\mathbf{x}) = \mathbf{0}$ has no solution set of dimension d that intersects the first d coordinate planes properly; otherwise
- if a d-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For algorithms to obtain the tropical prevariety, we refer to (19). See also Gfan (71), the Singular library `tropical.lib` (72) and Sage (121). In Chapter 3 and in (3), we described how we used Cayley embedding, see Definition 1.8, and software `cddlib` (45), to obtain pretropisms.

By applying the software `cddlib` (45) on the Cayley embedding, in Chapter 3 and in (3), we were able to obtain all the pretropisms of the cyclic 12-roots polynomial system.

In Chapter 3, we pointed out the similarities among the tropisms, which we encountered when we considered the cyclic 4, 8 and 12-roots polynomial systems. In particular, they were the tropisms for which we found an exact representation of the solution sets. In Example 4.18, we list the tropisms as an illustration. The Example 4.18 is taken from (134) (5).

Example 4.18.

$$\begin{aligned} C_{n=4}(\mathbf{x}): &\quad (1,-1,1-1) \\ C_{n=8}(\mathbf{x}): &\quad (1,-1,1-1,1,-1,1-1) \\ C_{n=12}(\mathbf{x}): &\quad (1,-1,1-1,1,-1,1-1,1,-1,1-1) \end{aligned} \tag{4.18}$$

As the Example 4.18 shows, there is a clear pattern among the tropisms of the cyclic 4, 8 and 12-roots polynomial systems. This tropism pattern can be applied to other cyclic n-roots polynomial systems, where $n = 4m$, for some integer m. By applying our polyhedral method to cyclic n roots polynomial systems, we are hence in a position to observe additional patterns, which can be exploited, in addition to the cyclic permutation of the variables, covered in Chapter 3. We will revisit these ideas in the subsequent sections, where we will present an interesting result regarding the cyclic n-roots problem.

We stated in Definition 1.11 that we are expanding our Puiseux series of the positive dimensional solution sets around zero. As we are using the coordinate hyperplane to intersect the solution set, by moving the first coordinate to zero, we require the first element of the first (pre)tropism to be positive. We make this convention our choice.

It is also possible to choose the first element of the first (pre)tropism to be negative and then move the first coordinate to infinity. In this way, the coordinate hyperplane would be intersecting the solution set at infinity.

4.18.1 From Cones of Pretropisms to Puiseux Series

We break down the development of a Puiseux series for general algebraic sets into four steps. We give the Algorithm Outline 4.19 here exactly as we gave it in (5).

Algorithm Outline 4.19. *(Polyhedral Method For Algebraic Sets)*

For every d-dimensional cone $\mathcal{C}$ of pretropisms:

1. *We select d linearly independent generators to form the d-by-n matrix A and the corresponding unimodular transformation $\boldsymbol{x} = \boldsymbol{z}^M$.*
2. *Because the matrix A contains pretropisms, the initial form system $in_{\boldsymbol{v}_0}(in_{\boldsymbol{v}_1}(\cdots in_{\boldsymbol{v}_{d-1}}(F)\cdots))(\boldsymbol{x}) = \boldsymbol{0}$ determined by the rows $\boldsymbol{v}_0$, $\boldsymbol{v}_1$, $\ldots$, $\boldsymbol{v}_{d-1}$ of A has at least two monomials in every equation. If the initial form system has no solution with all coordinates different from zero, then we move to the next cone $\mathcal{C}$ and return to step 1, else we continue with the next step.*
3. *Solutions of the initial form system found in the previous step may be leading coefficients in a potential Puiseux series with corresponding leading powers equal to the pretropisms. If the leading term satisfies the entire polynomial system, then we report an explicit solution of the system and we continue processing the next cone $\mathcal{C}$. Otherwise, we take the current leading term to the next step.*

4. If there is a second term in the Puiseux series, then we have computed an initial development for an algebraic set and report this development in the output.

Having solved the initial form system, we have obtained the first terms in what *may* be a Puiseux series of a general algebraic set. We say *may* as the solution of the initial form system may have provided us with just an *isolated* solution at infinity. In order to be certain that the solution set is a d-dimensional surface, we need to obtain the second term in its Puiseux series.

Instead of computing the second term for each of the d free parameters in the Puiseux series, it is enough to compute the second term for just one of them. In other words, it is enough to show that we can develop a space curve on the d-dimensional surface. If we are able to obtain the second term, using one of the free parameters, then, along with the other first terms in the Puiseux series, we can show that the solution set is a d-dimensional surface. We covered the development of a space curve in Chapter 3 and in (3). Furthermore, in Chapter 3 and in (3), we briefly mentioned the potential the Puiseux series carries in the computation of the witness sets (118) (134). The same potential applies here as well.

4.20 Our Polyhedral Method in Applications

When we apply our polyhedral method to the cyclic n-roots polynomial system (Equation 1.1), we are able to find exact representations of its d-dimensional solution sets. We now give several illustrations, which will later be incorporated into a more general result.

4.20.1 Cyclic 9-roots Problem

We first start with $n = 9$. The Lemma 1.2 of Backelin (11) tells us that we should expect the dimension of the solution set to be at least two. By applying our polyhedral method, we will obtain the solution and we will show that the solution we found can be put into the same form as what was given by Faugère in (43, Lemma 1.1) as the proof of Lemma 1.2 of Backelin (11). We want to point out that the cyclic 9-roots problem has been already solved by Faugère in (43).

In our search for the positive dimensional solution set of the cyclic 9-roots polynomial system, we started our polyhedral method in (5) by using the Cayley embedding and the `cddlib` (45) software to obtain the facet normals. Consecutive application of this approach, with additional processing to ensure that the facet normals are pretropisms, we obtained a cone generated by two pretropisms: $\mathbf{v}_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)$ and $\mathbf{v}_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1)$. This cone of pretropisms lead to the initial form system $in_{\mathbf{v}_0}(in_{\mathbf{v}_1}(F))(\mathbf{x}) = \mathbf{0}$, given in Example 4.21. The Example 4.21 is taken from (5).

Example 4.21.

$$\left\{\begin{array}{rcl}
x_2 + x_5 + x_8 & = & 0 \\
x_0x_8 + x_2x_3 + x_5x_6 & = & 0 \\
x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 & & \\
+\ x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_7 & & \\
+\ x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 & = & 0 \\
x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 & & \\
+\ x_0x_1x_2x_3x_7x_8 + x_0x_1x_2x_6x_7x_8 & & \\
+\ x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 & & \\
+\ x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 & & \\
+\ x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 & & \\
+\ x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 & & \\
+\ x_0x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 & = & 0.
\end{array}\right. \tag{4.19}$$

The initial form system (Equation 4.19) of Example 4.21 is considerably sparser and, hence, considerably easier to solve than the original cyclic 9-roots polynomial system. However, we do not proceed by solving (Equation 4.19) directly. Because (Equation 4.19) is an initial form system, with respect to the two pretropisms $\mathbf{v}_0$ and $\mathbf{v}_1$, we can use pretropisms $\mathbf{v}_0$ and $\mathbf{v}_1$ to eliminate the first two variables of $in_{\mathbf{v}_0}(in_{\mathbf{v}_1}(F))(\mathbf{x}) = \mathbf{0}$ by applying a unimodular coordinate transformation. As $\mathbf{v}_0$ and $\mathbf{v}_1$ have a particularly convenient form, we do not need to perform the Smith normal form or the Hermite normal form computation in order to obtain the unimodular matrix M. We covered that process in the subsection 4.5.1 <u>Unimodular Coordinate Transformations</u> of this Chapter.

In this particular case, we can simply form the unimodular matrix M by placing the pretropisms $\mathbf{v}_0$ and $\mathbf{v}_1$ in its first two rows. On the rest of the diagonal we place 1's and the rest we fill with 0's, such that M is a square matrix. When we are done, we get the matrix M, given Example 4.22, taken from (5):

Example 4.22 (Example 4.21 continued)**.**

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.20}$$

Clearly, $det(M) = 1$ and the unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ leads to the change of variables, given in Example 4.23, taken from (5).

Example 4.23 (Example 4.21 continued)**.**

$$\begin{array}{lll} x_0 = z_0 & x_3 = z_0 z_3 & x_6 = z_0 z_6 \\ x_1 = z_0 z_1 & x_4 = z_0 z_1 z_4 & x_7 = z_0 z_1 z_7 \\ x_2 = z_0^{-2} z_1^{-1} z_2 & x_5 = z_0^{-2} z_1^{-1} z_5 & x_8 = z_0^{-2} z_1^{-1} z_8. \end{array} \tag{4.21}$$

Using (Equation 4.21) to change the variables in (Equation 4.19), turns the initial form system (Equation 4.19) into a system consisting of 9 equations and now 7 variables. We refer to (118) for information on introduction of slack variables and computation of mixed volumes for polynomial systems. After the change of variables (Equation 4.21), the transformed initial form system is considerably easier to solve.

We used the blackbox solver of PHCpack (132) to solve the initial form system (Equation 4.19), after the coordinate transformation (Equation 4.21). Even though we used a numerical solver, we recognized the primitive roots of unity among the solutions. Upon return of the solutions to the original coordinates, via Equation 4.21, we found that they satisfy the original cyclic 9-roots polynomial system entirely.

Our goal was to develop the Puiseux series representation of a *two* dimensional solution set of the cyclic 9-roots polynomial system. Instead, we found its exact representation, as the entire system was satisfied by the first term of what was supposed to be a Puiseux series.

We next give the representation of the two dimensional solution set. We let t_0, t_1 be the free parameters and we let $u = e^{2\pi i/3}$ denote the third primitive root of unity, s.t. $u^3 - 1 = 0$. The the solution set of the cyclic 9-roots polynomial system that we obtained has an exact form, given in Example 4.24, taken from (5).

Example 4.24 (Example 4.21 continued)**.**

$$\begin{array}{lll} x_0 = t_0 & x_3 = t_0 u & x_6 = t_0 u^2 \\ x_1 = t_0 t_1 & x_4 = t_0 t_1 u & x_7 = t_0 t_1 u^2 \\ x_2 = t_0^{-2} t_1^{-1} u^2 & x_5 = t_0^{-2} t_1^{-1} & x_8 = t_0^{-2} t_1^{-1} u. \end{array} \tag{4.22}$$

Furthermore, if we perform an additional change of variables by letting $z_0 = t_0$, $z_1 = t_0 t_1$, and $z_2 = t_0^{-2} t_1^{-1} u^2$, then the solution has the form, given in Example 4.25, taken from (5).

Example 4.25 (Example 4.21 continued)**.**

$$\begin{array}{lll} x_0 = z_0 & x_3 = z_0 u & x_6 = z_0 u^2 \\ x_1 = z_1 & x_4 = z_1 u & x_7 = z_1 u^2 \\ x_2 = z_2 & x_5 = z_2 u & x_8 = z_2 u^2 \end{array} \tag{4.23}$$

The solution set (Equation 4.23) satisfies the cyclic 9-roots polynomial system, modulo $z_0^3 z_1^3 z_2^3 u^9 - 1 = 0$, on simple substitution, exactly as in Faugère's proof of (43, Lemma 1.1).

Since we have obtained an exact representation of the *two*-dimensional solution set (Equation 4.22) for the cyclic 9-roots polynomial system, we can now easily obtain its degree. In order to achieve that, we will intersect the *two*-dimensional surface with *two* random hyperplanes (134). The number of intersection points will equal the degree (134). See (134) for the computation of the degree in this manner. The setup leads to the system, given in Example 4.26, taken from (5).

Example 4.26 (Example 4.21 continued)**.**

$$\begin{cases} \alpha_1 t_0 + \alpha_{1,2} t_0 t_1 + \alpha_{-2,-1} t_0^{-2} t_1^{-1} = 0 \\ \beta_1 t_0 + \beta_{1,2} t_0 t_1 + \beta_{-2,-1} t_0^{-2} t_1^{-1} = 0 \end{cases} \tag{4.24}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are random complex numbers.

We can simplify the system (Equation 4.24), to obtain the smaller system, given in Example 4.27, taken from (5).

Example 4.27 (Example 4.21 continued)**.**

$$\begin{cases} t_0^{-3} t_1^{-1} - c_0 = 0 \\ t_1 - c_1 = 0 \end{cases} \tag{4.25}$$

where $c_0, c_1 \in \mathbb{C}$.

It is easy to see that the system (Equation 4.25) has three solutions. As a result, the *two*-dimensional solution set, shown in (Equation 4.22), is a surface of degree *three*.

Next, we will apply the cyclic permutation of the roots of unity to obtain an *orbit* of the *two*-dimensional solution sets of degree *three*. Permuting the coefficients of (Equation 4.23) cyclically, forwards and backwards, is illustrated in Example 4.28, taken from (5).

Example 4.28 (Example 4.21 continued).

$$\begin{array}{ccc} 1 & u & u^2 \\ u & u^2 & 1 \\ u^2 & 1 & u \\ u^2 & u & 1 \\ u & 1 & u^2 \\ 1 & u^2 & u \end{array} \tag{4.26}$$

From Example 4.28, we see that the cyclic permutation of the coefficients leads to an orbit of length $2 \times 3 = 6$. Hence, using one *two*-dimensional surface of degree *three*, we have obtained *six* such surfaces in total, all belonging to the same orbit. This result has also been obtained in (115), using a different method.

4.28.1 Cyclic m^2-roots Problem

The computation of pretropisms is the most difficult task in our polyhedral method. With regards to the cyclic n-roots problem, the application of the Cayley embedding to obtain the pretropisms is rather inefficient for values $n > 12$. However, when we consider polynomial systems, like the cyclic n-roots, that is, systems containing certain structure or symmetry, our polyhedral method is quite good in capturing that structure. Rather than attempting to compute all the pretropisms for the cyclic 16-roots problem, the problem next in line of interesting n-roots problems, we will use the tropisms and initial form solutions, that we obtained for the cyclic 9-roots problem, and use them to get the tropisms and the initial form solutions for the cyclic 16-roots problem.

Consider the Example 4.29, taken from (5). It shows the cone of tropisms of the cyclic 9-roots, which we used earlier

Example 4.29.

$$\begin{aligned} \mathbf{v}_0 &= (1, 1, -2, 1, 1, -2, 1, 1, -2) \\ \mathbf{v}_1 &= (0, 1, -1, 0, 1, -1, 0, 1, -1) \end{aligned} \tag{4.27}$$

We can now use the pattern, observed in Example 4.29, and extend it to find the correct cone of tropisms for the cyclic 16-roots problem. We give the cone of tropisms for the cyclic 16-roots polynomial system in Example 4.30, taken from (5).

Example 4.30.

$$\begin{aligned} \mathbf{v}_0 &= (1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3) \\ \mathbf{v}_1 &= (0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2) \\ \mathbf{v}_2 &= (0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1) \end{aligned} \tag{4.28}$$

Furthermore, the solutions of the initial form system, corresponding to the cone of tropisms generated by (Equation 4.28), are now the primitive *fourth* roots of unity $u^4 - 1 = 0$. Repeating here what we did in (Equation 4.22) and (Equation 4.23) for the cyclic 9-roots problem, it is easy to see that the exact solution we get for the cyclic 16-roots, with our polyhedral method, can be matched with what is given in Faugère's proof of (43, Lemma 1.1).

The Proposition 4.31 gives the general description of the solution set for the cyclic n-roots problem, when $n = m^2$. We give the Proposition 4.31 here exactly as we gave it in (5):

Proposition 4.31. *For $n = m^2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as*

$$\begin{array}{rcl} x_{km+0} & = & u_k t_0 \\ x_{km+1} & = & u_k t_0 t_1 \\ x_{km+2} & = & u_k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u_k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array} \tag{4.29}$$

for $k = 0, 1, 2, \ldots, m-1$ and $u_k = e^{i2k\pi/m}$.

Although we prefer to continue to work with the solution form, given in (Equation 4.29), it is possible to perform the following change of variables $t_0 = s_0$, $t_0 t_1 = s_1$, $t_0 t_1 t_2 = s_2$, $\ldots$, $t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} = s_0^{-1} s_1^{-1} \cdots s_{m-2}^{-1}$ and give (Equation 4.29) a more straight forward form. Note the effect the change of variables has on the generators of the cones of tropisms.

We next give the second result, regarding the cyclic n-roots problem, when $n = m^2$. We give the Proposition 4.32 and its proof here exactly as we gave it in (5):

Proposition 4.32. *The $(m-1)$-dimensional solution set in (Equation 4.29) has degree equal to m.*

Proof. To determine the degree of an $(m-1)$-dimensional algebraic set, we intersect the set with $m-1$ hyperplanes with random coefficients. In any linear equation we replace the x-variables using the equations in (Equation 4.29), dividing each equation by t_0 to obtain a nonzero constant coefficient. Because every x_j corresponds to one monomial in t_0, t_1, $\ldots$, t_{m-1}, bringing the coefficient matrix into a reduced row echelon form leads to a binomial system of $m-1$ equations in $m-1$ unknowns:

$$\left\{ \begin{array}{rcl} t_0^{-m} t_1^{-m+2} t_2^{-m+3} \cdots t_{m-2}^{-1} - c_0 & = & 0 \\ t_1 - c_1 & = & 0 \\ t_1 t_2 - c_2 & = & 0 \\ & \vdots & \\ t_1 t_2 t_3 \cdots t_{m-2} - c_{m-2} & = & 0 \end{array} \right. \tag{4.30}$$

Collecting the coefficients $(c_0, c_1, c_2, \ldots, c_{m-2})$ in $\mathbf{c}$ and the exponents in a matrix A, we denote the binomial system as $\mathbf{t}^A = \mathbf{c}$ with

$$A = \begin{bmatrix} -m & -m+2 & -m+3 & \cdots & -2 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{4.31}$$

The binomial system has $|\det(A)| = m$ solutions and therefore the degree equals m. $\square$

By applying cyclic permutation, as we applied it in (Equation 4.26), we can obtain $2m$ solution components with degree m.

4.33 Summary of the Chapter

In this chapter we covered the extension of ideas developed in Chapter 2 and Chapter 3. The main focus has been on extending our polyhedral method to develop general algebraic sets. Whereas in the previous chapters we dealt with single tropisms, in this chapter we showed how cones of tropisms arise when we consider general algebraic sets and how they lead to multivariate Puiseux series representations of such sets.

In this chapter, we initially focused on solving of binomial systems. For binomial systems, our approach is an algorithm. We next applied our polyhedral method to polynomial systems to obtain general algebraic sets. The most significant aspects of this chapter has been the construction of unimodular coordinate transformations, using the Smith and Hermite normal forms, the Proposition 4.16, Proposition 4.31, Proposition 4.32 and their proofs, and the Algorithm Outline 4.12 and Algorithm Outline 4.19.

Application of our polyhedral method to the cyclic n-roots problem lead to two interesting results, when $n = m^2$. We concluded this chapter by giving an exact representation of a solution set for the cyclic m^2-roots problem and the degree of that solution set.

CHAPTER 5

CONCLUSION

By developing our polyhedral method, we are aiming to generalize polyhedral homotopies. In particular, the start systems in polyhedral homotopies for isolated solutions, generalize in this thesis to start systems for the development of positive dimensional solution sets in form of a multivariate Puiseux series. As such, our polyhedral method generalizes the Newton-Puiseux method.

In his proof of the Theorem B, which he then used to prove his Theorem A, Bernshtein used the Puiseux series and a polyhedral homotopy. From the point of view of our polyhedral method, Bernshtein's Theorems A and B and Maurer's notion of tropism, are essential to the way we solve polynomial systems. Furthermore, our understanding of tropical algebraic geometry comes from understanding Bernshtein's work. Specifically, we see the fundamental theorem of tropical algebraic geometry as a generalization of Bernshtein's Theorem B. As a result, the polyhedral method of this thesis can be seen as the symbolic-numeric version of the fundamental theorem of tropical algebraic geometry.

Application of our polyhedral method to the computationally challenging cyclic n-roots polynomial systems produced several interesting results regarding their solutions sets. We give the success we had in obtaining the positive dimensional solutions sets of the cyclic n-roots polynomial systems as evidence that the concept behind our polyhedral method is valid.

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Publications and Preprints

Danko Adrovic and Jan Verschelde
Computing Puiseux Series for Algebraic Surfaces
Published in the Proceedings of the 37^{th} *International Symposium on Symbolic and Algebraic Computation (ISSAC 2012). ACM. 2012.*

Danko Adrovic and Jan Verschelde
Polyhedral Methods for Space Curves Exploiting Symmetry
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Tropical Algebraic Geometry in Maple: a preprocessing algorithm for finding common factors to multivariate polynomials with approximate coefficients. Published in the Journal of Symbolic Computation 46(7): 755-772, 2011.

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A tropical algorithm to find a common factor of multivariate polynomials with complex coefficients. Masters Thesis. Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago. 2009.

Conference Talks

Tropisms, Surfaces and the Puiseux Series
SIAM Conference on Discrete Mathematics
Combinatorial and Tropical Algebraic Geometry
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