Numerical Algebraic Geometry and Symbolic Computation

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ISSAC 2004, 4-7 July 2004, Santander

Plan of the Talk

1. Central Problem: compute an irreducible decomposition of the solution set of a polynomial system.
2. Key Data Structure: "witness set" uses notion of generic points in algebraic geometry.
3. Algorithms: embeddings and numerical homotopies to decompose and factor positive dimensional solution sets.
4. Two Connections with Symbolic Computation: straight-line programs and approximate multivariate factorization.
5. Applications from Mechanical Design.

Joint work with Andrew Sommese (University of Notre Dame) and Charles Wampler (General Motors Research Laboratories).

Overview of Data Structures

A **witness set** is a numerical representation of a k-dimensional solution set of degree d. It contains d generic points on k random hyperplanes.

A **sequence of witness sets** represents the decomposition of a solution set into components of the same dimension.

A **sequence of partitioned witness sets** is our data structure to represent a **numerical irreducible decomposition**.

Overview of Algorithms

To compute a witness set, we **embed** the polynomial system and apply a blackbox solver.

For sequences of witness sets, a **cascade of homotopies** allows to recycle solutions, pealing off hyperplane sections in going from the top to lower dimensions.

To partition witness sets, a factorization is predicted by **monodromy** and certified by **linear traces**.

Complete Intersections – Slicing

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_2 - x_1^2 \\ x_3 - x_1^3 \end{bmatrix} = \mathbf{0} \qquad \begin{array}{c} \textit{twisted} \\ \textit{cubic} \end{array}$$

general slice

$$\begin{cases} f(x_1, x_2, x_3) = \mathbf{0} \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \end{cases}$$

random c_0, c_1, c_2, c_3

three complex roots

special slice

$$\begin{cases} f(x_1, x_2, x_3) = \mathbf{0} \\ c_0 + c_1 x_1 + c_2 x_2 + 0 x_3 = 0 \end{cases}$$

random c_0, c_1, c_2

two complex roots

"numerical elimination methods"

Application: Spatial Six Positions

Planar Body Guidance (Burmester 1874)

Given five placements of a moving body in the plane, find the points of the moving body that lie on a common circle.

- 5 positions determine 6 circle-point/center-point pairs
- 4 positions give cubic circle-point & center-point curves

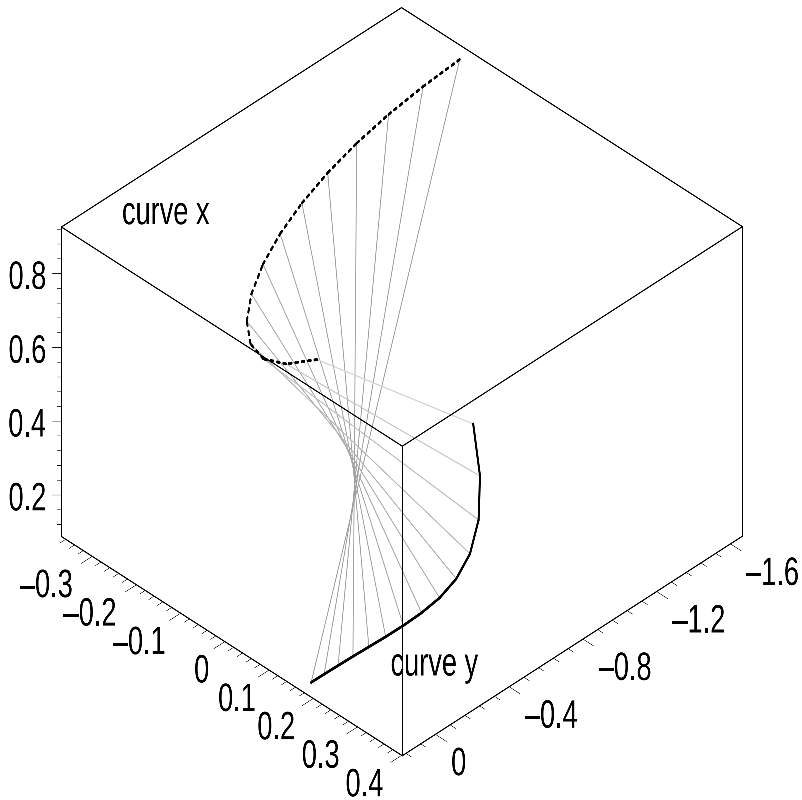
Spatial Body Guidance (Schoenflies 1886)

Given seven placements of a moving body in space, find the points of the moving body that lie on a common sphere.

- 7 positions determine 20 sphere-point/center-point pairs
- 6 positions give 10^{th}-degree sphere-point & center-point curves

Spatial Six Positions: Solution

Polynomial system of five quadrics in six unknowns $(\mathbf{x}, \mathbf{y})$ defines curve of degree 20.



Projection onto $\mathbf{x}$ or $\mathbf{y}$-space gives curves of degree 10.

General Intersections – Embedding

$$f(x_1, x_2, x_3) = \begin{bmatrix} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{bmatrix} = \mathbf{0} \qquad \begin{matrix} \textit{twisted cubic} \\ \textit{and} \\ \textit{four isolated points} \end{matrix}$$

Problem: Adding a hyperplane to $f \Rightarrow$ **overconstrained** system!

Solution: Use a slack variable z_1, with random $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$:

$$\mathcal{E}(f)(\mathbf{x}, z_1) = \begin{bmatrix} \begin{bmatrix} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{bmatrix} & + & \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} z_1 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 & + & z_1 \end{bmatrix} = \mathbf{0}$$

Solutions of $\mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$ with $z_1 = 0$ lie on the twisted cubic.

Solutions of $\mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$ with $z_1 \neq 0$ lead to the isolated points.

A Cascade of Homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde): *J. Complexity* 16(3):572–602, 2000

1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

solution paths defined by

$$H_i(\mathbf{x}, \mathbf{z}, t) = t\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + (1-t)\begin{pmatrix} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{pmatrix} = \mathbf{0}$$

starting at $t = 1$ with all solutions with $z_i \neq 0$ reach at $t = 0$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

A refined version of Bézout's theorem

Observe: The linear equations added to $f(\mathbf{x}) = \mathbf{0}$ in the cascade of homotopies do not increase the total degree.

Let $f = (f_1, f_2, \ldots, f_n)$ be a system of n polynomial equations in N variables, $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

$$\text{Bézout bound:} \quad \prod_{i=1}^{n} \deg(f_i) \geq \sum_{j=0}^{N} \mu_j \deg(W_j),$$

where W_j is a j-dimensional solution component of $f(\mathbf{x}) = \mathbf{0}$ of multiplicity μ_j.

Note: $j = 0$ gives the "classical" theorem of Bézout.

Example of a Homotopy in the Cascade

To compute numerical representations of the twisted cubic and the four isolated points, as given by the solution set of one polynomial system, we use the following homotopy:

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{ccc} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] & + & t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) & + & z_1 \end{array} \right] = \mathbf{0}$$

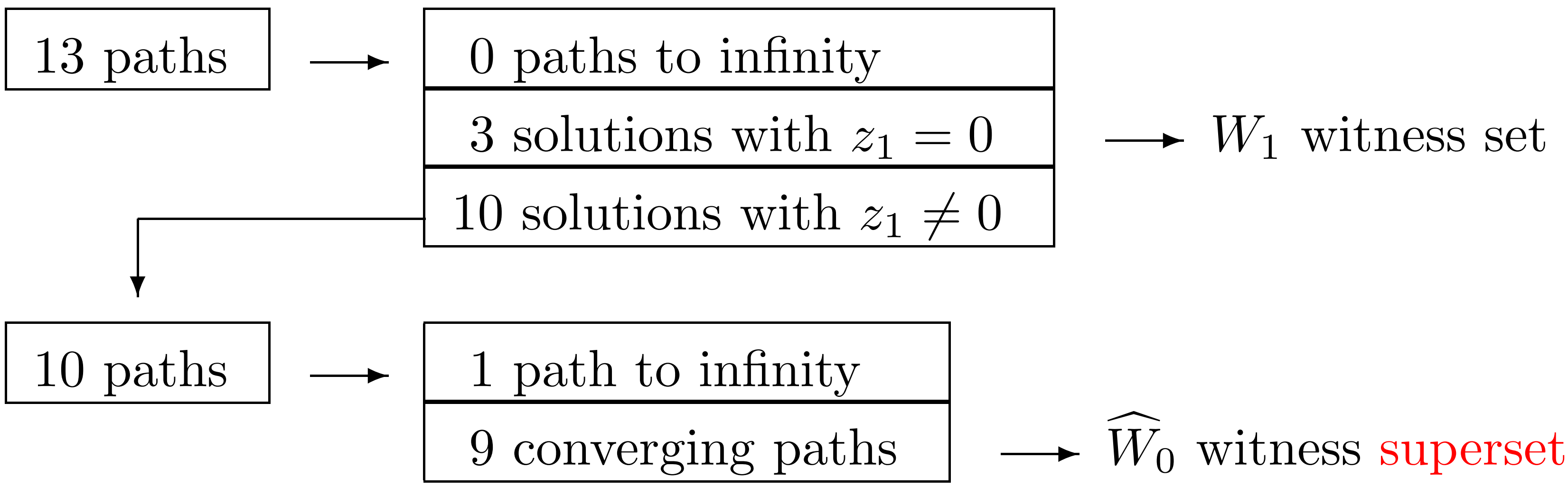
At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$.

At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the system, and z_1 is forced to zero.

#paths in twisted cubic + 4 isolated points example

The flow chart below summarizes the number of solution paths traced in the cascade of homotopies.



The set $\widehat{W_0}$ contains, in addition to the four isolated roots, also points on the twisted cubic. The points in $\widehat{W_0}$ which lie on the twisted cubic are considered junk and must be filtered out.

Membership Test

Does the point $\mathbf{p}$ belong to a component?

Given: a point in space $\mathbf{p} \in \mathbb{C}^N$; a system $f(\mathbf{x}) = \mathbf{0}$;

and a witness set W, $W = (Z, L)$:

for all $\mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0}$ and $L(\mathbf{w}) = \mathbf{0}$.

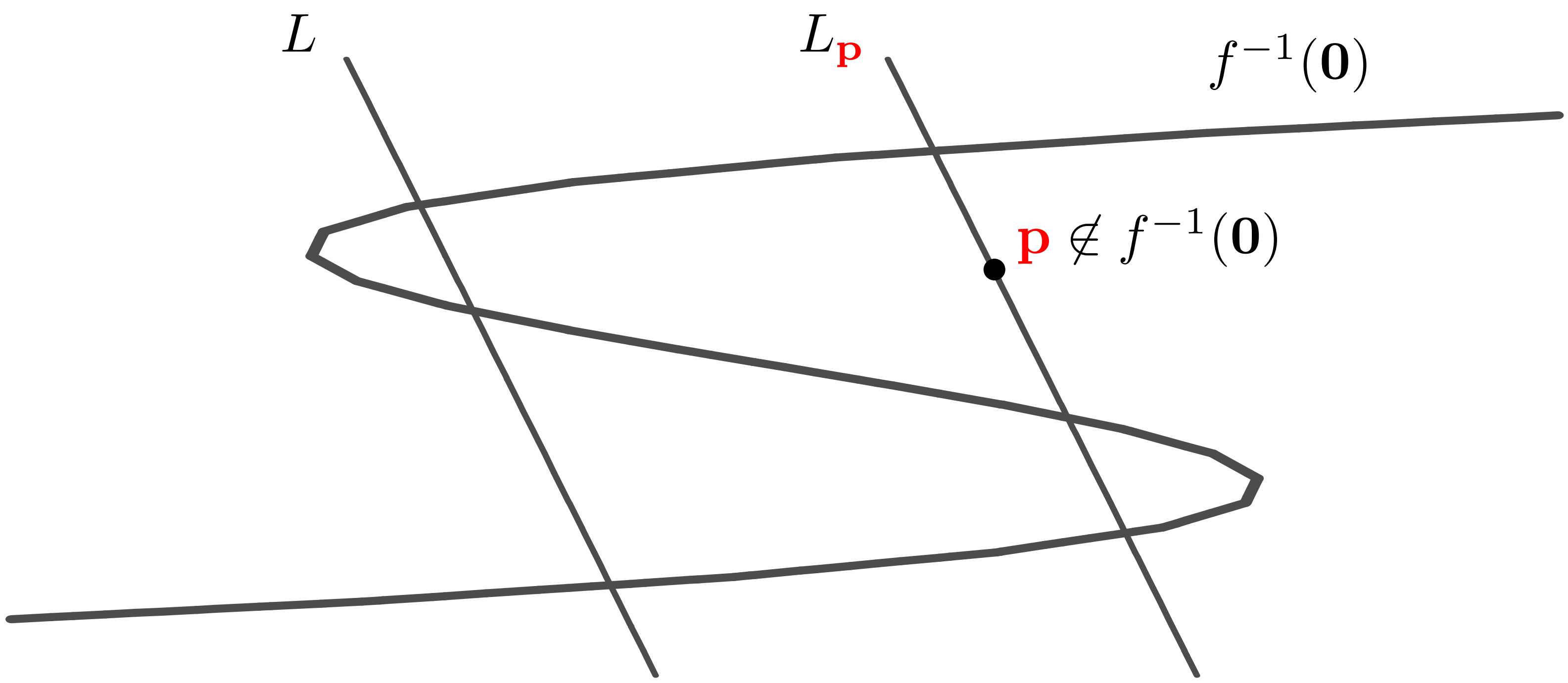
1. Let $L_{\mathbf{p}}$ be a set of hyperplanes through $\mathbf{p}$, and define

$$H(\mathbf{x}, t) = \left\{ \begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{p}}(\mathbf{x})t + L(\mathbf{x})(1 - t) = \mathbf{0} \end{array} \right.$$

2. Trace all paths starting at $\mathbf{w} \in Z$, for t from 0 to 1.

3. The test $(\mathbf{p}, 1) \in H^{-1}(\mathbf{0})$? answers the question above.

Membership Test – an example



$$H(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{p}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

Witness Sets

A **witness point** is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component

  equals the $\boxed{\textbf{dimension}}$ of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes

  equals the $\boxed{\textbf{degree}}$ of the solution component.

A **witness set** for a k-dimensional solution component consists of k random hyperplanes and the set of isolated solutions comprising the intersection of the component with those hyperplanes.

Witness Sets and Straight-line Programs

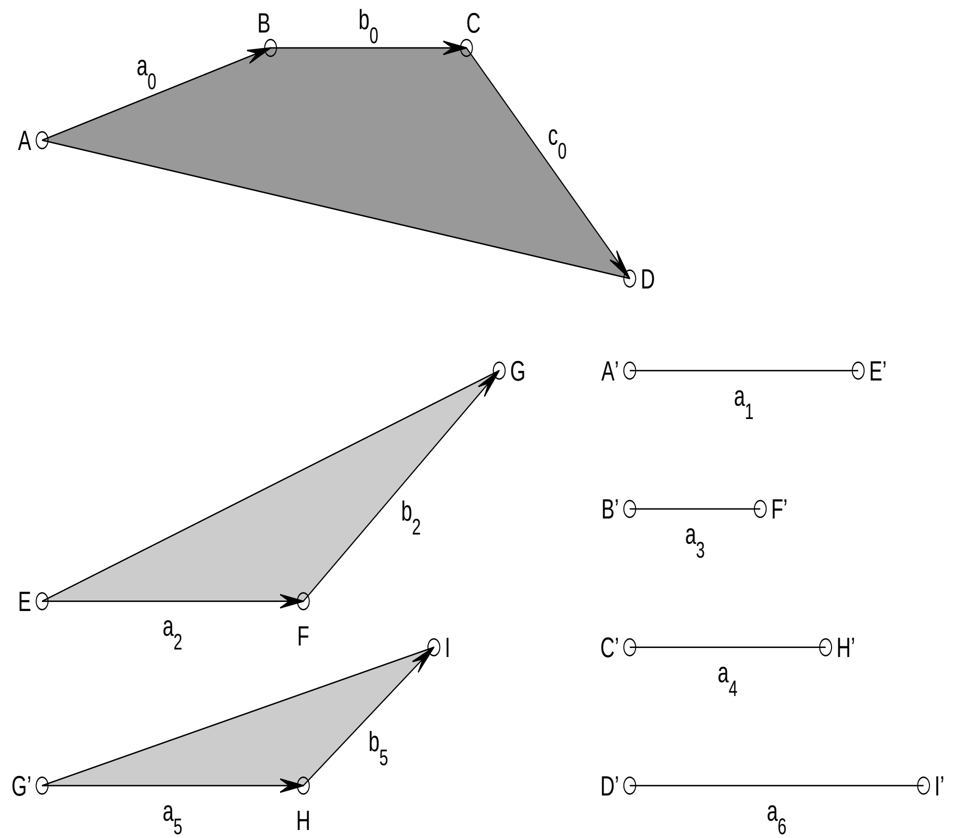
A witness set is data in the usual sense. To make the description of a positive dimensional solution set more complete (or better: more useful), we need to add two *functions* to the witness set:

1. **Functions to evaluate and differentiate the system** to sample the positive dimensional component.

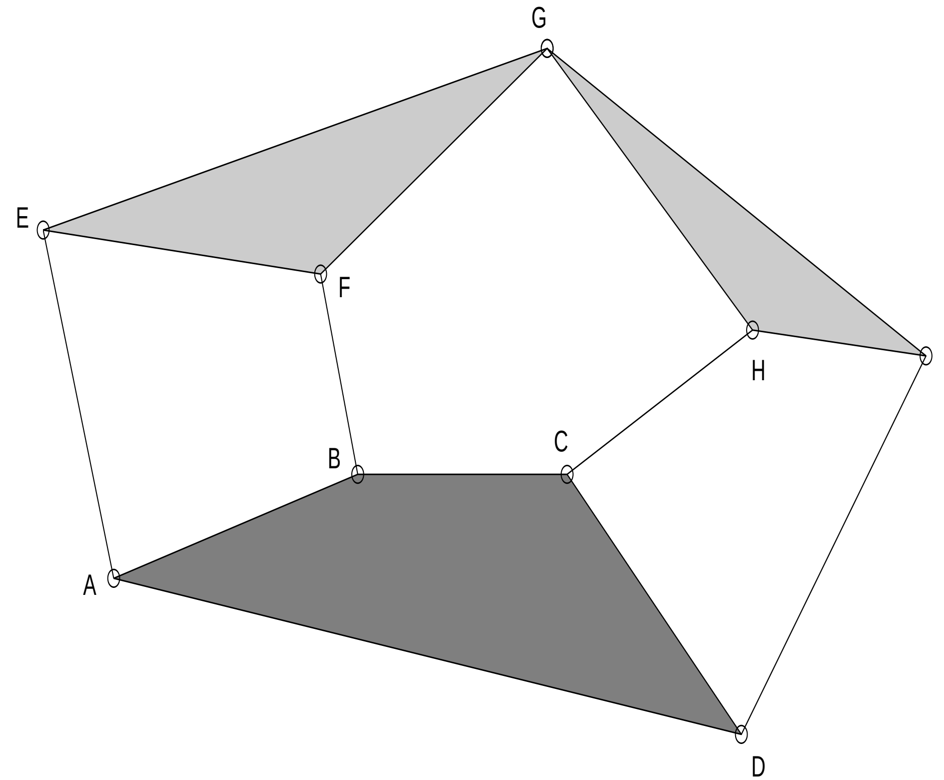
2. **The homotopy membership test** to determine whether a given point belongs to a component.

In some applications – like the determinantal conditions arising in the numerical Schubert calculus – we have better ways to evaluate the system, ways that are numerically better than just plugging in values in a sequence of expanded polynomials.

Application: Assembling a Seven-Bar Mechanism



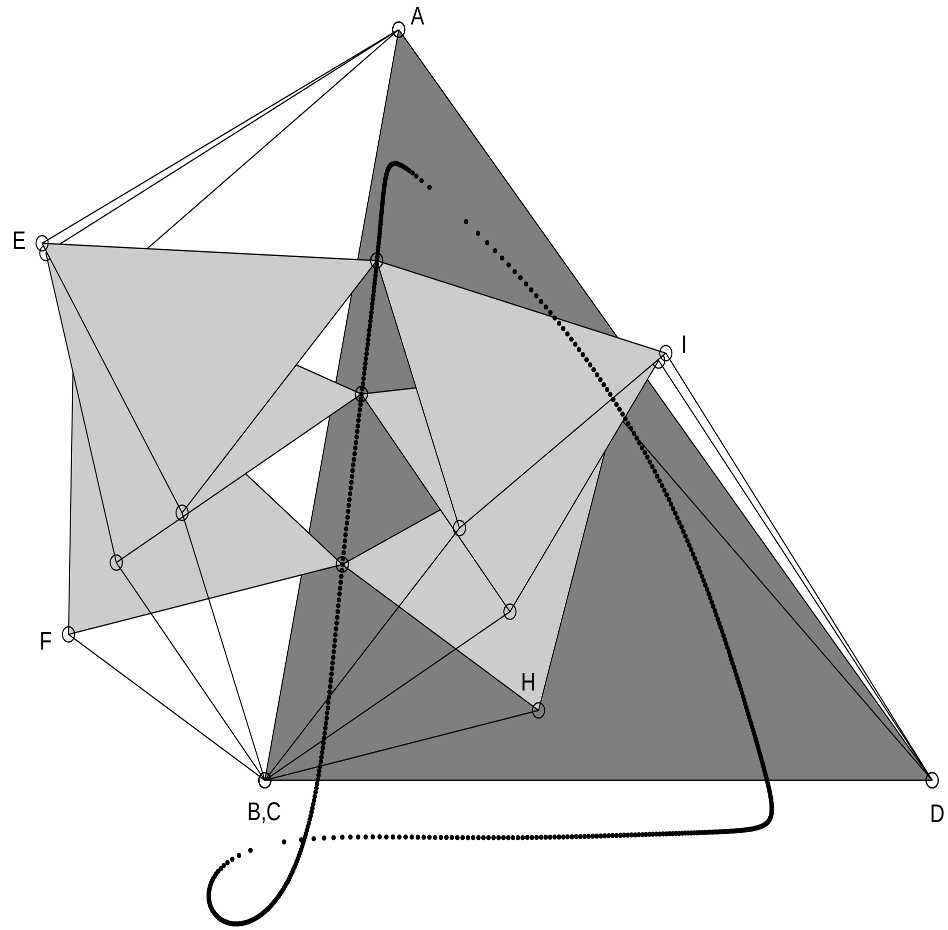
Problem: Find all possible assemblies of these pieces.



One possible assembly

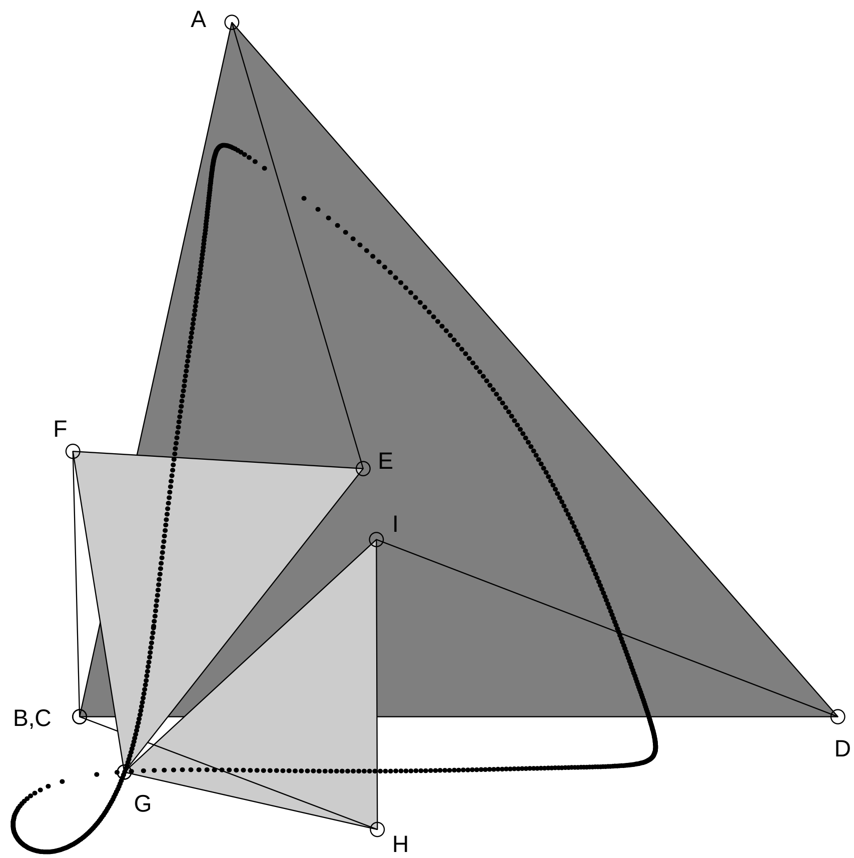
- Generally, 18 solutions. (This example, 8 real, 10 complex.)
- Intersection of two four-bar coupler curves.

A Moving Seven-Bar Mechanism



Roberts cognate 7-bar moves on a degree-6 curve (coupler curve)

AND . . .

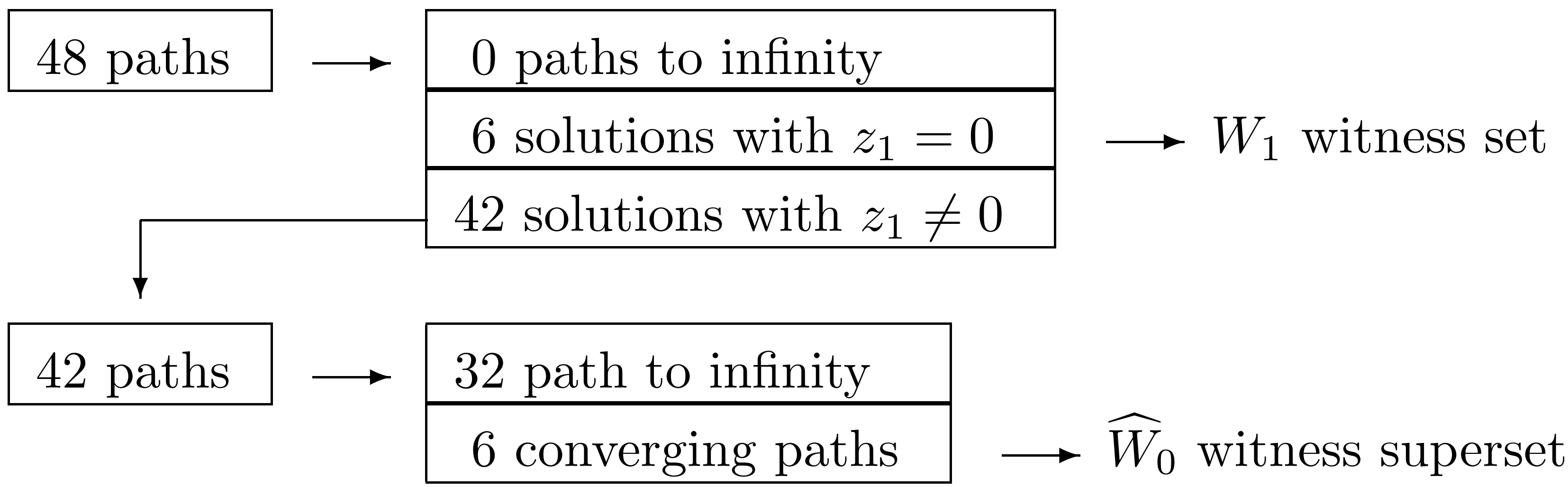


AND . . . has six isolated solutions

- two at each double point of coupler curve
- here, only 1 of 3 double points is real

Computational Summary for 7-bar Mechanism

On input is system of 12 equations in 12 unknowns.



1. solve top embedding — 8.2 cpu seconds $\leftarrow$ bottleneck!
2. run cascade of homotopies — 3.3 cpu seconds
3. filter $\widehat{W}_0$ to $\widehat{W}_0$ — 1.1 cpu seconds

on 1 Ghz PowerBook G4 Mac OS X 10.3.4 with gcc 3.3

Further Reading

We have new methods to compute witness sets faster.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Homotopies for intersecting solution components of polynomial systems.** To appear in *SIAM J. Numer. Anal.*

A.J. Sommese, J. Verschelde, and C.W. Wampler: **An intrinsic homotopy for intersecting algebraic varieties.** Accepted for publication in *J. Complexity*.

If we can intersect varieties, we will be able to solve systems equation by equation, in a similar fashion like in the software of Grégoire Lecerf.

Factoring Solution Components

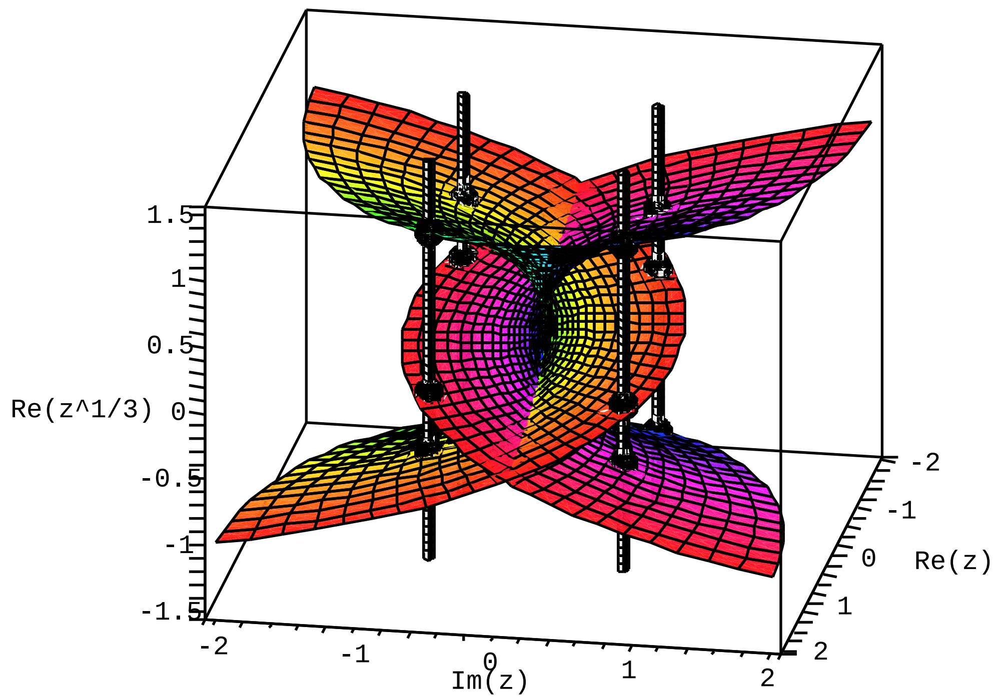
Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system with a positive dimensional solution component, represented by witness set.

coefficients of f known approximately, work with limited precision

Wanted: decompose the component into irreducible factors, for each factor, give its degree and multiplicity.

Symbolic-Numeric issue: essential numerical information (such as degree and multiplicity of each factor), is obtained much faster than the full symbolic representation.

The Riemann Surface of $z^3 - w = 0$:



R.M. Corless and D.J. Jeffrey: **Graphing elementary Riemann surfaces.** *SIGSAM Bulletin* 32(1):11–17, 1998.

Monodromy to Decompose Solution Components

Given: a system $f(\mathbf{x}) = \mathbf{0}$; and $W = (Z, L)$:

$$\text{for all } \mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0} \text{ and } L(\mathbf{w}) = \mathbf{0}.$$

Wanted: partition of Z so that all points in a subset of Z lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

Consider $H(x, y, \theta) = \begin{cases} xy - 1 = 0 \\ x + y = 4e^{i\theta} \end{cases}$ for $\theta \in [0, 2\pi]$.

For $\theta = 0$, we start with two real solutions. When $\theta > 0$, the solutions turn complex, real again at $\theta = \pi$, then complex until at $\theta = 2\pi$. Back at $\theta = 2\pi$, we have again two real solutions, but their order is permuted $\Rightarrow$ irreducible.

Connecting Witness Points

1. For two sets of hyperplanes K and L, and a random $\gamma \in \mathbb{C}$

$$H(\mathbf{x}, t, K, L, \gamma) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ \gamma K(\mathbf{x})(1-t) + L(\mathbf{x})t = \mathbf{0} \end{cases}$$

   We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, K, L, \gamma = \alpha) = \mathbf{0}$.

   For $\beta \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, L, K, \gamma = \beta) = \mathbf{0}$.

   Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other hyperplanes.

Linear Traces – an example

$$\text{Consider} \quad \begin{aligned} f(x, y(x)) &= (y - y_1(x))(y - y_2(x))(y - y_3(x)) \\ &= y^3 - t_1(x)y^2 + t_2(x)y - t_3(x) \end{aligned}$$

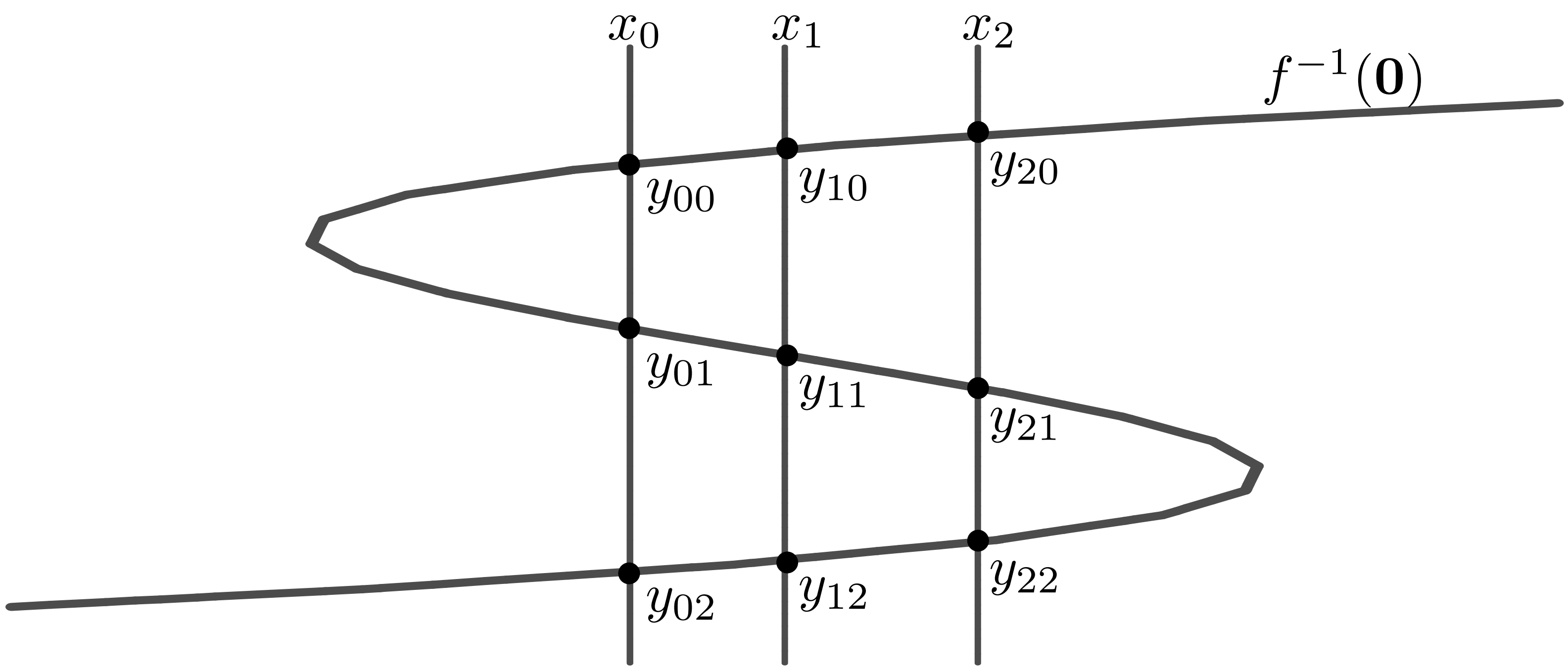
We are interested in the linear trace: $t_1(x) = c_1 x + c_0$.

Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \left\{ \begin{array}{l} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{array} \right. \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.

Linear Traces – example continued



Use $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$ to find the linear trace $t_1(x) = c_0 + c_1 x$.

At $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$: $c_0 + c_1 x_2 = y_{20} + y_{21} + y_{22}$?

Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- For a k-dimensional solution component, it suffices to consider a curve on the component cut out by $k-1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.

Notice: Instead of monodromy, we may enumerate all possible factors and use linear traces to certify. While the complexity of this enumeration is exponential, it works well for low degrees.

Adjacent minors of a general 2-by-$(n+1)$ matrix

$$n = 3 : \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \qquad f(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{cases}$$

P. Diaconis, D. Eisenbud, and B. Sturmfels. **Lattice walks and primary decomposition.** In *Mathematical Essays in Honor of Gian-Carlo Rota*, ed. B.E. Sagan and R.P. Stanley, pages 173–193, Birkhäuser, 1998.

S. Hoşten and J.Shapiro. **Primary decomposition of lattice basis ideals.** *J. Symbolic Computation* 29(4&5): 625–639.

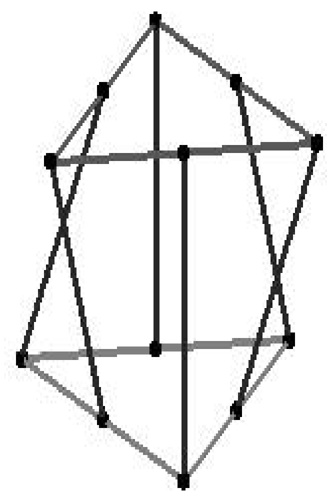
Computational results

n	d	$\#f$	witness set	#loops	factorization
3	8	3	1.4 s	9	6.8 s
4	16	5	4.5 s	3	9.4 s
5	32	8	23.9 s	4	41.6 s
6	64	13	56.4 s	2	1 m 17.0 s
7	128	21	3 m 39.5 s	4	6 m 42.0 s
8	256	34	8 m 22.6 s	5	16 m 54.7 s
9	512	55	25 m 19.2 s	7	1 h 48 m 52.9 s
10	1024	89	1 h 9 m 27.0 s	5	2 h 9 m 5.1 s

on 1 Ghz PowerBook G4 Mac OS X 10.3.4 with gcc 3.3

Application: Architecturally Singular Platforms

Special Griffis-Duffy type



- Base and endplate are equilateral triangles.
- Legs connect vertices to midpoints.

Results of Husty and Karger

Self-motions of Griffis-Duffy type parallel manipulators. In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

The special Griffis-Duffy platforms *move*:

- Case 1: Plates not equal, legs not equal.
  - Curve is degree 20 in Euler parameters.
  - Curve is degree 40 in position.
- Case 2: Plates congruent, legs all equal.
  - Factors are degrees $(4+4)+6+2=16$ in Euler parameters.
  - Factors are degrees $(8+8)+12+4=32$ in position.

Question: *Can we confirm these results numerically?*

Components of Griffis-Duffy Platforms

Solution components by degree

Husty & Karger		SVW	
Euler	Position	Study	Position
General Case			
20	40	28	40
Legs equal, Plates equal			
		6	8
4	8	6	8
4	8	6	8
6	12	6	12
2	4	4	4
16	32	28	40

Griffis-Duffy Platforms: Factorization

Case A: One irreducible component of degree 28 (general case).

Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

user cpu on 800Mhz	Case A	Case B
witness points	1m 12s 480ms	
monodromy breakup	33s 430ms	27s 630ms
Newton interpolation	1h 19m 13s 110ms	2m 34s 50ms

32 decimal places used to interpolate polynomial of degree 28

linear trace	4s 750ms	4s 320ms

Linear traces replace Newton interpolation:

$\Rightarrow$ **time to factor independent of geometry!**

Summary

- We can now deal **numerically** with positive dimensional solution sets – originally bad for Newton & path trackers – by embedding and cascade of homotopies.
- Numerical results can be certified
  + condition numbers
  + root counts
  + linear traces
- Promising performance on realistic applications.

→ **the software demo by Anton Leykin on PHCmaple!**