MATLAB Lecture 9. Linear Programming in MATLAB

One of the most widespread commercial applications of scientific computation is linear programming. This lecture is inspired on [1, Chapter 9].

9.1 An optimization problem

Suppose a farmer has 75 acres to plant and must decide how much farm land to devote to crop x or crop y. Crop x brings in more revenue than y, as can be seen from the profit function $P(x, y)$:

$$P(x, y) = 143x + 60y.$$

In maximizing the profit $P(x, y)$, all should be devoted to crop x, but farming is not that simple. There are additional constraints (other than $x + y \leq 75$). For instance, crop x requires more storage space than crop y, and if the total storage is limited to 4,000, then we have to take the following constraint

$$110x + 30y \leq 4,000$$

into account. The crops just do not grow for free. As the farmer cannot spend more than 15,000 to grow the crops, the third constraint takes the form

$$120x + 210y \leq 15,000.$$

In addition to the trivial constraints $x \geq 0$ and $y \geq 0$, our problem is summarized into:

$$\max_{x,y} 143x + 60y$$
$$\text{subject to} \begin{cases} x + y & \leq & 75 \\ 110x + 30y & \leq & 4,000 \\ 120x + 210y & \leq & 15,000 \\ x \geq 0, y \geq 0 \end{cases}$$

Since there are only two unknowns involved, we may graph all constraints:

```
>> x = 0:80;                          % range for graph
>> y1 = max(75 - x,0);                % x + y <= 75 farm land
>> y2 = max((4000-110*x)/30,0);       % 110x + 30y <= 4000 storage
>> y3 = max((15000 - 120*x)/210,0);   % 120x + 210y <= 15000 expenses
>> ytop = min([y1; y2; y3]);          % array of minima
>> area(x,ytop);                      % filled area plot
```

The shaded area enclosed by the constraints is called the *feasible region*, which is the set of points satisfying all the constraints. If this region is empty, then the problem is said to be *infeasible*, and it has no solution.

The lines of equal profit p are given by $p = 143x + 60y$. If we fix p to, say 50, then all points (x, y) which satisfy $143x + 60y$ yield the same profit 50.

```
>> hold on;
>> [u  v] = meshgrid(0:80, 0:80);
>> contour(u,v,143*u + 60*v);
>> hold off;
```

To find the optimal solution, we look at the lines of equal profit to find the corner of the feasible region which yields the highest profit. This corner can be found at the farthest line of equal profit which still touches the feasible region.

9.2 The command `linprog`

The command `linprog` from the optimization toolbox implements the simplex algorithm to solve a linear programming problem in the form

$$\min_{x} f * x$$
$$\text{subject to } A * x \leq b$$

where f is any vector and the matrix A and vector b define the linear constraints. So our original problem is translated into the format

$$\max_{x,y} 143x + 60y \quad \text{subject to} \quad \left\{ \begin{array}{rcl} x + y & \leq & 75 \\ 110x + 30y & \leq & 4,000 \\ 120x + 210y & \leq & 15,000 \\ x \geq 0, y \geq 0 & & \end{array} \right.$$

$$\min_{x,y} \; -143x - 60y \quad \text{subject to} \quad \left[\begin{array}{rr} 1 & 1 \\ 110 & 30 \\ 120 & 210 \\ -1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] \leq \left[\begin{array}{c} 75 \\ 4,000 \\ 15,000 \\ 0 \\ 0 \end{array} \right]$$

Observe the switching of signs to turn the max into a min and to deal with the $\geq$ constraints. Duality in linear programming is a very important concept, more than just a matter of formatting. The economical interpretation of duality can be simplified into the saying that minimizing the cost of production is equivalent to maximizing the profit.

Now we are ready to solve the problem. First we set up the vectors and matrices:

```
>> f = [-143  -60]
>> A = [120  210; 110  30; 1  1; -1  0; 0  -1]
>> b = [15000;  4000;  75;  0;  0]
```

The optimization toolbox has the command `linprog`:

```
>> linprog(f,A,b)     % optimize
>> -f*ans             % compute profit
```

The latest versions of MATLAB have the command `simlp`, which is very much like `linprog`.

9.3 Assignments

1. Suppose General Motors makes a profit of \$100 on each Chevrolet, \$200 on each Buick, and \$400 on each Cadillac. These cars get 20, 17, and 14 miles a gallon respectively, and it takes respectively 1, 2, and 3 minutes to assemble one Chevrolet, one Buick, and one Cadillac. Assume the company is mandated by the government that the average car has a fuel efficiency of at least 18 miles a gallon. Under these constraints, determine the optimal number of cars, maximing the profit, which can be assembled in one 8-hour day. Give all MATLAB commands and the final result.

2. Consider the following optimization problem:

$$\min_{x,y} 2x + y$$
$$\text{subject to} \quad \left\{ \begin{array}{rcl} x + y & \geq & 4 \\ x + 3y & \geq & 12 \\ x - y & \geq & 0 \\ x \geq 0, & & y \geq 0 \end{array} \right.$$

   (a) Use MATLAB to graph the constraints.

   (b) Use `linprog` to compute the solution.

3. Suppose an invester has a choice between three types of shares. Type A pays 4%, type B pays 6%, and type C pays 9% interest. The investor has \$100,000 available to buy shares and wants to maximize the interest, under the following constraints:
   (i) no more than \$20,000 can be spent on shares of type C;
   (ii) at least \$10,000 of the portfolio should be spent on shares of type A.

   (a) Give the mathematical description of the optimization problem.

   (b) Bring the problem into a form ready to call MATLAB's linprog command.

References

[1] B.R. Hunt, R.L. Lipsman, and J.M. Rosenberg. *A Guide to MATLAB, for beginners and experienced users*. Cambridge University Press, 2001.